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GEOMETRIC METHODS IN QUANTUM MECHANICS

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Abstract. This is a survey on geometric quantum mechanics and some of its implications on general issues in quantum theory.

Contents

1. Introduction

The present work, devoted to a survey of some topics in *Geometric Quantum Mechanics* (GQM), consists in a written version of a part of the lectures delivered at the XIII International Conference on Geometry, Integrability and Quantization held in Varna, from 3rd to 8th June 2011. Our discussion, which will be rather pedagogical, is mostly based on the papers [17, 18, 115] and the exposition of the results (essentially all known, possibly up to slight reformulation) will be complemented by comments and brief digressions, when needed, in order to enhance readability for a wider audience. After recalling some basic facts on symplectic and Kähler geometry and on the formalism of geometric quantization, we delve into the basic framework of GQM, which is actually *ordinary* quantum mechanics looked upon as a *classical dynamical system* on *a complex projective space*, together with:

- the *Kähler* structure of the latter, governing *uncertainty* of simultaneous measurements, and

- the geometry of the *tautological*, or, better, the *hyperplane section bundle* (see below), manifesting itself in the "universal" *geometric phase* of Aharonov-Anandan, the abstract counterpart of Berry's phase, cropping up in the context of adiabatic motions.

Subsequently, several implications will be discussed of the above geometric reinterpretation of quantum mechanics regarding particular issues such as *integrability*, *entanglement* and *quantum measurement*. We shall also address more speculative

items such as a link with *hydrodynamics*, giving an "Eulerian" counterpart to the "Lagrangian" view provided by the Schrödinger flow.

2. Preliminary Tools

In this Section we recall some basic notions and notation on symplectic geometry and geometric quantization needed throughout the paper, without aiming at depth and completeness, for which we refer to the given bibliography. The reader may well skip this section at all and refer to it whenever necessary.

2.1. Some Basic Symplectic Geometric Terminology

A symplectic manifold (M, ω) is a smooth manifold (necessarily of even dimension, in the finite-dimensional case) equipped with a closed non-degenerate two-form ω. Important examples are provided e.g. by the cotangent space T^*X associated to a manifold X, by Kähler manifolds, or coadjoint orbits of a Lie group G (see e.g. [1, 6, 7, 54–57, 68, 69, 77, 81, 109, 127] for details). The latter live in the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G and take the form $O_{f_0} \cong G/G_{f_0}$, with $f_0 \in \mathfrak{g}^*$ and G_{f_0} denoting the stabilizer of f_0 with respect to the group coadjoint action Ad^*. The (Kirillov) symplectic form B on O_{f_0}, evaluated on two generic fundamental vector fields induced by $\xi, \eta \in \mathfrak{g}$ reads, at $f \in O_{f_0}$

$$B_f(\mathrm{ad}^*_\xi f, \mathrm{ad}^*_\eta f) := \langle f, [\xi, \eta] \rangle$$

(here ad^* denotes (Lie algebra) coadjoint action, which dualizes the standard adjoint action $\mathrm{ad}_\xi \eta = [\xi, \eta]$). If the symplectic manifold (M, ω) is acted upon (symplectically) by a Lie group G, with Lie algebra $\mathfrak{g}$, a *G-equivariant moment map* $\mu : M \to \mathfrak{g}^*$ (existing under mild topological assumptions on M and G) is characterized by the property

$$\mu(g \cdot x) = \mathrm{Ad}^*(g)\, \mu(x), \qquad x \in M,\ g \in G.$$

Such a map yields, for each $\xi \in \mathfrak{g}$, a Hamiltonian $\lambda_\xi = \lambda_\xi(x) := \langle \mu(x), \xi \rangle$ (duality pairing), and the set of such functions yields indeed a Lie algebra isomorphic to $\mathfrak{g}$, via the Poisson bracket $\{\cdot, \cdot\}$ induced by the symplectic form

$$\{\lambda_\xi, \lambda_\eta\}(x) := \omega(\xi^\sharp, \eta^\sharp)(x) = \lambda_{[\xi,\eta]}(x)$$

(for all $x \in M$, with $\xi^\sharp$ denoting the fundamental vector field induced by $\xi \in \mathfrak{g}$).

2.2. Complex Polarizations and Kähler Manifolds

A *Kähler polarization* of the symplectic manifold (M, ω) consists of an endomorphism $I \in \Omega^0(\mathrm{End}\, TM)$ of the tangent bundle TM such that

$$\begin{aligned} I^2 &= -\mathrm{Id} \\ I[IX, IY] &= [X, Y] + I[IX, Y] + I[X, IY] \\ \omega(X, IY) &= -\omega(IX, Y) \end{aligned}$$

and for which $\omega(\cdot, I\cdot)$ is positive definite. By virtue of the theorem of Newlander and Nirenberg the first two conditions give M the structure of a complex manifold and the last two say that

$$g(X, Y) = \omega(X, JY)$$

defines a Kähler metric, with Kähler form $\omega \in \Omega^{1,1}(M)$ (cf. [33, 53, 61]).

2.3. A Digression on Geometric Invariant Theory

Given a Hamiltonian compact Lie group G-action on a Kähler manifold X (with Lie algebra $\mathfrak{g}$), one extends it to the complexification $G^{\mathbb{C}}$, with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \mathrm{i}\mathfrak{g}$ upon considering vector fields $\eta^\sharp = J\xi^\sharp$, with $\xi^\sharp$ a fundamental vector field associated to the above action. Obviously, such an extended action does not preserve the metric any longer. Under fairly general conditions (see [56]) one has an identification between *Marsden-Weinstein* and *Mumford quotients*, respectively

$$X_0/G \;\cong\; X_s/G^{\mathbb{C}}$$

with $X_0 = \mu^{-1}(0)$, $X_s := G^{\mathbb{C}} \cdot X_0$ (the stable points in Mumford's sense, see e.g. [11, 56, 71, 87]).

2.4. Completely Integrable Hamiltonian Systems

In this subsection we review some basic facts about completely integrable Hamiltonian systems. A Hamiltonian system (M, ω, h) (h (the Hamiltonian) is a smooth function on M), with n degrees of freedom, is said to be completely integrable if it admits n mutually Poisson-commuting first integrals, which are linearly independent almost everywhere in M and, restricting to regular fibres, the joint level sets of the first integrals (which are then Lagrangian, i.e., the restriction of ω vanishes thereon and have maximal dimension, namely n) are compact and connected. The celebrated *Liouville-Arnol'd Theorem* (see e.g. [6, 7, 43]) then says that the latter are actually n-dimensional tori foliating the manifold, labelled by (locally defined)

action variables $I = (I_1, I_2, \ldots, I_n)$, with *angle variables* $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ thereon in such a way that the coordinates (I, φ) on M are Darboux coordinates, that is

$$\omega = \sum_{i=1}^{n} \mathrm{d}I_i \wedge \mathrm{d}\varphi_i.$$

Thus, geometrically, M is a $\mathbb{T}^n$-bundle with Lagrangian fibres. The construction of the toric principal bundle or, equivalently, the existence of global action-angle coordinates is only (semi-)local. The quest for globality leads to the major issue of *monodromy* (cf. [41–43, 89]) which will not be addressed in the present paper.

2.5. A Glimpse at Geometric Quantization

We briefly review the basics of *Geometric Quantization* (GQ) and we refer to [28, 54, 56, 68, 69, 77, 109, 123, 127] for a complete account. It is a quite elegant and powerful method, essentially arising from generalizing Dirac's approach to the magnetic monopole and casting it into the appropriate mathematical framework of differential geometry and topology of complex line bundles, which allows for a neat geometrical understanding of important topics such as group representation theory (e.g. the Borel-Weil theorem and its extensions, see [56, 98]) that is crucial in many modern physical theories such as, among others, integral and fractional quantum Hall effects, conformal field theories and Chern-Simons-Witten theory [14, 73, 76, 98, 126]. We point out [104], [103] for a recent application to the issue of quantum monodromy. The full range of applications is however enormous, so we confine ourselves to mentioning in addition to the references already given, [47, 83–85, 107] for applications to Kepler-type systems, [21, 91–95, 114] for applications to fluid mechanics and [99, 112, 117–120, 128] for the geometry of infinite dimensional Grassmannians and related issues. These topics were indeed touched upon in the lectures. In the present paper GQ plays an ancillary yet most important role.

Roughly speaking, GQ aims at manufacturing a quantum Hilbert space from the geometry of the classical dynamical system, together with a consistent prescription for constructing quantum observables. It turns out that the natural candidate, namely the L^2-sections of an appropriate complex line bundle on the classical phase space, contains "wave functions" which are not physically acceptable: for instance, in dealing with the classical harmonic oscillator, one would encounter functions with arbitrarily "small" compact support on phase space, which violates the Heisenberg Uncertainty Principle. This is the reason why one needs to "polarize", namely, to stick either to wave functions defined solely in terms of positions *or* momenta (real polarization - one then abuts at the standard Schrödinger

representation) or using holomorphic functions (Bargmann's representation - complex polarization). The first strand leads in particular to the recovery of the Bohr-Sommerfeld conditions of "old" quantum theory, the "holomorphic way" is crucial in the above mentioned general theories. In a bit more detail, this goes as follows.

Recall that if (M,ω) is a symplectic manifold of (real) dimension $2n$ such that the ensuing cohomology class $\left[\frac{1}{2\pi}\omega\right] \in H^2(M,\mathbb{Z})$ (*integrality*, i.e., the integral of ω over any two-cycle is an integer) then the *Weil-Kostant theorem* states that there exists a complex line bundle (L,∇,h) over M equipped with a hermitian metric h and a compatible connection ∇ with curvature $F_\nabla = \omega$. Hence $[\omega] = c_1(L)$, the first Chern class of $L \to M$. The results holds in the pre-symplectic case as well, i.e., one may drop non-degeneracy. The integrality condition stems as a consistency condition coming from computing, via the Stokes theorem, the parallel transport of the connection on a loop in M bounded by two different two-chains building up a two-cycle.

Recall that the first Chern class of a complex line bundle, i.e., the Euler class of the associated real (oriented) vector bundle, arises as the obstruction to gluing together fibrewise angular forms on the corresponding principal S^1-bundle (cf. [24]). The connection form (global, when viewed on the total space), restricts fibrewise to the angular form on the S^1-fibres. Its differential is the pull-back of a global two-form on the base, i.e., the curvature form, equal (cohomologically) to (minus) the Euler class of the principal S^1-bundle $P \to M$ canonically associated to $L \to M$, which is the transgression of the angular form [24]. The Weil-Kostant theorem and the ensuing transgression interpretation can be generalized to the so-called n-gerbes ([28, 39, 46, 62, 97, 116]).

