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KILLING FORMS ON KERR-NUT-(A)dS SPACES OF EINSTEIN-SASAKI TYPE

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Abstract. In certain scaling limits the higher-dimensional Euclideanized Kerr-NUT-(A)dS metrics are related to the Einstein-Sasaki ones. The complete set of Killing forms of the Einstein-Sasaki spaces are presented. It is pointed out the existence of two additional Killing forms on these spaces associated with the complex volume form of the Calabi-Yau cone manifold. As a concrete example we present the complete set of Killing-Yano tensors on the five-dimensional Einstein-Sasaki $Y(p,q)$ spaces.

1. Introduction

In the last time the properties of higher-dimensional black holes have become of large interest. The most general known higher-dimensional metrics describing rotating black holes with NUT parameters in an asymptotically AdS spacetimes were described in [6]. The general Kerr-NUT-AdS metrics have $(2n-1)$ non-trivial parameters where the spacetime dimension is $(2n+1)$ in the odd-dimensional case and $(2n)$ in the even dimensional case.

In certain scaling limits [12, 13] these metrics are related to the Einstein-Sasaki ones. On the other hand the Einstein-Sasaki geometries have been the object of much attention in connection with the supersymmetric backgrounds relevant to the AdS/CFT correspondence.

The Kerr-NUT-(A)dS metrics possess isometries and hidden symmetries encoded in a series of Killing vectors and Stäckel-Killing tensors [6]. These symmetries are connected with a set of conserved quantities which are functionally independent, in involution, and guarantee the complete integrability of the geodesic motions [11, 15, 18].

In the case of Sasaki spaces the hidden symmetries are derived from the characteristic Sasakian one-form and a tower of Killing-Yano and conformal Killing-Yano tensors can be constructed [12].

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The main purposes of this paper is to point out the special case of the higher dimensional Kerr-NUT-(A)dS metrics which are related to the Einstein-Sasaki ones. In this instance there are two additional Killing-Yano tensors taking into account that the metric cone is Calabi-Yau [5,17]. These two exceptional Killing forms can be also described using the Killing spinors of an Einstein-Sasaki manifold [2].

In Section 2 we review some basic facts about the Stäckel-Killing and Killing-Yano tensors. In Section 3 we present the close connection between Einstein-Sasaki and Einstein-Kähler geometries. In the next Section we discuss the Killing forms on Einstein-Sasaki spaces which proceed from Euclideanized Kerr-NUT-(A)dS metrics in certain scaling limits. We identity two new Killing forms associated with the complex volume form of the cone manifold. In Section 5 we restrict to the five-dimensional $Y(p, q)$ manifolds and present the complete set of Killing forms. Finally we give our conclusions in Section 6.

2. Killing Tensors

Let (M, g) be a n-dimensional differentiable manifold equipped with a (pseudo)-Riemannian metric

$$\mathrm{d}s^2 = g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j. \tag{1}$$

Definition 1. *A vector field X on M is said to be a Killing vector if the Lie derivative with respect to X of the metric g vanishes*

$$\mathcal{L}_X g = 0. \tag{2}$$

In coordinates this means that

$$X_{(i;j)} = 0 \tag{3}$$

where a semicolon precedes an index i of covariant differentiation associated with the Levi-Civita connection and a round bracket denotes symmetrization over the indices within.

A symmetric generalization of the Killing vectors is that of Stäckel-Killing tensors.

Definition 2. *A symmetric tensor $K_{(i_1 \cdots i_r;j)}$ of rank $r > 1$ satisfying the generalized Killing equation*

$$K_{(i_1 \cdots i_r;j)} = 0 \tag{4}$$

is called a Stäckel-Killing tensor.

From the generalized Killing equation (4) we get that for any geodesic γ with tangent vector $\dot{\gamma}^i$

$$Q_K = K_{i_1 \cdots i_r} \dot{\gamma}^{i_1} \cdots \dot{\gamma}^{i_r} \tag{5}$$

is constant along γ.

Antisymmetric Killing-Yano tensors [21] are a different generalization of the Killing vectors.

Definition 3. *A Killing-Yano tensor is a p-form $f(p \leq n)$ which satisfies*

$$\nabla_X f = \frac{1}{p+1} X \lrcorner \, \mathrm{d}\omega \tag{6}$$

for any vector field X, where 'hook' operator $\lrcorner$ is dual to the wedge product.

