JGSP **28** (2012) 105–112

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

# CLAIRAUT'S THEOREM IN MINKOWSKI SPACE

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Communicated by Gregory L. Naber

**Abstract.** We consider some aspects of the geometry of surfaces of revolution in three-dimensional Minkowski space. First, we show that Clairaut's theorem, which gives a well-known characterization of geodesics on a surface of revolution in Euclidean space, has an analogous result in three-dimensional Minkowski space. We then illustrate the significant differences between the two cases which arise in spite of their formal similarity.

## 1. Introduction

The relationship between Euclidean and Minkowskian geometry has many intriguing aspects, one of which is the manner in which formal similarity can co-exist with significant geometric disparity. There has been considerable interest in the comparison of these two geometries, as we see from the lecture notes of López [3]. In particular, aspects of surfaces of revolution in Minkowski space have been considered, *e.g.* in [2]. There is an elegant characterization of godesics on surfaces of revolution due to Clairaut–see, for example, Pressley's differential geometry textbook [7], which is a valuable tool in the study of such surfaces in the Euclidean context [1, 4–6]. Our purpose here is to see how this characterization carries over to Minkowski space, and how it can be used to investigate the difference between the two situations.

## 2. Euclidean Geometry

We begin by recalling the situation in Euclidean space, the better to see how closely the situation in Minkowski space parallels this one.

Let $\Sigma$ be a surface of revolution, obtained by rotating the profile curve $x = \rho(u)$, $y = 0$, $z = h(u)$ about the axis of symmetry, where we assume that $\rho > 0$ and

doi: 10.7546/jgsp-28-2012-105-112

that $\rho'(u)^2 + h'(u)^2 = 1$. Then $\Sigma$ is parameterized by

$$\mathbf{x}(u,v) = \begin{bmatrix} \rho(u)\cos(v) \\ \rho(u)\sin(v) \\ h(u) \end{bmatrix}$$

so that

$$\mathbf{x}_u = \begin{bmatrix} \rho'(u)\cos(v) \\ \rho'(u)\sin(v) \\ h'(u) \end{bmatrix} = \mathbf{n}_u$$

where $\mathbf{n}_u$ is the unit vector pointing along meridians of $\Sigma$ and

$$\mathbf{x}_v = \begin{bmatrix} -\rho(u)\sin(v) \\ \rho(u)\cos(v) \\ 0 \end{bmatrix} = \rho \begin{bmatrix} -\sin(v) \\ \cos(v) \\ 0 \end{bmatrix} = \rho\mathbf{n}_v$$

where $\mathbf{n}_v$ is the unit vector pointing along parallels of $\Sigma$.

The first fundamental form of $\Sigma$ is

$$I = \begin{bmatrix} \mathbf{x}_u.\mathbf{x}_u & \mathbf{x}_u.\mathbf{x}_v \\ \mathbf{x}_v.\mathbf{x}_u & \mathbf{x}_v.\mathbf{x}_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \rho(u)^2 \end{bmatrix}.$$

Since $\mathbf{n}_u.\mathbf{n}_v = 0$, the two form an orthonormal basis, and any unit vector $\mathbf{t}$ tangent to $\Sigma$ is of the form $\mathbf{n}_u\cos\theta + \mathbf{n}_v\sin\theta$ where $\theta$ is the angle between $\mathbf{t}$ and $\mathbf{n}_u$.

First let $\gamma(s)$ be a geodesic on $\Sigma$, parameterized by arc length, and given by $u(s)$ and $v(s)$, so that

$$\gamma(s) = \begin{bmatrix} \rho(u(s))\cos(v(s)) \\ \rho(u(s))\sin(v(s)) \\ h(u(s)) \end{bmatrix}.$$

From the first fundamental form, we have the Lagrangian

$$\dot{u}^2 + \rho^2\dot{v}^2$$

and so the Euler-Lagrange equations, whose solutions are arc-length parameterised geodesics, are

$$\ddot{u} = \rho\rho'\dot{v}^2$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\rho^2\dot{v}\right) = 0.$$

But we also have

$$\dot{\gamma} = \dot{u}\mathbf{x}_u + \dot{v}\mathbf{x}_v = \dot{u}\mathbf{n}_u + \rho\dot{v}\mathbf{n}_v = \mathbf{n}_u\cos\theta + \mathbf{n}_v\sin\theta$$

where $\theta$ is the angle between $\dot{\gamma}$ and a meridian.