The connection ∇ is called a *prequantum connection* and $L \to M$ the *prequantum line bundle*. The different choices of $L \to M$ and ∇ are parameterized by the first cohomology group $H^1(M,S^1)$ (see e.g. [127], Ch.8). In particular, if M is simply connected, this cohomology group it trivial and the connection is unique.

We also recall for completeness that the prequantum connection ∇ allows the construction of the (Hermitian) *prequantum* observables $Q(\cdot)$, acting on the space of smooth (complex valued) sections $\Gamma(L)$ of $L \to M$ via the formula

$$Q(f) = -\mathrm{i}\nabla_{X_f} + f = -\mathrm{i}X_f - i_{X_f}\theta + f$$

and this fulfils Dirac's prescription

$$[Q(f),Q(g)] = \mathrm{i}Q(\{f,g\}), \qquad Q(1) = \mathrm{Id}$$

(where $\{,\}$ again denotes the Poisson bracket pertaining to the symplectic structure, θ is a (local) symplectic potential, i.e., $\omega = \mathrm{d}\theta$ and we take $\hbar = 1$). The

specific expression for $Q(f)$ is natural since it is the lifting to the total space L of the action of f on the base manifold M via its Hamiltonian vector field X_f, and turns out to be closely related to the path integral formalism ([127]). In general the connection is determined up to a closed one-form, yielding a corresponding ambiguity in the definition of the quantum observable $Q(f)$ attached to f. This fact turns out to be important in dealing with quantum monodromy ([104], [103]).

2.6. The Bohr-Sommerfeld Condition

Coming back to the specific geometric quantization setting, consider a Lagrangian submanifold Λ of the symplectic manifold M so that, any (semi-local) symplectic potential θ becomes a closed form thereon, defining a (semi-local) connection form pertaining to the restriction of the prequantum connection ∇ and denoted by the same symbol. The latter is a *flat* connection and a global covariantly constant section s of the restriction of the prequantum line bundle (namely one has $\nabla s = 0$) exists if and only if it has trivial holonomy, that is, the induced character $\chi : \pi_1(\Lambda) \longrightarrow \mathrm{U}(1)$ is trivial (see e.g. [123, 127]), or, equivalently, that the *Bohr-Sommerfeld* condition is fulfilled

$$\left[\frac{\theta}{2\pi}\right] \in H^1(M, \mathbb{R}) \qquad \text{i.e.,} \qquad \int_\gamma \theta \in 2\pi\mathbb{Z}$$

for any closed loop γ in Λ.

Further on we shall discuss this condition in relation with second quantization in Subsection 4.7.

2.7. Holomorphic Geometric Quantization

Given a Kähler polarization, we can endow the complex line bundle $L \to M$ with the structure of a *holomorphic* line bundle by considering the differential operator $\nabla^{0,1} : \Omega^0(M, L) \to \Omega^{0,1}(M, L)$ defined by

$$\nabla^{0,1} = (1 + \mathrm{i}I)\nabla.$$

Let us remind that the complex forms of type $(0, 1)$ are those acted upon by I via multiplication by $-\mathrm{i}$. In local terms, this is a differential operator of the form

$$\sum_i \left(\frac{\partial f}{\partial \bar{z}_i} + \theta_i f\right) \mathrm{d}\bar{z}_i$$

and by the Dolbeault lemma, a local solution to the l.h.s $= 0$ exists if and only if

$$\bar{\partial}\left(\sum \theta_i \mathrm{d}\bar{z}_i\right) = 0.$$

But the l.h.s. is the $(0,2)$ component of the curvature of ∇, which, in the present situation, is the Kähler form of the manifold, which is then of type $(1,1)$. Thus the integrability condition is satisfied and the equation

$$\nabla^{0,1}s = 0$$

has local non-vanishing solutions. If s and s' are two such solutions with $s' = gs$, then one immediately checks that $\frac{\partial g}{\partial \bar{z}_i} = 0$, so g is a holomorphic transition function for the line bundle L, and this gives rise to its holomorphic structure. The connection becomes the Chern-Bott one (see also below). Thus, in the Kähler case one can perform the so-called *holomorphic quantization*, whereby one takes the space of holomorphic sections $H^0(L,J)$ of a holomorphic prequantum line bundle, provided it is not trivial, (this being achieved by Kodaira vanishing conditions) as the Hilbert space of the theory (J denotes a complex structure on M, see e.g. [61] for details). As a holomorphic line bundle, $L \to M$ varies with J, whilst its topological type is fixed. In this case there is a canonically defined connection, called the *Chern,* or *Chern-Bott connection*, compatible with both the hermitian and the holomorphic structure (cf. [53]). Independence of polarization (i.e., of the complex structure, in this case) is achieved once one finds a (projectively) flat connection on the vector bundle $V \to \mathcal{T}$ with fibre $H^0(L,J)$ (of constant dimension, under suitable assumptions provided by the Kodaira vanishing theorem) over the (Teichmüller) space of the complex structures $\mathcal{T}$ (see [61]). The classical theory of theta-functions and their heat-equation related properties provides a most beautiful example of this phenomenon (see [14] and [61]). For an application to quantum monodromy see [104]. See also [110] for an early discussion of the link between geometric quantization, canonical commutation relations and theta functions.

2.8. A Hydrodynamical Intermezzo

In this section, based on [115], we recall some results valid for Killing vector fields on a (connected) Riemannian manifold (M,g) (i.e., those generating infinitesimal isometries which always exist, at least locally). As general references we may quote [31,48,75]. For hydrodynamics we refer, among others, to [1,8,45,80,121]. The Levi-Civita connection of (M,g) will be denoted by ∇. We shall employ the notation $\langle X,Y\rangle := g(X,Y)$, for $X,\ Y \in \Gamma(TM)$ (vector fields on M). Upon freely using the musical isomorphism notation ($\sharp$ = vector field, $\flat$ = one-form, corresponding to index raising and lowering, respectively, so, for instance, $(X^\flat, Y) = \langle X,Y\rangle$, with $(\cdot,\cdot)$ being the pairing between one-forms and vector fields), we first recall the following basic identity (cf. [1], 5.5.8, p.474, or [8], Ch. IV, p. 202, Theorem 1.17)

$$\mathcal{L}_Y Y^\flat = (\nabla_Y Y)^\flat + \frac{1}{2}\,\mathrm{d}\,\langle Y, Y\rangle$$

($\mathcal{L}$ is the Lie derivative), which easily yields the following

Lemma 1. *Let X be a Killing vector field on a Riemannian manifold (M, g). Then*

$$\mathcal{L}_X X^\flat = 0.$$

Proof: If X is Killing, then for any vector field Y, one has

$$\mathcal{L}_X(Y^\flat) = (\mathcal{L}_X Y)^\flat$$

which yields immediately

$$\mathcal{L}_X(X^\flat) = (\mathcal{L}_X X)^\flat = [X, X]^\flat = 0. \qquad \blacksquare$$

Recall that the *Euler equation* on a Riemannian manifold reads, among others, in the following equivalent guises, in terms of one-forms

$$\frac{\partial X^\flat}{\partial t} + (\nabla_X X)^\flat = -\mathrm{d}\,p$$

or

$$\frac{\partial X^\flat}{\partial t} \,+\, \mathcal{L}_X X^\flat = \mathrm{d}\,(\frac{1}{2}\langle X, X\rangle \,-\, p)$$

(p being the *pressure*) together with $\mathrm{div} X = 0$ (see e.g. [8] or [121], Ch.17, 1.15, p. 469). One immediately establishes the following

Lemma 2. *A divergence-free vector field Y on a (finite dimensional, connected) Riemannian manifold (M, g) satisfies the* stationary Euler equation*, with* pressure $p = \frac{1}{2}\langle Y, Y\rangle$ *(up to a constant) if and only if $\mathcal{L}_Y Y^\flat = 0$.*

Let us also notice the general identity, valid for a Killing vector field X

$$(\mathcal{L}_X g)(Y, Z) \,=\, \langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$$

which implies, setting $Z = Y$

$$\langle Y, \nabla_Y X\rangle = 0$$

and, setting further $Y = X$

$$\langle X, \nabla_X X\rangle = 0\,.$$

The main result of this section is the following

Theorem 3 ([115]). *Let X be a Killing vector field on a finite dimensional, connected Riemannian manifold (M, g). Then*

i) *the* (*necessarily divergence-free*) *vector field X fulfils the* stationary Euler equation*, with* pressure *given by $p = \frac{1}{2}\langle X, X\rangle$* (*up to a constant*)

ii) *the* vorticity *form of the* (*stationary*) *Euler equation reads* (*with $w = \mathrm{d}X^\flat$ the* vorticity *two-form*)
$$\mathcal{L}_X w = 0$$

iii) *the* (*Riemannian*) *gradient of the pressure, $(\mathrm{d}p)^\sharp$, is orthogonal to X*

iv) *if γ is an integral curve of X starting from a point $m \in M$, then γ is a geodesic if and only if $\mathrm{d}p = 0$ (at m and hence along γ).*

Sketch of Proof: The first three assertions are easily established with the aid of the previous lemmata, so let us comment on iv). Let $\gamma : s \mapsto \gamma(s)$ denote the integral curve of X starting from a point m. Then, due to the stationary Euler equation fulfilled by X, one has
$$(\nabla_{\dot\gamma}\dot\gamma)^\flat = -\mathrm{d}p\mid_{\gamma(s)} .$$
Thus γ is a geodesic if and only if $\mathrm{d}p\mid_{\gamma(s)} = 0$ for all s. On the other hand, p, and hence $\mathrm{d}p$ are invariant under the flow of X, by iii), whence $\mathrm{d}p\mid_{\gamma(s)} = 0$ for all s if and only if it holds at $m = \gamma(0)$, this yielding iv). ■

Remark 4. *Notice that for one-sided invariant metrics on Lie groups, even in the finite dimensional case, geodesics do not correspond to one-parameter Lie subgroups* (*see e.g.* [48]) *so, even ignoring the subtleties of the infinite dimensional situation, one cannot directly conclude that a divergence-free vector field on a* (*compact, say*) *Riemannian manifold* (*i.e., an element of the "Lie algebra" of the group* $\mathrm{SDiff}(M)$ *of measure preserving diffeomorphisms of M*) *automatically yields a solution of the* (*stationary*) *Euler equation, i.e., a geodesic of the natural right-invariant* (*but not bi-invariant*) *metric induced by the kinetic energy* [8], [45].

In this paper, we shall ultimately just treat the simple, yet fundamental example given by the projective space $\mathbb{P}(V)$ - to be presently reviewed - equipped with the Fubini-Study Kähler form, whose prequantization bundle is unique and is given by the hyperplane section bundle $\mathcal{O}(1) \to \mathbb{P}(V)$, endowed with the Chern-Bott connection (see below, Subsection 3.2). It possesses an extremely rich mathematical structure at the crossroads of many fields, which is most interesting in itself.