This definition is equivalent with the property that $\omega_{i_1 \cdots i_p; j}$ is totally antisymmetric or, in components

$$\omega_{i_1 \cdots i_{p-1}(i_p; j)} = 0. \tag{7}$$

It was observed that Killing-Yano tensors generate *non-standard supersymmetries* in the dynamics of pseudo-classical spinning particles being the natural objects to be coupled with the fermionic degrees of freedom. At the quantum level, Killing-Yano tensors generate conserved *non-standard Dirac operators* which commute with the standard one.

These two generalizations of the Killing vectors could be related. Given two Killing-Yano tensors $\omega^{i_1, \ldots, i_k}$ and $\sigma^{i_1, \ldots, i_k}$ it is possible to associate with them a Stäckel-Killing tensor of rank two

$$K_{ij}^{(\omega, \sigma)} = \omega_{i i_2 \ldots i_k} \sigma_j^{\ i_2 \ldots i_k} + \sigma_{i i_2 \ldots i_k} \omega_j^{\ i_2 \ldots i_k}. \tag{8}$$

Therefore a method to generate higher order integrals of motion is to identify the complete set of Killing-Yano tensors. The existence of enough integrals of motion leads to complete integrability or even superintegrability of the mechanical system when the number of functionally independent constants of motion is larger than its number of degrees of freedom.

The conformal extension of the Killing vectors is given by the conformal Stäckel-Killing and conformal Killing-Yano tensors.

Definition 4. *A conformal Killing-Yano tensor of rank p is a p-form ω which satisfies*

$$\nabla_X \omega = \frac{1}{p+1} X \lrcorner \, \mathrm{d}\omega - \frac{1}{n-p+1} X^* \wedge \mathrm{d}^* \omega \tag{9}$$

for any vector field X on M, where X^* is the one-form dual to the vector field X with respect to the metric g, and d^* is the adjoint of the exterior differential d.

If ω is co-closed in (9), then we obtain the definition of a Killing-Yano tensor [21]. We mention that Killing-Yano tensors are also called Yano tensors or Killing forms, and conformal Killing-Yano tensors are sometimes referred as conformal Yano tensors, conformal Killing forms or twistor forms.

Definition 5. *A Killing form ω is said to be a special Killing form if it satisfies for some constant c the additional equation*

$$\nabla_X(\mathrm{d}\omega) = cX^* \wedge \omega \tag{10}$$

for any vector field X on M.

3. Kähler and Sasakian Manifolds

The Sasakian geometry, defined on an odd dimensional manifold, is the closest possible analogue of the Kähler geometry of even dimension.

There are several equivalent definitions of the Sasakian structure and for describing the problems of interest here it is more convenient to use the following definition:

Definition 6. *A compact Riemannian manifold (S, g) is Sasakian if and only if the metric cone*

$$C(S) = \mathbb{R}_{>0} \times S, \qquad \bar{g} = \mathrm{d}r^2 + r^2 g \tag{11}$$

is Kähler.

A Sasakian manifold inherits a number of geometrical structures from the Kähler structure of its cone. Let us note that if the odd dimension of the Sasaki space is $(2n+1)$, the Kähler cone has the complex dimension $(n+1)$. In local holomorphic coordinates $(z^1, ..., z^{n+1})$ the associated Kähler form Ω can be written as

$$\Omega = \mathrm{i}g_{j\bar{k}}\,\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k = \sum X_j^* \wedge Y_j^* = \frac{\mathrm{i}}{2}\sum Z_j^* \wedge \bar{Z}_j^* \tag{12}$$

where $(X_1, Y_1, ..., X_{n+1}, Y_{n+1})$ is an adapted local orthonormal field (i.e., such that $Y_j = JX_j$), and $(Z_j, \bar{Z}_j)$ is the associated complex frame given by

$$Z_j = \frac{1}{2}(X_j - \mathrm{i}Y_j), \qquad \bar{Z}_j = \frac{1}{2}(X_j + \mathrm{i}Y_j). \tag{13}$$

There is an intimate connection between its Kähler form and the volume form (which is just the Riemannian volume form determined by the metric) as follows

$$\mathrm{d}\mathcal{V} = \frac{1}{(n+1)!}\,\Omega^{n+1} \tag{14}$$

where $\mathrm{d}\mathcal{V}$ denotes the volume form of $C(S)$, Ω^{n+1} is the wedge product of Ω with itself $n+1$ times [1]. Hence the volume form is a real $(n+1, n+1)$-form on $C(S)$. On the other hand, if the volume of a Kähler manifold is written as

$$\mathrm{d}\mathcal{V} = \frac{\mathrm{i}^{n+1}}{2^{n+1}}(-1)^{n(n+1)/2}\,\mathrm{d}V \wedge \overline{\mathrm{d}V} \tag{15}$$

then $\mathrm{d}V$ is the complex volume holomorphic $(n+1, 0)$ form of $C(S)$.