Equating the components of $\mathbf{n}_v$ in the latter two expressions, we see that $\rho\dot{v} = \sin\theta$, so that $\rho^2\dot{v} = \rho\sin\theta$, and hence the second Euler-Lagrange equation is equivalent to $\rho^2\sin\theta$ being a constant along $\gamma$.

Conversely, suppose that $\gamma$ is a unit speed curve with $\rho\sin\theta$ constant, so that the second Euler-Lagrange equation is satisfied, and with $\dot{u} \neq 0$. Then since

$$\dot{u}^2 + \rho^2\dot{v}^2 = 1$$

we have

$$\dot{u}\ddot{u} + \rho\rho'\dot{u}\dot{v}^2 + \rho^2\dot{v}\ddot{v} = 0$$

Substituting into this the second Euler-Lagrange equation

$$2\rho\rho'\dot{u}\dot{v} + \rho^2\ddot{v} = 0$$

and dividing by $\dot{u}$ yields

$$\ddot{u} = \rho\rho'\dot{v}^2$$

which is the first Euler-Lagrange equation.

We see, then, that curve which is parameterized by arc length and has $\dot{u} \neq 0$ is a geodesic if and only if $\rho^2\sin(\theta)$ is constant. This establishes Clairaut's theorem, and we observe in passing that all meridians are geodesics.

## 3. Clairaut's Theorem in Minkowski Space

We now consider the situation of a surface generated by a curve rotated about the $t$-axis in Minkowski space, which we take to have the usual coordinates $(x, y, t)$ with metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2.$$

So let the generating curve be given by $x = \rho(u) > 0$, $y = 0$, $t = h(u)$, and we assume that $\rho'(u)^2 - h'(u)^2 = -1$, so that the curve is timelike, and parameterised by proper time.

We then find that the surface $\Sigma$ is parameterised by

$$\mathbf{x}(u,v) = \begin{bmatrix} \rho(u)\cos(v) \\ \rho(u)\sin(v) \\ h(u) \end{bmatrix}$$

giving

$$\mathbf{x}_u = \begin{bmatrix} \rho'(u)\cos(v) \\ \rho'(u)\sin(v) \\ h'(u) \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_v = \begin{bmatrix} -\rho(u)\sin(v) \\ \rho(u)\cos(v) \\ 0 \end{bmatrix}$$

and resulting in the first fundamental form

$$I = \begin{bmatrix} \mathbf{x}_u.\mathbf{x}_u & \mathbf{x}_u.\mathbf{x}_v \\ \mathbf{x}_u.\mathbf{x}_v & \mathbf{x}_v.\mathbf{x}_v \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \rho(u)^2 \end{bmatrix}$$

which gives a Lorentz metric on $\Sigma$.

We also note that $\mathbf{x}_u = \mathbf{n}_u$ is a unit timelike vector pointing along the meridians, while $\mathbf{x}_v = \rho\mathbf{n}_v$, where $\mathbf{n}_v$ is a unit spacelike vector pointing along the parallels. Again, $\mathbf{n}_u.\mathbf{n}_v = 0$, so we have an orthonormal basis, and hence a unit timelike vector $\mathbf{t}$ tangent to $\Sigma$ can be written $\mathbf{n}_u \cosh\theta + \mathbf{n}_v \sinh\theta$ where $\theta$ is the hyperbolic angle between $\mathbf{t}$ and $\mathbf{n}_u$.

This time the Lagrangian is

$$-\dot{u}^2 + \rho^2\dot{v}^2$$

giving Euler-Lagrange equations

$$\ddot{u} = -\rho\rho'\dot{v}^2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho^2\dot{v}\right) = 0.$$

Now let $\gamma$ be a timelike geodesic on $\Sigma$, parameterized by proper time, and again given by $u(s), v(s)$. Then as before we have

$$\dot{\gamma} = \dot{u}\mathbf{x}_u + \dot{v}\mathbf{x}_v = \dot{u}\mathbf{n}_u + \rho\dot{v}\mathbf{n}_v.$$

In the Minkowski setting, however, this gives

$$\dot{\gamma} = \mathbf{n}_u \cosh\theta + \mathbf{n}_v \sinh\theta$$

where $\theta$ is now the hyperbolic angle between $\dot{\gamma}$ and $\mathbf{n}_u$, i.e., between $\dot{\gamma}$ and the meridian.