3. Geometric Quantum Mechanics: the Basic Formalism

We shall mostly refer to [17], but see also [9,25–27,34–38,40,60,63,67,100,111].

The basic idea of the geometric approach to quantum mechanics roughly consists, in a first instance, in regarding it as a classical mechanics on the projective Hilbert space associated to the quantum system, considered as given *a priori*, its dynamics being governed by a special class of Hamiltonians, namely those arising as mean values of self-adjoint operators (see Subsection 3.1).

Given such a Hamiltonian (confining ourselves to the finite dimensional and non-degenerate case), there is a natural toral action leaving it invariant and foliating the projective space into Lagrangian (or isotropic) tori, thereby yielding *complete integrability* of the associated classical mechanical system (Subsection 3.3). The ensuing action-angle variables receive a natural interpretation, the former being, in particular, transition probabilities. This has been already shown in greater generality in [38] using different techniques. Actually, the above theorem (in finite dimensions) can be also regarded as a consequence of a much more general result by Thimm [122] stating that $\mathrm{U}(n)$- or $\mathrm{O}(n)$-invariant Hamiltonian systems on symmetric spaces are completely integrable. Furthermore, projective spaces provide the basic examples of Hamiltonian toric manifolds (see. e.g. [10], [55] or the textbooks [57], [13], [81]) and we shall sketch some explicit arguments.

We then discuss the differential geometric properties of the Schrödinger vector field (showing that a suitable restriction thereof gives rise to a Jacobi field), and we elaborate on the relationship between uncertainty and curvature (Subsection 3.4). Also, in Subsection 3.5, still acting within a Riemannian geometrical framework, we discuss a possibly useful hydrodynamical interpretation recently set forth by the present author [115]. Our geometric approach is basically finite-dimensional. However, this is far from being devoid of physical significance: indeed, one often works with a finite dimensional approximation, namely in quantum chemistry (Hartree-Fock), see e.g. [57] and another important example is provided by *quantum computation*, see e.g. [40, 72].

3.1. Projective Space and its Symplectic and Kähler Geometry

Throughout the paper we assume $\hbar = 1$. Let V be a complex Hilbert space of finite dimension $n+1$, for simplicity, with scalar product $\langle\cdot|\cdot\rangle$, linear in the second variable. Let $\mathbb{P}(V)$ denote its associated projective space, of complex dimension n, which represents the space of (pure) states in quantum mechanics. We make free use of Dirac's bra-ket notation, we can identify a point in $\mathbb{P}(V)$, which is, by

definition, the ray (i.e., one-dimensional vector space) $<v>$ pertaining to (respectively generated by) a non zero vector $v \equiv |v\rangle$ - and often conveniently denoted by $[v]$- with the projection operator onto that line, namely

$$[v] = \frac{|v\rangle\langle v|}{\| v \|^2}$$

(actually, the above identification can be interpreted in terms of the moment map defined below). For the sequel, we notice that, upon choosing an orthonormal basis $(e_0, e_1, \dots e_n)$ of V, and setting, for a *unit* vector $v = \sum_{i=0}^{n} \alpha_i e_i$, the above projection can be written as a density matrix ([23], [82])

$$|v\rangle\langle v| \leftrightarrow (\bar{\alpha}_i \alpha_j)$$

(with $\sum_{i=0}^{n} |\alpha_i|^2 = 1$). If $\mathrm{U}(V)$ denotes the unitary group pertaining to V, with Lie algebra $\mathfrak{u}(V)$, consisting of all skew-hermitian endomorphisms of V which we call observables, with a slight abuse of language and then the projective space $\mathbb{P}(V)$ is a $\mathrm{U}(V)$-homogeneous Kähler manifold. The isotropy group (stabilizer) of a point $[v] \in \mathbb{P}(V)$ is isomorphic to $\mathrm{U}(V') \times \mathrm{U}(1)$, with V' the orthogonal complement to $<v>$ in V, the $\mathrm{U}(1)$ part coming from phase invariance: $[\mathrm{e}^{\mathrm{i}\alpha} v] = [v]$. Hence

$$\mathbb{P}(V) \cong \mathrm{U}(V)/(\mathrm{U}(V') \times \mathrm{U}(1)) \cong \mathrm{U}(n+1)/(\mathrm{U}(n) \times \mathrm{U}(1)).$$

The fundamental vector field $A^\sharp$ associated to $A \in \mathfrak{u}(V)$ reads (evaluated at $[v] \in \mathbb{P}(V)$, $\| v \| = 1$)

$$A^\sharp|_{[v]} = |v\rangle\langle Av| + |Av\rangle\langle v|.$$

In view of homogeneity, these vectors span the tangent space of $\mathbb{P}(V)$ at each point. The (action of the) complex structure J reads, accordingly

$$J|_{[v]} A^\sharp_{[v]} = |v\rangle\langle \mathrm{i}Av| + |\mathrm{i}Av\rangle\langle v|.$$

Next we are going to write down the expression for the natural (i.e., Fubini-Study) metric g and the Kähler form ω (recalling that, if Tr denotes the trace on $\mathrm{End}(V)$, then clearly $\mathrm{Tr}(|v\rangle\langle w|) = \langle w|v\rangle$) which are essentially the real and imaginary part (respectively) of the hermitian form $\langle \mathrm{d}v|\mathrm{d}v\rangle$. Explicitly

$$g_{[v]}(A^\sharp|_{[v]}, B^\sharp|_{[v]}) = \mathrm{Re}\{\langle Av|Bv\rangle + \langle v|Av\rangle\langle v|Bv\rangle\}$$

and

$$\omega_{[v]}(A^\sharp|_{[v]}, B^\sharp|_{[v]}) = g_{[v]}(J|_{[v]}(A^\sharp|_{[v]}, B^\sharp|_{[v]}) = \frac{\mathrm{i}}{2}\langle v|[A,B]v\rangle.$$

Actually, our discussion can be conveniently rephrased in terms of the *moment map*

$$\mu : \mathbb{P}(V) \ \rightarrow \ \mathfrak{u}(V)^* \cong \mathfrak{u}(V)$$

$$\mu([v]) \ = \ -\mathrm{i}|v\rangle\langle v|$$

(the last isomorphism coming from the Killing-Cartan metric on $\mathfrak{u}(V)$ given by $(A, B) := -\frac{1}{2}\mathrm{Tr}(AB)$, for A, $B \in \mathfrak{u}(V)$). The Hamiltonian algebra corresponding to μ consists, accordingly, of the real smooth functions

$$\mu_A([v]) = (\mu, A) = \frac{\mathrm{i}}{2}\langle v|Av\rangle, \quad A \in \mathfrak{u}(V)$$

i.e., up to a constant, the mean values of the observables. It follows immediately that ω emerges as the canonical Kirillov symplectic form pertaining to $\mathbb{P}(V)$ looked upon (via μ) as a $\mathrm{U}(V)$-coadjoint orbit (see e.g. [57], [81]). Clearly, $A^\sharp$ becomes the Hamiltonian vector field associated to $A \in \mathfrak{u}(V)$, i.e., one has

$$\mathrm{d}\mu_A = i_{A^\sharp}\omega.$$

The Poisson bracket $\{\cdot, \cdot\}$ defined by ω is of course

$$\{\mu_A, \mu_B\} := \omega(A^\sharp, B^\sharp) = \mu_{[A,B]}.$$

We also notice, that with the present conventions (i.e., those of [17]), if A, $B \in \mathfrak{u}(V)$, then

$$[A^\sharp, B^\sharp] = -[A, B]^\sharp$$

where the l.h.s. commutator refers to vector fields, the r.h.s. one is the Lie algebraic one. The latter identity can be directly checked by evaluating both sides on a Hamiltonian μ_C.

From this point of view we may characterize Fubini-Study Killing vector fields as the infinitesimal generators of unitary one-parameter groups, i.e., with the Hamiltonian vector fields $A^\sharp$ (cf. also [34]).

Let us finally quote the following elementary but important result.

Theorem 5 (cf. [25], [27], [63]). *Given two distinct points* (*quantum states*) $[\xi]$ *and* $[\eta]$ *in* $\mathbb{P}(V)$*, with representative* (*ket*) *vectors* ξ *and* η*, and given their respective orthogonal states* $[\xi^\perp]$ $[\eta^\perp]$ *on the projective line* $\overline{[\xi][\eta]}$ *they determine, then the cross-ratio* $k^2 := ([\xi], [\eta], [\eta^\perp], [\xi^\perp])$ *equals the transition probability between* $[\xi]$ *and* $[\eta]$*, namely*

$$([\xi], [\eta], [\eta^\perp], [\xi^\perp]) = \frac{|\langle\xi|\eta\rangle|^2}{\langle\xi|\xi\rangle\langle\eta|\eta\rangle}.$$

Notice that if $\overline{[\xi][\eta]}$ is regarded as a sphere, then $[\xi]$ and $[\xi^\perp]$, and $[\eta]$, $[\eta^\perp]$, respectively, become *antipodal* points thereon.

As a further remark, we may observe that a quantum observable induces a projective reference frame built from its eigenstates, and a choice of phase of their representing vectors amounts at fixing its *unit point*.

3.2. The Chern-Bott Connection

We now wish to compute a (local) symplectic potential θ for ω, i.e., a one-form such that $\mathrm{d}\theta = \omega$. The one-form θ cannot be global since a symplectic form on a compact manifold cannot be exact: indeed, it generates the one-dimensional second cohomology group $H^2(\mathbb{P}(V), \mathbb{Z})$ and gives rise to the first Chern class of the hyperplane section bundle $\mathcal{O}(1) \to \mathbb{P}(V)$, whose space of holomorphic sections is canonically (conjugate linear) isomorphic to V (see also [53]).

We may take (for $\| v \| = 1$)

$$\theta = -\mathrm{i}\langle v|\mathrm{d}v\rangle.$$

Up to a constant, θ is just the canonical (Chern-Bott) connection form (with respect to a hermitian local frame) on $\mathcal{O}(1)$, governing the so-called Berry (or, rather Aharonov-Anandan) phase ([2, 3, 19, 40, 53] see also Section 4). Geometrically, it just represents the infinitesimal angle variation of v (relative to the complex plane it generates) upon an infinitesimal (norm-preserving) displacement. This will be crucial for the sequel.