An Einstein-Sasaki manifold is a Riemannian manifold (S, g) that is both Sasaki and Einstein, i.e., a Sasakian manifold satisfying the Einstein condition

$$\mathrm{Ric}_g = \lambda g \tag{16}$$

for some real constant λ, where Ric_g denotes the Ricci tensor of g. Einstein manifolds with $\lambda = 0$ are called Ricci-flat manifolds. Similarly, an Einstein-Kähler manifold is a Riemannian manifold that is both Kähler and Einstein. An important subclass of Einstein-Kähler manifolds are the Calabi-Yau manifolds which are Kähler and Ricci-flat.

A simple calculation shows that

Corollary 7. *A Sasakian metric g is Einstein with*

$$\mathrm{Ric}_g = 2ng \tag{17}$$

if and only if the cone metric $\bar{g}$ is Ricci flat, i.e., Calabi-Yau.

Suppose we have an Einstein-Sasaki metric g_{ES} on a manifold S_{2n+1} of odd dimension $2n+1$. An Einstein-Sasaki manifold can always be written as a fibration over an Einstein-Kähler manifold M_{2n} with the metric g_{EK} twisted by the overall U(1) part of the connection [10]

$$\mathrm{d}s^2_{ES} = (\mathrm{d}\psi_n + 2A)^2 + \mathrm{d}s^2_{EK} \tag{18}$$

where $\mathrm{d}A$ is given as the Kähler form of the Einstein-Kähler base. This can be easily seen when we write the metric of the cone manifold $M_{2n+2} = C(S_{2n+1})$ as

$$\mathrm{d}s^2_{\text{cone}} = \mathrm{d}r^2 + r^2\,\mathrm{d}s^2_{ES} = \mathrm{d}r^2 + r^2\left((\mathrm{d}\psi_n + 2A)^2 + \mathrm{d}s^2_{EK}\right). \tag{19}$$

The cone manifold is Calabi-Yau and its Kähler form can be written as

$$\Omega_{\text{cone}} = r\,\mathrm{d}r \wedge (\mathrm{d}\psi_n + 2A) + r^2\Omega_{EK} \tag{20}$$

and the Kähler condition $\mathrm{d}\Omega_{\text{cone}} = 0$ implies

$$\mathrm{d}A = \Omega_{EK} \tag{21}$$

where Ω_{EK} is Kähler form of the Einstein-Kähler base manifold M_{2n}.

The Sasakian one-form of the Einstein-Sasaki metric is

$$\eta = 2A + \mathrm{d}\psi_n \tag{22}$$

which is a special unit-norm Killing one-form obeying for all vector fields X [17]

$$\nabla_X \eta = \frac{1}{2} X \lrcorner \, \mathrm{d}\eta, \qquad \nabla_X(\mathrm{d}\eta) = -2X^* \wedge \eta. \tag{23}$$

4. Killing Forms on Kerr-NUT-(A)dS Space in a Certain Scaling Limit

In recent time new Einstein-Sasaki spaces have been constructed by taking certain BPS [7] or scaling limits [12, 13] of the Euclideanized Kerr-de Sitter metrics.

In even dimensions, performing the scaling limit on the Euclideanized Kerr-NUT-(A)dS spaces, the Einstein-Kähler metric g_{EK} and the Kähler potential A are [12]

$$g_{EK} = \frac{\Delta_\mu \, \mathrm{d}x_\mu^2}{X_\mu} + \frac{X_\mu}{\Delta_\mu} \left(\sum_{j=0}^{n-1} \sigma_\mu^{(j)} \, \mathrm{d}\psi_j \right)^2 \tag{24}$$

with

$$X_\mu = -4 \prod_{i=1}^{n+1} (\alpha_i - x_\mu) - 2b_\mu, \qquad A = \sum_{k=0}^{n-1} \sigma^{(k+1)} \mathrm{d}\psi_k \tag{25}$$

and

$$\Delta_\mu = \prod_{\nu \neq \mu} (x_\nu - x_\mu), \quad \sigma_\mu^{(k)} = \sum_{\substack{\nu_1 < \cdots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1} \dots x_{\nu_k}, \quad \sigma^{(k)} = \sum_{\nu_1 < \cdots < \nu_k} x_{\nu_1} \dots x_{\nu_k}. \tag{26}$$

Here, coordinates $x_\mu \, (\mu = 1, \dots, n)$ stands for the Wick rotated radial coordinate and longitudinal angles and the Killing coordinates $\psi_k \, (k = 0, \dots, n-1)$ denote time and azimuthal angles with Killing vectors $\xi^{(k)} = \partial_{\psi_k}$. Also α_i, $i = 1, \dots, n+1$ and b_μ are constants related to the cosmological constant, angular momenta, mass and NUT parameters [6].