We then see that the second Euler-Lagrange equation is equivalent to $\rho\sinh\theta$ being constant.

Conversely, let $\gamma$ be a proper-time parameterized timelike curve such that $\rho\sinh\theta = \rho^2\dot{v}$ is constant. Since $\gamma$ is timelike, it necessarily follows that $\dot{u}^2 > 1$, and so $\dot{u} \neq 0$. We then have

$$\dot{u}^2 - \rho^2\dot{v}^2 = 1 \quad \text{and} \quad \rho^2\dot{v} = \text{constant}.$$

Differentiating this gives

$$\dot{u}\ddot{u} - \rho\rho'\dot{u}\dot{v}^2 - \rho^2\dot{v}\ddot{v} = 0$$
$$2\rho\rho'\dot{u}\dot{v} + \rho^2\ddot{v} = 0.$$

Multiplying the second equation by $\dot{v}$ and substituting into the first gives

$$\dot{u}\ddot{u} + \rho\rho'\dot{u}\dot{v}^2 = 0$$

and since $\dot{u} \neq 0$ we have

$$\ddot{u} = -\rho\rho'\dot{v}^2$$

which is the second Euler-Lagrange equation. It follows that the curve $\gamma$ is a timelike geodesic.

We thus see that Clairaut's theorem has a Minkowski space analogue with $\rho \sinh\theta$ replacing $\rho \sin\theta$ as the quantity conserved along a timelike geodesic. As before, we can immediately deduce that all meridians are geodesics.

We note in passing that for small values of $\theta$, $\sin(\theta) \approx \sinh(\theta)$ and hence the geodesics will be close to those for the Euclidean case.

## 4. Comparison of Geodesics

Let us now examine an explicit example, to investigate the difference between the two cases. We consider the simplest non-trivial case: the surface of revolution generated by a straight line, given by $z = 2x$ in the Euclidean, and $t = 2x$ in the Minkowski case, restricted to positive values of $x$.

In this case the surface of rotation is actually flat. Nevertheless, the geodesics display distinct quantitative and qualitative behaviour, as we now see.

First, we find the equation of an arc-length parameterized geodesic in the Euclidean case.

An arc-length parameterization of the generator is given by $x = u/\sqrt{5}, t = 2u/\sqrt{5}$, which gives the metric

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \frac{u^2}{5}\mathrm{d}v^2$$

and so the Lagrangian

$$L = \dot{u}^2 + \frac{u^2}{5}\dot{v}^2.$$

A geodesic is then completely determined by the value of $u^2\dot{v} = \Omega$ and the condition that $L = 1$. Subtsituting for $\Omega$ in $L$ gives

$$\dot{u}^2 = 1 - \frac{\Omega^2}{5u^2}$$

and hence

$$\frac{\dot{u}^2}{\dot{v}^2} = \frac{u^2}{\Omega^2}\left(u^2 - \frac{\Omega^2}{5}\right)$$

so that the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \pm\frac{u}{\Omega}\sqrt{u^2 - \Omega^2/5}$$

describes the curve in the $(v, u)$ plane which gives a geodesic in $\Sigma$.

In the Minkowskian case, we have

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + \frac{u^2}{3}\mathrm{d}v^2$$

and

$$L = -\dot{u}^2 + \frac{u^2}{3}\dot{v}^2.$$

The analogous calculation with $L = -1$ then gives

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \pm\frac{u}{\Omega}\sqrt{u^2 + \Omega^2/3}.$$

We are interested in timelike geodesics, for which $\mathrm{d}s^2 < 0$ and thus we must have

$$\mathrm{d}u^2 > \frac{u^2}{3}\mathrm{d}v^2$$

or, equivalently

$$\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 > \frac{u^2}{3}.$$

We can therefore ensure that a geodesic is timelike by insisting that its initial value of $\mathrm{d}u/\mathrm{d}v$ satisfy this criterion.