The Chern-Bott connection (actually the covariant derivative) on the hyperplane section bundle on $\mathbb{P}(V)$ can be exhibited in the following slightly sloppy but physically vivid form (evaluating on the fundamental vector field $A^\sharp$ pertaining to $A \in \mathfrak{u}(V)$)

$$\nabla_{A^\sharp}|v\rangle = A|v\rangle - \langle A\rangle_{[v]}|v\rangle$$

that is, one just removes from $A|v\rangle$ its component along the ray $[v]$. GQ, when applied to the present setting, yields the so-called *Aharonov-Anandan* (*AA*) *phase* ([2, 3, 40]) in the form

$$\int_\Sigma \Omega_{FS} = \varphi_{AA}(\mathcal{C})$$

(if $\partial\Sigma = \mathcal{C}$), via the parallel transport of the Chern-Bott connection, showing the universal character of the latter, since it intrinsically crops up in any quantum system.

Remark 6. *From the preceding observations it is clear that quantum mechanics, even when looked upon geometrically,* cannot *be reduced to* (*a sort of*) *classical*

mechanics: *the Schrödinger equation involves, in fact, state* vectors *and non just states* (*i.e., rays*)*, imposing, as a consequence, phase evolutions as well. Also, we have the appearance of the AA-phase* (*a universal Berry's phase*)*, due to non flatness of the Chern-Bott connection and, more precisely, to the non triviality of the tautological or hyperplane bundle. Thus we may say that quantum mechanics, formally, appears as classical mechanics on a projective space, together with the geometry and topology of the hyperplane section bundle, equipped with the Chern-Bott connection* (*and this requires* geometric prequantization *applied to the symplectic manifold* $(\mathbb{P}(V), \omega_{FS})$*). Also, the Riemannian metric governs uncertainty of outcomes in simultaneous measurements.*

In fact, more can be said: the vectors in V arise as *coherent state vectors* after the GQ-reinterpretation, either in the Kählerian sense [101] or in the group theoretical sense [90, 96], the group in question being $\mathrm{U}(V)$. Roughly speaking, it just describes the quantum Hilbert space via probability amplitudes ($|v\rangle \mapsto \langle v| \equiv |w\rangle \mapsto \langle v|w\rangle$). Incidentally, this actually yields compatibility between geometric quantization (applied to $\mathbb{P}(V)$) and geometric quantum mechanics. We do not further delve into this important topic, referring, among others, to [90, 111, 113]. Also notice that obviously $h^0(\mathcal{O}(1)) = n+1 = \dim V$. When GQ is applied to a classical n-oscillator system, followed by a reduction to a fixed energy level, it yields its "traditional" quantization, which can be used, in turn, to produce the "correct" GQ of the Kepler problem (see [47]) for details. This formal similarity (together with integrability, see below) can be put to use in interpreting second quantization as a sort of Bohr-Sommerfeld quantization, see Subsection 4.7 of this paper.

3.3. Toral Actions and Integrability

Let us now consider a non degenerate quantum Hamiltonian

$$H = \sum_{j=0}^{n} \lambda_j P_j = \sum_{j=0}^{n} \lambda_j |e_j\rangle\langle e_j|$$

i.e., $\lambda_i \neq \lambda_j$, if $i \neq j$, and (e_j) is an orthonormal basis of eigenvectors, with $P_j := |e_j\rangle\langle e_j|$ being the orthogonal projection operator onto the line $<e_j>$. Without loss of generality we assume $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$, so that

$$H = \sum_{j=1}^{n} \lambda_j P_j.$$

The *Schrödinger equation* is given by (recall that $\hbar = 1$)

$$\frac{\partial}{\partial t}|v\rangle = -\mathrm{i}H|v\rangle$$

inducing its projective space version ([25–27, 63], in which the spinor formalism is used)

$$|\frac{\partial}{\partial t}v\rangle\langle v| + |v\rangle\langle\frac{\partial}{\partial t}v| = \mathrm{i}|Hv\rangle\langle v| - \mathrm{i}|v\rangle\langle Hv|$$

(here $\| v \| = 1$). Its mean value on a state $[v]$ yields a "classical" Hamiltonian h on $\mathbb{P}(V)$. With the above notation we have

$$h([v]) = \frac{\langle v|Hv\rangle}{\langle v|v\rangle} = \frac{\sum_{j=0}^n \lambda_j|\alpha_j|^2}{\sum_{j=0}^n |\alpha_j|^2} = \sum_{j=1}^n \lambda_j|\alpha_j|^2$$

and the last equality holds for $\| v \| = 1$, $\lambda_0 = 0$. Consequently

$$h([v]) = \mu_{(-2\mathrm{i}H)}.$$

The critical points of h are given by the zeros of $(-\mathrm{i}H)^\sharp$ (symplectic gradient) or equivalently $J(-\mathrm{i}H)^\sharp = H^\sharp$ (Riemannian gradient), and these, in turn correspond to the states $[e_j]$ determined by the eigenvectors e_j. This can be seen in various ways, for instance via the immediately checked formula for the dispersion (variance) of an observable $A \in u(V)$ in a state $[v]$, see also, e.g. [3, 36, 37, 111]

$$\Delta_{[v]}A = \| Av - \langle v|Av\rangle v \| = \| A^\sharp_{[v]} \|_{FS} := \sqrt{g_{[v]}(A^\sharp_{[v]}, A^\sharp_{[v]})} = \| J_{[v]}A^\sharp_{[v]} \|_{FS}.$$

The nature of the critical point $[e_j]$ can be ascertained via the formula (resorting to normalized vectors and then to obviously defined real coordinates)

$$h([v]) = \lambda_j + \sum_{k=0}^n (\lambda_k - \lambda_j)|\alpha_k|^2 = \lambda_j + \sum_{k=0}^n (\lambda_k - \lambda_j)(x_k^2 + y_k^2)$$

showing, in particular, that h is a perfect Morse function, i.e., the index of the jth critical point, namely $2j$, equals the Betti number $b_{2j}(\mathbb{P}(V))$.

Now let $v = \sum_{j=0}^n \alpha_j e_j$, with $\alpha_j \neq 0$ for all $j = 0, \ldots, n$. The submanifold consisting of such $[v]$'s is open and dense in $\mathbb{P}(V)$. The torus $\mathbb{T}^{n+1}$ acts on $\mathbb{P}(V)$ via the position $e_j \mapsto \mathrm{e}^{\mathrm{i}\beta_j}e_j$, $\beta_j \in [0, 2\pi)$, but actually, in view of global phase arbitrariness this action descends to an effective action of $G := \mathbb{T}^n$ and this is clearly seen in the density matrix formalism

$$(\bar\alpha_i\alpha_j) \mapsto (\bar\alpha_i\alpha_j \mathrm{e}^{\mathrm{i}(\beta_j - \beta_i)})$$

(we shall resume this particular formalism when discussing quantum measurement). We set $\beta_0 = 0$ in order to be specific. The generators of the torus action are the (mutually commuting) operators $\mathrm{i}P_j$, $j = 1, 2, \ldots, n$. Their associated Hamiltonians $p_j := \langle\cdot|P_j\cdot\rangle = \mu_{(-2\mathrm{i}P_j)}$ give rise to n constants of motion (first integrals)

in involution, with respect to the Poisson bracket induced by the Fubini-Study form, which turn out to be the action variables (see below). In the complement we have a stratification of toral orbits of dimensions $k = 0, 1, \ldots, n-1$ (isotropic tori), but the basic picture persists. Precisely, we may state the following

Theorem 7 ([17]). i) *Under the above assumptions, the "classical" Hamiltonian system* $(\mathbb{P}(V), \omega, h)$ (*actually an open dense set thereof*) *is completely integrable. The Lagrangian tori are provided by the orbits* $G \cdot [v]$ *of the* n*-dimensional torus* G*-action above. The action variables* I_j *coincide with the transition probabilities* $|\alpha_j|^2 = p_j([v])$, $j = 1, 2, \ldots, n$.

ii) *Indeed, the full system remains integrable, allowing isotropic tori, and the orbit space can be identified with the standard* n*-simplex in the Euclidean space* $\mathbb{R}^n$.

Proof: Ad i). We compute the action variables I_j, $j = 1, 2, \ldots, n$ in the standard fashion [6].

If ϑ is a (local) potential of the symplectic form, they read, upon choosing a homology basis (γ_j) for a fixed Lagrangian torus

$$I_j = \frac{1}{2\pi} \int_{\gamma_j} \vartheta.$$

In our case, considering a generic orbit $G \cdot [v]$ (which is topologically an n-dimensional torus itself and it is clearly Lagrangian, since $\omega_{|_{G\cdot[v]}} \equiv 0$) we may take as γ_j the curves

$$[0, 2\pi) \ni \beta_j \mapsto [\sum_{h \neq j} \alpha_h e_h + \alpha_j \mathrm{e}^{\mathrm{i}\beta_j} e_j] \in \mathbb{P}(V)$$

so we easily get

$$I_j = \frac{1}{2\pi} \int_0^{2\pi} |\alpha_j|^2 (-\mathrm{i} \langle \mathrm{e}^{\mathrm{i}\beta_j} e_j | \mathrm{d}\mathrm{e}^{\mathrm{i}\beta_j} e_j \rangle) = |\alpha_j|^2.$$

The Schrödinger evolution reads, in coordinates (taking as before $\lambda_0 = 0$)

$$v = \sum_{i=0}^{n} \alpha_i e_i \mapsto \sum_{i=0}^{n} \alpha_i \mathrm{e}^{-\mathrm{i}\lambda_i t} e_i = \alpha_0 e_0 + \sum_{i=1}^{n} \alpha_i \mathrm{e}^{-\mathrm{i}\lambda_i t} e_i$$

and induces an obvious evolution on the torus $G \cdot [v]$.

Ad ii). The action variables I_j, $j = 1, 2, \ldots$, are globally defined, and collectively they give rise to the convex polytope (in $\mathbb{R}^n$)

$$0 \leq \sum_{j=1}^{n} I_j = 1 - |\alpha_0|^2 \leq 1$$

which is actually the standard n-simplex Δ_n in $\mathbb{R}^n$. Thus, the orbit space is just Δ_n, the singular k-toral orbits, $0 \leq k < n$ corresponding to its k-faces. ■

Remarks 8. 1. *As we have already pointed out, this result is known in different guises, though possibly not so directly* (cf. [10, 12, 38, 55, 57, 70, 71, 81]). *This concerns, in particular, the identification of action variables with transition probabilities, which is important for the sequel, in particular, in our approach to the quantum measurement problem.*

2. *The simplest case* $\dim V = 2$*, i.e.,* $\mathbb{P}(V) \cong \mathbb{P}(\mathbb{C}^2) \cong S^2$ *is already interesting*: *Schrödinger's dynamics takes place on parallels* (*associated to the "poles"* $[e_0] \equiv [0], [e_1] \equiv [1]$) *parameterized by an appropriate transition probability*: *this geometric picture has proved to be crucial for establishing a possible link with elliptic function theory* [16, 18]. *We shall say a bit more about this in the next* Subsection.