We mention that in the case of odd-dimensional Kerr-NUT-(A)dS spaces the appropriate scaling limit leads to the same Einstein-Sasaki metric (18).

The hidden symmetries of the Sasaki manifold M_{2n+1} are described by the special Killing $(2k+1)-$forms [17]

$$\Psi_k = \eta \wedge (\mathrm{d}\eta)^k, \qquad k = 0, 1, \dots, n-1. \tag{27}$$

In [17] Semmelmann has proved that special Killing forms on a Riemannian manifold M are exactly those forms which translate into parallel forms on the metric cone $C(M)$. Therefore, the metric cone being either flat or irreducible, the problem of finding all special Killing forms is reduced to a holonomy problem [4]. In the case of holonomy U(n+1), i.e., the cone $M_{2n+2} = C(M_{2n+1})$ is Kähler, or equivalently M_{2n+1} is Sasaki, it follows that all special Killing forms are spanned by the forms Ψ_k (27). Besides these Killing forms, there are n closed conformal Killing forms (also called $*$-Killing forms)

$$\Phi_k = (\mathrm{d}\eta)^k, \qquad k = 1, \ldots, n. \tag{28}$$

In the case of holonomy SU(n+1), i.e., the cone $M_{2n+2} = C(M_{2n+1})$ is Kähler and Ricci-flat, or equivalently M_{2n+1} is Einstein-Sasaki, there are *two additional* Killing forms of degree n on the manifold M_{2n+1}.

In order to write explicitly these additional Killing forms, we introduce the complex vierbeins on the Einstein-Kähler manifold M_{2n}. First of all we shall write the metric g_{EK} in the form

$$g_{EK} = o^{\hat{\mu}} o^{\hat{\mu}} + \tilde{o}^{\hat{\mu}} \tilde{o}^{\hat{\mu}} \tag{29}$$

and the Kähler two-form

$$\Omega = \mathrm{d}A = o^{\hat{\mu}} \wedge \tilde{o}^{\hat{\mu}} \tag{30}$$

where

$$o^{\hat{\mu}} = \sqrt{\frac{\Delta_\mu}{X_\mu(x_\mu)}}\, \mathrm{d}x_\mu, \qquad \tilde{o}^{\hat{\mu}} = \sqrt{\frac{X_\mu(x_\mu)}{\Delta_\mu}} \sum_{j=0}^{n-1} \sigma_\mu^{(j)}\, \mathrm{d}\psi_j. \tag{31}$$

We introduce the following complex vierbeins on Einstein-Kähler manifold M_{2n} [20]

$$\zeta_\mu = o^{\hat{\mu}} + \mathrm{i}\tilde{o}^{\hat{\mu}}, \qquad \mu = 1, \ldots, n. \tag{32}$$

On the Calabi-Yau cone manifold M_{2n+2} we take $\Lambda_\mu = r\zeta_\mu$ for $\mu = 1, \ldots, n$ and

$$\Lambda_{n+1} = \mathrm{d}r + \mathrm{i}r\eta. \tag{33}$$

The standard complex volume form of the Calabi-Yau cone manifold M_{2n+2} is [20]

$$\mathrm{d}V = \Lambda_1 \wedge \Lambda_2 \wedge \ldots \wedge \Lambda_{n+1}. \tag{34}$$

The real Killing forms are given the real respectively the imaginary part of the complex volume form.

The additional Killing forms on the Einstein-Sasaki spaces are connected with the parallel forms on the metric cone. For this purpose we make use of the fact that for any p-form ω^M on the space M_{2n+1} we can define an associated $(p+1)$-form ω^C on the cone $C(M_{2n+1})$

$$\omega^C := r^p \,\mathrm{d}r \wedge \omega^M + \frac{r^{p+1}}{p+1}\,\mathrm{d}\omega^M. \tag{35}$$

Moreover ω^C is parallel if and only if ω^M is a special Killing form (10) with constant $c = -(p+1)$ [17]. The one-to-one-correspondence between special Killing p-forms on M_{2n+1} and parallel $(p+1)$-forms on the metric cone $C(M_{2n+1})$ allows us to describe the additional Killing forms on Einstein-Sasaki spaces.