An immediate qualitative difference is that in the Euclidean case, all geodesics except the generators can be continued for arbitrarily large positive or negative values of the parameter, have a closest point of approach to the origin determined by $\Omega$, and are symmetric about this point. This is a consequence of the fact that since $\rho \sin\theta$ is constant, and $\sin\theta$ is bounded above by one, there is a minimum possible value of $\rho$. In the Minkowskian case, we have $\dot{u} > 0$, so a timelike geodesic cannot bounce away from the origin: but since there is no upper bound on $\sinh\theta$, the radial distance $\rho$ can become arbitrarily small. Hence all timelike geodesics approach the origin arbitrarily closely, and can be continued only a finite amount of proper time into the past.

To illustrate this behaviour, we consider geodesics starting at $v = 0, u = 1$ with initial gradient $\mathrm{d}u/\mathrm{d}v = -1$, so that the Minkowskian geodesic is timelike. In the Euclidean case, we obtain $\Omega = \sqrt{5/6}$, and in the Minkowski case, $\Omega = \sqrt{3/2}$.

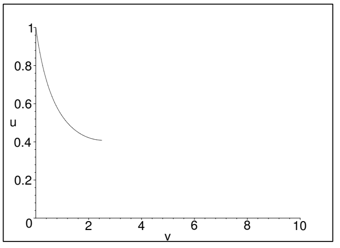


**Figure 1.** Geodesic in $\mathbb{E}^3$.

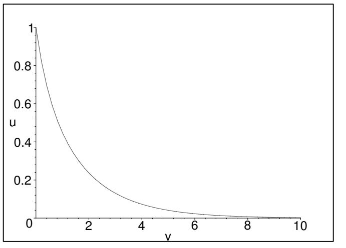


**Figure 2.** Geodesic in $\mathbb{M}^{2,1}$.

The geodesics cannot be obtained in an instructive closed form, but can be found numerically. The results are shown in Figures 1 and 2.

We see in Figure 1 how the downward geodesic in the Euclidean case has a minimum value of $u$ at $u = 1/\sqrt{6}$ and after this it proceeds back up, with the sign changed in the differential equation.

By contrast, Figure 2 shows that in the Minkowskian case, the geodesic passes down arbitrarily close to the origin.

It is clear that this difference is generic. In any surface of revolution (other than the cylinder) there will be geodesics in the Euclidean case which bounce away from regions of sufficiently small $\rho$; on the other hand, in the Minkowskian case, since $\dot{u}^2 - \rho(u)^2\dot{v}^2 = 1$, it follows that $|\dot{u}| > 1$, so no such bouncing can take place, and timelike geodesics will generally reach every value of $u$ in the domain of the generating curve.

## 5. Conclusion

The characterisation of geodesics in surfaces of revolution looks formally identical in the Euclidean and Minkowskian case: in each case geodesics are completely characterized by $\rho^2\dot{v}$ being a conserved quantity. In spite of this, the difference in signature results in entirely different qualitative behaviour of the geodesics in these surfaces.

## Acknowledgement

Anis Saad thanks Omar Al-mukhtar University in Albaida - Libya for the financial support.

## References

[1] Alexander J., *Closed Geodesics on Certain Surfaces of Revolution*, J. Geom. Symmetry Phys. **8** (2006) 1–16.

[2] Kazan A. and Karadag H., *A Classification of Surfaces of Revolution in Lorentz-Minkowski Space*, Int. J. Contemp. Math. Sciences **6** (2011) 1915–1928.

[3] López R., *Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space*, Preprint arxiv:0810.3351v1 [math.DG] (2008).

[4] Mladenov I. and Oprea J., *The Mylar Ballon Revisited*, Am. Math. Monthly **110** (2003) 761–784.

[5] Mladenov I. and Oprea J., *Unduloids and Their Closed Geodesics,* In: Geometry, Integrability and Quantization IV, Coral Press, Sofia 2003, pp. 206–234.

[6] Mladenov I. and Oprea J., *The Mylar Balloon*: *New Viewpoints and Generalizations,* In: Geometry, Integrability and Quantization VII, SOFTEX, Sofia 2006, pp. 246–263.

[7] Pressley A., *Elementary Differential Geometry*, Springer, London 2010.

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