3.4. Uncertainty and Jacobi Fields

In this Subsection we show, closely following [18], that the fundamental vector field induced by the Schrödinger Hamiltonian, when restricted to a minimal geodesic connecting two orthogonal eigenstates pertaining to different energy levels is a Jacobi field thereon. We first observe that we may confine ourselves to the case of a two level system with non degenerate Hamiltonian $H = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|$ with $\lambda_0 < \lambda_1$ and $\delta h := \lambda_1 - \lambda_0$. The result we are going to discuss will hold even in the infinite dimensional case, for two different eigenvalues of H (when present).

Also recall from the preceding Subsection that $\Delta_{[v]}H = \|JH^\sharp|_{[v]}\|$. The vector field $\mathcal{J} := JH^\sharp$, taken along a (minimal) geodesic curve joining two orthogonal eigenstates of H (this is just a half-meridian in $S^2 \cong \mathbb{P}(\mathbb{C}^2)$ viewed as a totally geodesic submanifold of the full projective space) is perpendicular to it at every point (see also below).

Theorem 9 ([18]). i) *The dispersion* $\Delta_{[v]}H$ *equals* $\delta h \cdot r_\vartheta$*, with* r_ϑ *the radius of the parallel with colatitude* ϑ *pertaining to the sphere with radius* $\frac{1}{2}$.

ii) *The vector field* $\mathcal{J}$ *is a Jacobi vector field when restricted to a geodesic connecting two orthogonal eigenstates corresponding to different energy levels.*

Proof: Assertion i) amounts to state that the Fubini-Study metric for $\mathbb{P}(\mathbb{C}^2)$ coincides with the standard metric on a sphere of radius $1/2$ (whose curvature is $K = 4$), however we can directly check our assertion as follows. First one has $r_\vartheta =$

$\frac{1}{2}\sin\vartheta = \sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}$ and then we explicitly compute the dispersion. We have, setting as usual (see below, Remark 2) $|v\rangle = z_0|0\rangle + z_1|1\rangle = \cos\frac{\vartheta}{2}|0\rangle + e^{\mathrm{i}\varphi}\sin\frac{\vartheta}{2}|1\rangle$, with $\vartheta \in [0,\pi]$ the "colatitude" taken along a "meridian" (on the standard S^2), and $\varphi \in [0, 2\pi)$ the "longitude"

$$\Delta_{[v]}H = (\lambda_1 - \lambda_0)|z_1|(1-|z_1|^2)^{\frac{1}{2}} \equiv \delta h\,(|z_1|(1-|z_1|^2)^{\frac{1}{2}}) = \delta h \cdot r_\vartheta$$

as desired, and $s = \frac{\vartheta}{2}$ is then the geodesic parameter along a meridian with respect to the Fubini-Study metric. Then assertion ii) is also immediate

$$\frac{\mathrm{d}^2\mathcal{J}}{\mathrm{d}s^2} + 4\mathcal{J} = 0$$

which is indeed the Jacobi equation since the sectional curvature of the appropriate plane is $K = 4$ (i.e., the constant holomorphic sectional curvature, see [31], [48]). ■

Another quick proof of ii) is the following: a rotation around the "polar axis" induces a geodesic variation with fixed extremities, which, when infinitesimalized, gives rise to a Jacobi field (cf. e.g. [74]). In our case, it is immediately verified via elementary geometry that the Jacobi field is, up to a constant, the one given above.

Remarks 10. 1. *We clearly see that the Heisenberg Uncertainty Principle is essentially a manifestation of curvature.*

2. *Notice that the standard parametrization of S^2 with "half-angles" comes from stereographic projection, and the inhomogeneous complex variable $\zeta := \frac{z_1}{z_0} = \mathrm{e}^{\mathrm{i}\varphi}\tan\frac{\vartheta}{2}$ is just the coordinate of the projection onto the "equatorial" plane - taken from the "south pole"* $[1]$ *- of the above point* $[v]$ *on the sphere, having colatitude* ϑ *(i.e., the angle between* $[0]$ *and* $[v]$*) and longitude* φ*, measured from one fixed meridian. In the projective line picture,* $[0]$ *is the origin of a projective frame in which* $[1]$ *is the point at infinity and* $[|0\rangle + |1\rangle]$ *is the unit point.*

We may call the quantity $r_\vartheta^2 = \frac{\Delta H^2}{\delta h^2}$ the *geometrical uncertainty*, in view of its purely geometric origin.

Also recall, in passing, that the quantity $\tau_Z = 1/\Delta_{[v]}H$ is called the *Zeno time* and plays an important role in quantum measurement theory [49].

3.5. Hydrodynamical Aspects of Geometric Quantum Mechanics

We now show that, in the framework of finite dimensional geometric quantum mechanics, the Schrödinger velocity field on projective Hilbert space is divergence-free (being Killing with respect to the Fubini-Study metric) and fulfils the stationary Euler equation, with pressure proportional to the Hamiltonian uncertainty

(squared). We explicitly compute the pressure gradient of this "Schrödinger fluid" and determine its critical points. Its vorticity is also calculated and shown to depend on the spacings of the energy levels.

The vector field $X =: (-\mathrm{i}H)^\sharp$ is called the Schrödinger vector field on $\mathbb{P}^n$ (the Schrödinger equation reads, of course, $\partial_t \,|\, v\rangle = -\mathrm{i}H \,|\, v\rangle$) and is Killing thereon (hence divergence-free). It is also stationary since the Hamiltonian H is time independent.

We shall use the representation $\mathbb{P}^n \equiv \mathbb{P}(\mathbb{C}^{n+1}) \cong S^{2n+1}/S^1$, where S^{2n+1} is the $2n+1$-dimensional sphere in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2(n+1)}$.

Then Theorem 3 immediately implies part of the following

Theorem 11 ([115]). i) *If $(M, g) = (\mathbb{P}^n, g_{FS})$, and X is the Schrödinger vector field pertaining to the Hamiltonian H, then X fulfils the* stationary Euler equation *with $2p = (\Delta H)^2$.*

ii) *The critical points of the* pressure*, in the Schrödinger case, are given by the energy eigenstates (minima, zero pressure) and by the equal probability superpositions of pairs thereof.*

iii) *The* vorticity *two-form $w = \mathrm{d}X^\flat$, evaluated on the geodesic sphere S_{ij} - with area two-form $\mathrm{d}\sigma$ and colatitude ϑ - determined by the superpositions of two energy eigenstates, reads* (*see below for details*)

$$w|_{S_{ij}} = 2\,(\delta h)_{ij}\,\cos\vartheta\,\mathrm{d}\sigma\,.$$

Proof: Ad i). This is just an application of Theorem 3, i). Of course, the remaining assertions of that result hold in the present case. As a consistency check (see also the third remark in the preceding section) observe that, in the projective line (Riemann sphere) case, on the equator one has critical (actually maximal) uncertainty and the Schrödinger trajectory is a geodesic.

Ad ii). In order to determine the critical points of the quantum mechanical pressure field explicitly, we proceed as follows.

Set $\varrho_i^2 = |\alpha_i|^2$ and $f = (\Delta H)^2$ as a function of the ϱ_i, namely

$$f = \sum_{i=0}^{n} \lambda_i^2 \varrho_i^2 - (\sum_{i=0}^{n} \lambda_i \varrho_i^2)^2$$

and introduce the constraint $g = \sum_{i=0}^n \varrho_i^2 \; - \; 1 = 0$. Then the critical points of f, subject to $g = 0$, are given by the solutions of the (Lagrange) system

$$\mathrm{d}f = \mu\,\mathrm{d}g, \qquad g = 0$$

namely

$$(\lambda_i^2 - 2\,\langle vHv\rangle\,\lambda_i - \mu)\varrho_i = 0, \qquad i = 0, ..., n.$$

Upon defining $\mathbb{P}(\lambda) = \lambda^2 - 2\langle v|Hv\rangle - \mu$, we see that, if we have a solution with $\varrho_k \neq 0$, then λ_k must be a root of P. Therefore, since the eigenvalues are all distinct, there are at most two indices i_1, i_2 for which $\varrho_i \neq 0$, and this leads to $\langle v|Hv\rangle = \lambda_{i_1}\varrho_{i_1}^2 + \lambda_{i_2}\varrho_{i_2}^2 = \frac{1}{2}(\lambda_{i_1} + \lambda_{i_2})$, whence from it follows that $\varrho_{i_1}^2 = \varrho_{i_2}^2 = \frac{1}{2}$, and $\mu = -\lambda_{i_1}\,\lambda_{i_2}$. The remaining possibility, that only one $\varrho_i \neq 0$, yields the eigenstates of H.

Ad iii). In computing the vorticity two-form $\mathrm{d}X^\flat$ pertaining to the Schrödinger velocity one-form $X^\flat$, we first notice that in view of the previous discussion, it is enough, in order to grasp its physical meaning, to restrict to the (totally) geodesic spheres S_{ij}, say, determined by superpositions of two energy eigenstates. The Schrödinger motion is, as already noticed, just a uniform rotation about the axis whose poles are given by the eigenstates in question. The angular velocity $\omega \equiv (\delta h)_{ij}$ equals $\lambda_i - \lambda_j$ $(i > j)$, the difference of the energy levels. We find (ϑ is the colatitude, measured appropriately, and $\mathrm{d}\sigma$ is area two-form) that the radius $R = \frac{1}{2}$, cf. [18], [40], and therefore

$$\begin{aligned} w|_{S_{ij}} = \mathrm{d}X^\flat|_{S_{ij}} &= \mathrm{d}(\omega\ R^2\ \sin^2\vartheta \mathrm{d}\varphi) \\ &= 2\,\omega\,\cos\vartheta\,(R^2 \sin\vartheta\,\mathrm{d}\vartheta \wedge \mathrm{d}\varphi) = 2\,\omega\,\cos\vartheta\,\mathrm{d}\sigma \end{aligned}$$

whence the vorticity vanishes on the equator (maximal uncertainty) and it is maximal (with opposite signs) at the poles (zero uncertainty). Notice, as a further check, that the *scalar* vorticity function $\widetilde{w} := 2\,\omega\,\cos\vartheta$ does indeed satisfy the 2d-vorticity equation on S_{ij} (obvious notation, cf. [121], Ch.17, p. 470, (1.27))

$$\frac{\partial\widetilde{w}}{\partial t} + \mathrm{grad}\,\widetilde{w}\cdot X = 0. \qquad \blacksquare$$

Remarks 12. 1. *In geometric terms, the critical points are given by the vertices and the midpoints of the Atiyah - Guillemin-Sternberg convex polytope arising from the standard moment map.*

2. *In essence, we provided, as already anticipated, an "Eulerian" counterpart to the "Lagrangian" portrait inherent to the geometric interpretation of the Schrödinger flow.*