Therefore in order to find the additional Killing forms on the manifold M_{2n+1} we must identify the ω^M form in the complex volume form of the Calabi-Yau cone. An explicit example is presented in the next Section.

5. $Y(p,q)$ Manifolds

Recently an infinite family $Y(p,q)$ of Einstein-Sasaki metrics on $S^2\times S^3$ have been discovered [7–9, 13]. Such manifolds provide supersymmetric backgrounds relevant to the AdS/CFT correspondence. The total space $Y(p,q)$ of an S^1-fibration over $S^2\times S^2$ with relative prime winding numbers p and q is topologically $S^2\times S^3$.

The starting point is the explicit local metric of the five-dimensional $Y(p,q)$ manifold given by the line element [8, 9, 16]

$$\begin{aligned}\mathrm{d}s^2_{ES} = {}& \frac{1-y}{6}(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2) + \frac{1}{p(y)}\,\mathrm{d}y^2 + \frac{1}{36}p(y)(\mathrm{d}\beta+\cos\theta\,\mathrm{d}\phi)^2\\ & + \frac{1}{9}[\mathrm{d}\psi' - \cos\theta\,\mathrm{d}\phi + y(\mathrm{d}\beta+\cos\theta\,\mathrm{d}\phi)]^2\end{aligned} \tag{36}$$

with

$$p(y) = \frac{2(a-3y^2+2y^3)}{1-y} \tag{37}$$

and a is a constant.

From (22) in the case of the $Y(p,q)$ space the Sasakian one-form is

$$\eta = \frac{1}{3}\,\mathrm{d}\psi' + 2A \tag{38}$$

with

$$A = \frac{1}{6}\left(-\cos\theta\,\mathrm{d}\phi + y(\mathrm{d}\beta+\cos\theta\,\mathrm{d}\phi)\right) \tag{39}$$

connected with local Kähler form Ω_{EK} as in (21).

The form of the metric (36) with the one-form (38) is the standard one for a locally Einstein-Sasaki metric with $\frac{\partial}{\partial\psi'}$ the Reeb vector field. Note also that the holomorphic $(2,0)$-form of the Einstein-Kähler base manifold is

$$\mathrm{d}V_{EK} = \sqrt{\frac{1-y}{6p(y)}}(\mathrm{d}\theta + \mathrm{i}\sin\theta\,\mathrm{d}\phi)\wedge\Big(\mathrm{d}y + \mathrm{i}\frac{p(y)}{6}(\mathrm{d}\beta + \cos\theta\,\mathrm{d}\phi)\Big). \qquad (40)$$

From the isometries SU(2) × U(1) × U(1) the momenta P_ϕ, P_ψ, P_α and the Hamiltonian describing the geodesic motions are conserved [3, 16]. P_ϕ is the third component of the SU(2) angular momentum, while P_ψ and P_α are associated with the U(1) factors. Additionally, the total SU(2) angular momentum given by

$$J^2 = P_\theta^2 + \frac{1}{\sin^2\theta}(P_\phi + \cos\theta P_\psi)^2 + P_\Psi^2 \qquad (41)$$

is also conserved.

In what follows we are looking for further conserved quantities specific for motions in Einstein-Sasaki spaces. First of all, according to (27), the Killing one-form η (38) together with the third rank form

$$\begin{aligned}\Psi &= \eta\wedge\mathrm{d}\eta\\ &= \frac{1}{9}\big((1-y)\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi' + \mathrm{d}y\wedge\mathrm{d}\beta\wedge\mathrm{d}\psi' + \cos\theta\,\mathrm{d}y\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi'\\ &\quad - \cos\theta\,\mathrm{d}y\wedge\mathrm{d}\beta\wedge\mathrm{d}\phi + (1-y)y\sin\theta\,\mathrm{d}\beta\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\big)\end{aligned} \qquad (42)$$

are special Killing forms (10) with constants $c=-2$ and $c=-4$ respectively. Let us note also that Φ_k (28) with $k=1,2$ are closed conformal Killing forms.