3. *Notice that the Schrödinger motion itself can be viewed as a coadjoint orbit motion for the group* $\mathrm{U}(n+1)$. *On the other hand, the vorticity form of the Euler equation is a manifestation of a coadjoint orbit motion relative to the group of measure preserving diffeomorphisms* [8], [80]. *In our case we deal with a stationary fluid, and we get the previous equation.*

3.6. Aharonov-Anandan Phase for Mixed States

The whole framework can be extended to the case of mixed states (see e.g. the textbook [40]). We give a rapid sketch of what is going on, possibly deferring a fuller discussion elsewhere. This should be compared with the analysis of [32, 50]. In the general case the $\mathrm{U}(V)$- coadjoint orbit $\mathcal{O}_\rho$ of a density matrix ρ is a *flag manifold*, and in the non-degenerate spectrum situation, it specialises to

$$\mathcal{O}_\rho \cong \mathrm{U}(n+1)/(\,\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)\,)$$

($n+1$ factors), having real dimension

$$\dim \mathcal{O}_\rho = [2(1+2+...+n)+n+1]-(n+1)] = n(n+1).$$

The orbit $\mathcal{O}_\rho$ is equipped with the KKS-symplectic form, which, when integral, will give rise to a (holomorphic) line bundle equipped with the Chern-Bott connection, with curvature equal to the KKS form itself, and this line bundle is again the receptacle of a generalised Aharonov-Anandan-Berry phase coming out from parallel transport. Without invoking the full strength of the Borel-Hirzebruch theory, we just observe that the cohomology group $H^2(\mathcal{O}_\rho,\mathbb{Z}) \cong \mathbb{Z}^{n+1}$, is coming from transgressing the first cohomology of the fibre in $H \to G \to G/H$ in the general homogeneous space picture (see e.g. [22], Ch. 8). This is consistent with the theory set up in [32] where the Berry phase stemming from the KKS form is singled out for its naturality among other possible choices.

The Chern-Bott connection has the same expression as before. The complex lines of the GQ-line bundle sit inside $\mathfrak{u}(n+1)^{\mathbb{C}}$. This provides a quite concrete and physically relevant manifestation of the Borel-Weil theory. Flag manifolds could play also a relevant role in quantum measurement (see below).

Remark 13. *The following important remark is in order. There are of course various non-abelian generalisations of Berry's phase, e.g. the Wilczek-Zee phase* [125] *which has been exploited, via the Riemannian geometry of projective space, in the so-called holonomic quantum computation, starting from the seminal paper* [129] (*see the textbook* [40] *for further discussion*).

4. Some Applications of Geometric Quantum Mechanics

In this section we shall see how geometric quantum mechanics affects (and, in our opinion, enlightens) the treatment of many important topics in quantum mechanics, such as Berry's phase, entanglement and the measurement problem. Here is

an overview of the material. We discuss various implications of integrability [17] by looking anew at quantum adiabaticity and at the emergence of Berry's phase ([19], [108]). In view of classical complete integrability we can interpret this problem both quantum mechanically (Berry [19]) and classical mechanically Hannay ([58] and [86]), showing compatibility of the two pictures. Moreover it is interesting, in view of the statistical interpretation of quantum mechanics, to compute the partition function of the classical canonical ensemble explicitly (cf. [25–27]). This can be immediately achieved by resorting to the Duistermaat-Heckman formula ([13, 44, 57, 81]) exploiting the toral action. It is also possible to give a direct elementary computation as well (see [17]). In accordance with the suggestion of [25–27], we find that the partition function indeed differs from the standard quantum mechanical one by certain weights depending on the energy level spacings and reflecting the topological structure of the projective space as a CW-complex.

Subsequently (Subsection 4.3), we address quantum entanglement, which is, actually, the characteristic feature of quantum mechanics [106]. We discuss, in a geometric fashion via Segre maps, an entanglement criterion (Theorem 8). The variety X of disentangled states emerges as an intersection of quadrics. A recursive "localization" procedure is devised which produces a (minimal) set of equations locally cutting out X. This construction can be extended to encompass "partial" entanglement (see [18]).

Furthermore, in the spirit of [27], we build up (see Theorem 10) a geometrical portrait of the two-qubit space and its unitary operators (quantum gates) which could turn out to be useful in discussing quantum teleportation, see e.g. [40] and references therein or [66] and [72]. Intersecting the (Segre) quadric Q parametrizing disentangled states in $\mathbb{P}^3$ with a suitable "plane at infinity" yields an "absolute" conic (classical terminology) which is the image of the one-qubit space under the Veronese map. Any diagonalization of the quadric Q is obtained via a universal two-quantum gate (by the Brylinskis' theorem [30]). A natural choice is provided by the unitary R-matrix of [66].

Moreover (Subsection 4.5), we give a geometric interpretation of some aspects of the theory of quantum measurement (see e.g. [124], [49]) in the version developed, e.g. in [23] (we stress the fact that we act within orthodox quantum mechanics). The passage from a pure state to a mixture after interaction with a measuring apparatus can be described in "classical" terms as averaging over the ("fast") angle variables. One gets, as a by-product, a version of the averaging theorem (time averaging = angle averaging, [6]). The collapse of the wave function can also be described (though by no means "explained away") by resorting to basic geometric invariant theory ([56], [87]), by letting unitarity (but not linearity) be violated

during the measurement process. The latter can be "visualized" geometrically in terms of a natural polytope (parametrizing toral orbits) emerging from convexity properties of the relevant moment map (cf. [10, 12, 13, 55–57, 70, 71, 81]).

In Subsection 4.6, we discuss Aravind's intriguing idea of connecting quantum states with knots and links ([5]), further pursued e.g. in [66]. We discuss successive measurements of particular entangled states made up of identical particles (specifically photons, but also fermions can be accommodated, after taking the Pauli Exclusion Principle into due account), generalizing the so-called GHZ states [52], by resorting to standard $\mathrm{SU}(2)$ (actually $\mathrm{U}(2)$) - representation theory (Clebsch-Gordan decomposition). We obtain a clear-cut, systematic description of these successive measurements via suitable trees, determined by simplices and subsimplices in certain complex vector spaces - in passing, the whole set-up illustrates finite $\mathbb{F}_2$-projective geometries - whose leaves can be associated to Brunnian or Hopf links. The upshot is that, in agreement with Aravind and the "quantum knot" picture of Kauffman-Lomonaco [65], links are related to measurements rather than to states alone.

Then, in Subsection 4.7, we show that the second quantization can be realized via Bohr-Sommerfeld quantization (cf. also [9] and [105]).

4.1. Berry and Hannay Angles

Here we wish to reinterpret the emergence of Berry's geometric phase ([19, 20, 40, 108]) after cyclic adiabatic perturbations of the Hamiltonian within the classical interpretation of the quantum mechanical formalism outlined in the previous section. In an adiabatic evolution of a non-degenerate Hamiltonian $H = H(R)$ depending on a point $R \in \mathcal{R}$ (parameter space, of dimension ≥ 2) the eigenvectors evolve into eigenvectors (see e.g. [23], [82] and particularly [108] for a careful discussion of the "quantum adiabatic theorem") and, if the evolution is also cyclic, a final eigenvector differs from the initial one by a phase factor (Berry's phase), which can be ascribed to the parallel transport via the Chern-Bott connection on $\mathcal{O}(1)$. In what follows we shall neglect the so-called dynamical phase, coming from Schrödinger's equation. Explicitly, if $C : [0, T] \to \mathcal{R}$ denotes a closed oriented circuit in the parameter space

$$e_j(C(T)) = \mathrm{e}^{\mathrm{i}\int_C -\mathrm{i}\langle e_j(R)|\, d_{\mathcal{R}}\, e_j(R)\rangle} \cdot e_j(C(0)) =: \mathrm{e}^{\mathrm{i}\Delta\vartheta_j^B} \cdot e_j(C(0))$$

since, again by the very definition of the Chern-Bott connection, the infinitesimal angle variation, say $\mathrm{d}\vartheta_j$, of $e_j(R)$ in the complex plane in V it determines is

$-\mathrm{i}\langle e_j(R)|\,\mathrm{d}_{\mathcal{R}}\,e_j(R)\rangle$ (the differential being now taken with respect to the parameter space $\mathcal{R}$, pulling back everything from $\mathbb{P}(V)$ to $\mathcal{R}$). We have tacitly assumed that in our evolution $e_0(C(t)) \equiv e_0$ for all $t \in [0,T]$.

Now, the adiabatic perturbation induces a *migration* of the Lagrangian tori (and isotropic ones) pertaining to the quantum system, happening on the trivial fibration $\mathcal{R} \times \mathbb{P}(V) \to \mathcal{R}$, but this is exactly the framework leading the appearance of Hannay's angles ([40, 58, 86]). The migration is governed by Montgomery's connection (given by averaging over tori [86]). The two pictures are compatible - upon computing the relevant Hannay's angles $\Delta\vartheta_j^H$, we recover Berry's phases $\Delta\vartheta_j^B$.

Theorem 14 ([17]). *With the above notations, we have*

$$\Delta\vartheta_j^H = \Delta\vartheta_j^B, \qquad j = 1, 2, \ldots, n.$$

Proof: Averaging $\mathrm{d}\vartheta_j$ over the torus G, with respect to its normalized Haar measure $\mathrm{d}g$, leaves it unaltered: $<\mathrm{d}\vartheta_j>_G = \mathrm{d}\vartheta_j$ by virtue of its geometric significance. The full Hannay angle $\Delta\vartheta_j^H$ is obtained by integrating along the closed oriented circuit C in the parameter space, yielding

$$\Delta\vartheta_j^H = \int_C (-\mathrm{i}\langle e_j(R)|\,d_{\mathcal{R}}\,e_j(R)\rangle) = \Delta\vartheta_j^B, \qquad j = 1, 2, \ldots, n. \qquad \blacksquare$$

The geometric phase phenomenon can be synthetically described via the density matrix formalism. The off-diagonal (interference) terms have indeed been detected in specific experiments.

4.2. The Partition Function

In classical statistical mechanics it is natural to consider, among others, the canonical ensemble partition function pertaining to a classical Hamiltonian system. This is particularly relevant in our case in view of the statistical interpretation of quantum mechanics. So in this section we are going to compute the canonical ensemble partition function $Z = Z(\beta)$ associated to the Hamiltonian system $(\mathbb{P}(V), \omega, h)$ explicitly, slightly improving some results in [25–27]. Recall that

$$Z(\beta) = \int_{\mathbb{P}(V)} \mathrm{e}^{-\beta h([v])}\,\frac{1}{n!}\omega^n$$

where $\beta \in \mathbb{R}$ (actually, the formula holds for β complex, after suitable interpretation), and ω is, in this section, *one-half* of the previous one, in order to comply with the convention adopted in [81]. So we resort to the *Duistermaat-Heckman formula* [44, 56, 81], concerning exactness of the stationary phase approximation.

This is possible in view of the toral invariance of the classical Hamiltonian h and proper handling of square roots eventually yields the following

Theorem 15 ([17]). *The partition function Z pertaining to $\mathbb{P}(V)$ and to the G-invariant Hamiltonian h reads, explicitly*

$$Z(\beta) = (\frac{\pi}{\beta})^n \sum_{j=0}^{n} w_j \mathrm{e}^{-\beta\lambda_j}$$

where

$$w_j = \prod_{i\neq j} (\lambda_i - \lambda_j)^{-1}.$$

We notice that, as a retrospective check, one gets $\lim_{\beta\to 0}$ r.h.s. $= \frac{\pi^n}{n!} = \mathrm{vol}(\mathbb{P}(\mathrm{V}))$. So, following [27], we may assert that the canonical partition function differs from the standard quantum mechanical one in that the presence of the weights w_j encodes information about energy level spacings, this being related to the Hessian of the Hamiltonian at critical points, which, in turn, is related to the topology of $\mathbb{P}(V)$ as a CW-complex via Morse theory. Recall that $\mathbb{P}(V)$ is made up of $2k$-dimensional cells, one for each $k = 0, \ldots, n$, this being also reflected by the (de Rham) cohomology algebra, which is generated by the Fubini-Study form, whose various exterior products yield the appropriate Poincaré-Cartan invariants (see e.g. [53]).

It has been advocated in [25] and [27] that this "classical" partition function is more natural than the standard quantum mechanical one since it does not stick to stationary states from the outset. We have shown that nevertheless the latter naturally arise via Duistermaat-Heckman, and this somehow reconciles the two perspectives.

4.3. Geometric Entanglement Criteria

In this section we review one general entanglement criterion given in [18]. We resort to the Segre embedding, familiar from classical algebraic geometry (see e.g. [59] and [15]. This approach is also briefly outlined in [29], but it will be useful to discuss it more explicitly. More general results are given in [18], see [51] as well.

Let us review the Segre and Veronese embeddings, referring to [59] for full details. Given (complex) vector spaces V and W of respective dimensions $n+1$ and $m+1$, the Segre map $S : \mathbb{P}(V)\times\mathbb{P}(W) \to \mathbb{P}(V\otimes W)$ (the latter space has then dimension $(n+1)(m+1)-1$) is intrinsically given by $([v],[w]) \mapsto [v\otimes w]$. In terms of

homogeneous coordinates, it reads (with obvious notation)

$$\begin{aligned} S:\ \mathbb{P}^n \times \mathbb{P}^m &\to \mathbb{P}^{(n+1)(m+1)-1} \\ ([Z_i],[W_j]) &\mapsto [Z_i W_j] \end{aligned}$$

where $i = 0, ..., n$, $j = 0, ..., m$ and lexicographic ordering is adopted.

The Veronese map $\nu_d : \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^d V) \hookrightarrow \mathbb{P}(V^{\otimes d})$ is intrinsically given by $[v] \mapsto [v \otimes .... \otimes v] \equiv [v^d]$. Here $\mathrm{Sym}^d V$ denotes the dth-symmetric tensor power of V. If $\dim V = 2$, we get a curve in $\mathbb{P}^d$, called the *rational normal curve*. It is immediately checked that the image of ν_d is given by the common zero locus of the polynomials $Z_i Z_j - Z_{i-1} Z_{j+1}$, $1 \leq i \leq j \leq d-1$.

Here $(V, \langle\cdot|\cdot\rangle)$ will be again a Hilbert space of dimension two, with a choice of an orthonormal basis $\{|0\rangle, |1\rangle\}$, with one-dimensional associated complex projective space $\mathbb{P}(V) \cong \mathbb{P}^1 \cong S^2$. Concretely, and also in view of further analysis later on, one may consider the space of polarization states for a monocromatic electromagnetic wave. The chosen orthonormal basis may represent the (right and left-handed) circularly polarized states, yielding the eigenstates of the helicity operator $\mathcal{H}$ (the analogue of spin for photons, see [40] and below for further discussion of this point). Thus V can be regarded as the carrier of the fundamental representation of $\mathrm{U}(2) \cong \mathrm{SU}(2) \times \mathrm{U}(1)$.

Let $V^{\otimes n}$ denote the n-fold tensor product of V (the n-qubit space). In view of enforcement of Bose-Einstein statistics, we are also interested in $\mathrm{Sym}^n V$ the *fully symmetric* part of $V^{\otimes n}$, which, upon resorting to the Clebsch-Gordan theory (see e.g. [88], [78], [82]), is given by $V_{\frac{n}{2}}$, the $(n+1)$-dimensional space pertaining to the $\frac{n}{2}$-spin representation (of $\mathrm{SU}(2)$).

A state in $\mathbb{P}(V^{\otimes n})$ (which has dimension $2^n - 1$) is (completely) disentangled if it is of the form $[\xi_1 \otimes ... \otimes \xi_n]$, i.e., if it comes from a decomposable vector $|\xi_1 ... \xi_n\rangle$. These states build up the (generalized) Segre variety $X \subset \mathbb{P}^{2^n-1}$. The corresponding Veronese curve describes the completely symmetric and disentangled states. Since it is *nonlinear*, it is not physically realizable (*no cloning theorem*). In particular, in the one-qubit space case only the chosen basis vectors $|0\rangle$ and $|1\rangle$ can be copied and $\mathbb{P}(V)$ is embedded via ν_2 into $\mathbb{P}(\mathrm{Sym}^2 V)$ as a conic C (whose only physically realizable states are then $|00\rangle$ and $|11\rangle$).

Although the following theorem can be subsumed by a more general result (see e.g. [29] and [18]), it is possibly useful to discuss it separately, in view of its special importance, and for the explicit proof we give. The notation is as follows: the projective space (homogeneous) coordinates of a point in $\mathbb{P}^{2^n-1}$ can be represented as $[Z_\gamma]$, $\gamma = 0, ..., 2^n - 1$, with γ written in binary form, so, for instance, if $n = 3$ one has $[Z_{000}, Z_{001}..., Z_{111}]$ and the suffix $\alpha 0_k$ - with $\alpha = 0, 1, \ldots, 2^{n-1} - 1$ -

is just a string of n binary digits given by the ones of α, with the kth position occupied by 0 (so they are $n-1$). A similar meaning is attached to $\alpha 1_k$. Thus, for example, if $n=4$, $\alpha=5$, $k=3$, one has $\alpha 0_k = 1001$.

Theorem 16 ([18]). i) *The set of completely disentangled states is an algebraic subvariety* (*generalized Segre variety*) $X \subset \mathbb{P}^{2^n-1}$ *of dimension* n *and degree* $n!$ *cut out set-theoretically by the family of quadratic polynomials*

$$Q_{\alpha,k} = Z_{00_k} Z_{\alpha 1_k} - Z_{01_k} Z_{\alpha 0_k}$$

where $\alpha = 1, ...2^{n-1}-1$ *and* $k = 1, 2, ...n-1$*, i.e.,* X *is the common zero locus of the* $(n-1)\cdot(2^{n-1}-1)$ *polynomials* $Q_{\alpha,k}$*. Geometrically,* X *is the intersection of the quadric hypersurfaces* $Q_{\alpha,k}=0$*. Equivalently,* X *is the common zero locus of the polynomials*

$$Q_{\alpha,\beta,k} = Z_{\alpha 0_k} Z_{\beta 1_k} - Z_{\alpha 1_k} Z_{\beta 0_k}$$

where $\alpha, \beta = 0, 1, ..., 2^{n-1}-1$ $(\alpha \neq \beta)$ *and* $k = 1, 2, ...n-1$.

ii) *A recursive change of coordinates procedure can devised so as to produce an "optimal" set of* $2^n - n - 1$ *equations.*

Proof: The (necessary and sufficient) disentanglement conditions for the first particle state read

$$\frac{\alpha_0^{(1)}}{\alpha_1^{(1)}} = \frac{Z_{0\beta}}{Z_{1\beta}}, \qquad \beta = 0, 1, \ldots, 2^{n-1}-1.$$

Thus we get $2^{n-1}-1$ equations for the Z's. The fact that k ranges from 1 to $n-1$ is clear since the conditions for $k=n$ are automatically fulfilled if the preceding ones are (if $n-1$ states are disentangled, the remaining one is such). Thus we obtain $(n-1)\cdot(2^{n-1}-1)$ equations, which can be put in the form $Q_{\alpha,k}=0$. Vanishing denominator situations are easily handled.

Now, if we denote the homogeneous coordinates of $\mathbb{P}^{2^{n-1}-1}$ collectively by Z', we get, for the embedding $\mathbb{P}^1 \times \mathbb{P}^{2^{n-1}-1} \hookrightarrow \mathbb{P}^{2^n-1}$ the equations

$$Z_{0\beta} = \alpha_0^{(1)} \cdot Z'_\beta, \qquad Z_{1\beta} = \alpha_1^{(1)} \cdot Z'_\beta$$

which enable us to compute Z'_β. The special case in which one of the α's vanishes is easily settled, and correspond to a disentangled state containing one of the basis vectors in the first copy of V. Then, proceeding inductively, we get $(2^{n-1}-1) + (2^{n-2}-1) + \cdots + (2^0-1) = 2^n - 1 - n$ equations *locally* cutting out, set theoretically, the variety X (this number equals the codimension of X). The above procedure can be easily algorithmically implemented. ■

Remark 17. *The above proof can be used to check partial entanglement conditions as well, i.e., whether a certain "particle" is disentangled from the others.*

As a simple application of the above criterion, we observe that the *symmetry* (*or antisymmetry*) *operator is in general entangling*, i.e., transforms a disentangled quantum state into an entangled one. Specifically, we consider the following example: take the n-particle state vector $\Psi = |0\,\alpha\rangle$, $\alpha \neq 0...0$ ($n-1$ binary digits). Then its symmetrization $S|\psi\rangle$ induces an entangled state. Indeed, the initial state has just one non vanishing component $Z_{0\,\alpha} = 1$. In view of the above assumption, $S\Psi$ is a supersposition of the states labelled by the appropriately permuted digits containing $|1\,\beta\rangle$, for some β. Then $Z_{1\,\beta} = 1$ (it is not necessary to normalize). But clearly $Z_{1\,\alpha} = Z_{0\,\beta} = 0$, whence $Z_{0\,\alpha}\,Z_{1\,\beta} - Z_{1\,\alpha}\,Z_{0\,\beta} = 1 \neq 0$, yielding the conclusion. Actually, one has the following

Theorem 18 ([18]). *Any symmetric disentagled state must be of the form* $[\xi^n]$, $\xi \neq 0$, *i.e., it is a point on the Veronese curve. The latter can be cut out by the above quadrics* $Q_{\alpha,\beta,k} = 0$, *in addition to the hyperplanes* $Z_\gamma - Z_{\sigma\cdot\gamma} = 0$, *with* σ *denoting any permutation from the symmetric group* S_n *acting on* $\gamma \in \{0, ..., 2^n - 1\}$, *written in binary form* (*redundancies occur*). *Thus one abuts again at an intersection of quadrics.*

4.4. On the Geometry of Quantum Two-Gates

This Subsection furnishes an illustration of the preceding techniques and it is quite close to the discussion of spin one-systems given in [27], see also [72], [40].