On the Calabi-Yau manifold $C(M_{2n+1})$ the Kähler form (20) with the Sasakian one-form (38) is

$$\begin{aligned}\Omega_{\mathrm{cone}} &= r^2\frac{1-y}{6}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi + \frac{r^2}{6}\mathrm{d}y\wedge(\mathrm{d}\beta+\cos\theta\,\mathrm{d}\phi)\\ &\quad + \frac{1}{3}r\,\mathrm{d}r\wedge\big(y\,\mathrm{d}\beta + \mathrm{d}\psi' - (1-y)\cos\theta\,\mathrm{d}\phi\big)\,.\end{aligned} \qquad (43)$$

The complex volume holomorphic $(3,0)$ form on the metric cone is [14]

$$\begin{aligned}\mathrm{d}V_{\mathrm{cone}} &= \mathrm{e}^{\mathrm{i}\psi'}r^2\,\mathrm{d}V_{EK}\wedge(\mathrm{d}r + \mathrm{i}r\eta)\\ &= \mathrm{e}^{\mathrm{i}\psi'}r^2\sqrt{\frac{1-y}{6p(y)}}(\mathrm{d}\theta + \mathrm{i}\sin\theta\,\mathrm{d}\phi)\\ &\quad\wedge\left(\mathrm{d}y + \mathrm{i}\frac{p(y)}{6}(\mathrm{d}\beta+\cos\theta\,\mathrm{d}\phi)\right)\\ &\quad\wedge\Big(\mathrm{d}r + \mathrm{i}\frac{r}{3}(y\,\mathrm{d}\beta + \mathrm{d}\psi' - (1-y)\cos\theta\,\mathrm{d}\phi)\Big)\,.\end{aligned} \qquad (44)$$

Extracting from the complex volume (44) the form ω^M on the Einstein-Sasaki space according to (35) for $p = 2$ we get the following additional Killing 2-forms of the $Y(p,q)$ spaces written as real forms [19]:

$$\begin{aligned}
\Xi = \Re\,\omega^M = &\sqrt{\frac{1-y}{6\,p(y)}}\\
&\times\Bigg(\cos\psi'\Big(-\mathrm{d}y\wedge\mathrm{d}\theta + \frac{p(y)}{6}\sin\theta\,\mathrm{d}\beta\wedge\mathrm{d}\phi\Big)\\
&-\sin\psi'\Big(-\sin\theta\,\mathrm{d}y\wedge\mathrm{d}\phi - \frac{p(y)}{6}\,\mathrm{d}\beta\wedge\mathrm{d}\theta + \frac{p(y)}{6}\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi\Big)\Bigg)\\
\Upsilon = \Im\,\omega^M = &\sqrt{\frac{1-y}{6\,p(y)}}\\
&\times\Bigg(\cos\psi'\Big(-\sin\theta\,\mathrm{d}y\wedge\mathrm{d}\phi - \frac{p(y)}{6}\,\mathrm{d}\beta\wedge\mathrm{d}\theta + \frac{p(y)}{6}\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi\Big)\\
&+\sin\psi'\Big(-\mathrm{d}y\wedge\mathrm{d}\theta + \frac{p(y)}{6}\sin\theta\,\mathrm{d}\beta\wedge\mathrm{d}\phi\Big)\Bigg).
\end{aligned} \tag{45}$$

The Stäckel-Killing tensors associated with the Killing forms $\Psi\,,\Xi\,,\Upsilon$ are constructed as in (8). The list of the non vanishing components of these Stäckel-Killing tensors is quite long and will be given elsewhere. Together with the Killing vectors P_ϕ, P_ψ, P_α and the total angular momentum J^2 (41) these Stäckel-Killing tensors provide the superintegrability of the $Y(p,q)$ geometries.

6. Concluding Remarks

In general it is a hard task to find solutions of the Killing-Yano equation (6) or conformal Killing-Yano equation (9). However in the case of spaces endowed with special geometrical structures, the existence of Killing forms and their explicit construction is granted.

In this paper we presented the complete set of Killing forms on Einstein-Sasaki spaces associated with Euclideanized Kerr-NUT-(A)dS spaces in a certain scaling limit. The multitude of Killing-Yano and Stäckel-Killing tensors makes possible a complete integrability of geodesic equations.

As an exemplification of the general framework we have presented the complete set of Killing forms on five-dimensional Einstein-Sasaki $Y(p,q)$ spaces. The multitude of Stäckel-Killing tensors associated with these Killing forms implies the superintegrability of the geodesic motions.

These remarkable properties of the Killing forms offer new perspectives in the investigation of the supersymmetries, separability of Hamilton-Jacobi, Klein-Gordon and Dirac equations on Einstein-Sasaki spaces.

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