Consider the so-called Bell basis in $V \otimes V$ given by $(\varphi^+, \varphi^-, \psi^+, \psi^-)$, with

$$|\varphi^+\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\varphi^-\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\psi^+\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \qquad |\psi^-\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

We have the following

Theorem 19 ([18]). *The basis* $(\psi^-, \psi^+, \varphi^+, \varphi^-)$ *of* $V \otimes V \cong \mathbf{C}^4$ (*made up of entangled states*), *gives rise, projectively, to a self-polar tetrahedron in* $\mathbb{P}^3$ (*with respect to the polarity induced by the* (*Segre*) *quadric* Q *of disentangled states*), *namely, the equation of the quadric* Q *takes* (*after appropriate adjustment*) *the* projective *canonical form*

$$\xi_0{}^2 + \xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2 = 0.$$

Taking the plane $\pi_\infty : \xi_3 = 0$ as the plane at infinity, the conic $C = \pi_\infty \cap Q$ is the image of the Bloch sphere $\mathbb{P}(V)$ under the Veronese map.

Proof: Consider the following modified Bell basis $(\widetilde{\varphi}^-, \widetilde{\varphi}^+, \widetilde{\psi}^+, \widetilde{\psi}^-)$, with $\widetilde{\varphi}^+ = \varphi^+$, $\widetilde{\varphi}^- = -\mathrm{i}\,\varphi^+$, $\widetilde{\psi}^+ = -\mathrm{i}\,\psi^+$, $\widetilde{\psi}^- = \psi^-$ (they give rise to the same states), with respective coordinates $(\xi_0, \xi_1, \xi_2, \xi_3)$. One has (obvious notation)

$$\xi_0 = \tfrac{1}{\sqrt{2}}(x_{00} + x_{11}), \qquad \xi_1 = \tfrac{\mathrm{i}}{\sqrt{2}}\,(x_{00} - x_{11})$$
$$\xi_2 = \tfrac{\mathrm{i}}{\sqrt{2}}\,(x_{01} + x_{10}), \qquad \xi_3 = \tfrac{1}{\sqrt{2}}(x_{01} - x_{10}).$$

Therefore, the equation of Q, becomes

$$\xi_0{}^2 + \xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2 = 2\,(x_{00}x_{11} - x_{01}x_{10}) = 0$$

as claimed. Intersecting it with π_∞, we see that C coincides with the Veronese curve on that plane (indeed $\xi_3 = 0$ enforces the symmetry condition $x_{01} = x_{10}$). The geometrical assertions come from rephrasal in classical algebro-geometric language. Also, the points $[\varphi^+]$ and $[\varphi^-]$ lie on the polar of $[\psi^+]$ with respect to C, and, together with $[|00\rangle]$ and $[|11\rangle]$, belonging to C, give rise to a harmonic quadruple (in an approriate order), whereas the tangents drawn therefrom meet in $[\psi^+]$. ■

Remark 20. *By virtue of a theorem of J. and R. Brylinski* [30]*, the above change of basis yields a universal quantum gate, in quantum computing terminology.*

4.5. On the Quantum Measurement Problem

The quantum measurement problem is actually the most tantalizing problem concerning the interpretation of quantum mechanics (we refer to [124], [49] for a thorough discussion). In this section we just make some remarks aiming at reinterpreting (part of) the treatment of the measurement problem given by D. Bohm in his "orthodox" book [23], following [17].

The upshot of his fairly detailed analysis (based on the Stern-Gerlach experiment and generalizations thereof) is that upon measuring an observable, say the energy H, a superposition of its eigenstates goes to a different superposition characterized by *uncontrollable* (relative) phase shifts (in view of the Heisenberg Uncertainty Principle)

$$\sum_j \alpha_j e_j \to \sum_j \alpha_j \mathrm{e}^{\mathrm{i}\beta_j} e_j.$$

We take, for definiteness, $\alpha_j \neq 0$ for all j, and, as before, we may arrange things so that $\beta_0 = 0$. We consider the Schrödinger-von Neumann quantum evolution as taking place on the space of density matrices (mixed states), which may be identified, up to an i factor, with a submanifold of $\mathfrak{u}(V)$. More explicitly, let ϱ be a density matrix, i.e., a positive operator ($\varrho \geq 0$) on V with $\mathrm{Tr}\varrho = 1$. Its evolution is governed by the von Neumann's equation

$$\frac{\partial \varrho}{\partial t} = -\mathrm{i}[H, \varrho]$$

which, when applied to a pure state, reproduces the (projective form of the) Schrödinger equation given above. Furthermore, in the notation of Section 3, see also above, one has $\varrho = |v\rangle\langle v| = (\bar{\alpha}_i \alpha_j)$.

Geometrically, the above equation says that the (undisturbed) evolution of a density matrix takes place on a $\mathrm{U}(V)$-coadjoint orbit (with the customary identification of adjoint and coadjoint action via, e.g. the Killing metric on $\mathfrak{u}(V)$ and up to an "i" factor), which is a symplectic manifold. This picture can be naturally supplemented by a C*-algebraic one. Indeed, the density matrices constitute precisely the *state space* of the finite dimensional C*-algebra $B(V)$ consisting of all linear operators on the finite dimensional space V (so they are necessarily bounded), see e.g. [79]. This space is closed under *convex* combinations, and this will be crucial for what follows.

Now, roughly speaking for the moment, the point is that, upon averaging over the phases (i.e., over a (long) series of measurements), one gets a diagonal density matrix $\rho := (|\alpha_j|^2 \delta_{ij})$ giving rise to a statistical ensemble in which an assembly of equal systems is partitioned in subsystems with energy values λ_j in proportions $|\alpha_j|^2$. In view of the classical interpretation of the quantum formalism outlined above, we can rephrase the preceding description by saying that the measurement process gives rise to an *adiabatic* perturbation (since the action variables, i.e., the transition probabilities, do not change). Hence, as in perturbation theory in classical mechanics, one averages over the "fast" (i.e., angle) variables, namely, over an n-dimensional torus (since a global phase change yields nothing), this boiling down to the mixed state above. More precisely, we may state the following kind of *averaging theorem* (cf. [6]) (valid in the non degenerate case), whose proof is straightforward

Theorem 21 ([17]). *In terms of density matrices, the following formula holds*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \mathrm{e}^{-\mathrm{i}Ht} \cdot [(\bar{\alpha}_i \alpha_j)]\, \mathrm{d}t = \int_G g \cdot [(\bar{\alpha}_i \alpha_j)]\, \mathrm{d}g = (|\alpha_j|^2 \delta_{ij}) = \rho$$

where $\mathrm{e}^{-\mathrm{i}Ht\cdot}$ denotes the standard Schrödinger evolution, $g\cdot$ stands for the toral action, whereas $\mathrm{d}g$ again denotes the normalized Haar measure on G. Notice that both integrals make sense since they both represent generalized convex combinations of (pure) states, so they still define density matrices.

This "phase wash-out" or "decoherence" (see e.g. [124], [49]) can be described geometrically by saying that the torus action determines a transition from the pure state $\mathrm{U}(V)$-coadjoint orbit given by $\mathbb{P}(V)$ to the (mixed state) one labelled by ρ, i.e., another *flag manifold* (of maximal dimension), cf. Subsection 3.6. The Hamiltonian, clearly, does not change.

Next we would like to present a geometric description of the "collapse of the wave function", which should supplement the preceding mechanism, in terms of basic notions from geometric invariant theory ([87], [55]). Upon resuming the discussion in Subsection 2.3, we take, in our case $X = \mathbb{P}(V)$, $G = \mathbb{T}^n$, $\mathfrak{g} = \mathrm{i}\mathbb{R}^n$, $G^{\mathbb{C}} \cong (\mathbb{C}^*)^n$, $\mathfrak{g}^{\mathbb{C}} = \mathrm{i}\mathbb{R}^n \oplus \mathbb{R}^n$, $\mu([v]) = (I_1, I_2, \ldots, I_n)$ (here μ denotes the toral moment map naturally induced from the $u(V)$ one, up to a scalar) and the above quotients are both reduced to the point $[e_0]$. The vertices of the polytope also correspond to the absolute minima ($[e_0]$) and maxima ($[e_j], j = 1, 2, \ldots, n$) of the norm square of the toral moment map μ. The slightly asymmetrical role of the critical points $[e_j]$ just stems from our initial conventions. The action of the complex torus is no longer unitary (it is indeed a Lie subgroup of the full linear group $\mathrm{GL}(V)$).

In view of the basic formalism applied to P_j (complexification of $-\mathrm{i}P_j$) we get the following geometric portrait: upon measuring the energy $H = \sum_j \lambda_j |e_j\rangle\langle e_j| = \sum_j \lambda_j P_j$ (we always require non-degeneracy of the energy levels), the system undergoes a gradient flow motion (with respect to the Fubini-Study metric) starting from an initial state $[v]$ with velocity field $P_j^\sharp$ with probability $I_j = |\alpha_j|^2$. The velocity diminishes by gradual loss of uncertainty provided by the measurement until in the limit $t \to +\infty$, one gets for the energy the exact value λ_j, corresponding to the critical point $[e_j]$ of the Hamiltonian.

It is indeed easy to check that, under the evolution $[v] \mapsto \mathrm{e}^{tP_j} \cdot [v]$, one has, provided $\alpha_j \neq 0$

$$\lim_{t \to +\infty} \mathrm{e}^{tP_j} \cdot [v] = [e_j]$$

yielding the desired collapse, or reduction, of the superposition $[v]$ to the stationary state $[e_j]$.

The "dissipative" process in question involves a violation of unitarity - this is mathematically clear, as we have seen, and it is physically reasonable as well, since we discuss the system evolution alone, neglecting both the measuring apparatus and the environment, cf. [49] - but linearity is retained. Resorting to the geometric picture of the orbit space, we may also say that the collapse of the wave function

consists, geometrically, in a point in the polytope being "forced" onto one of its vertices, with probabilities given by its $\mathbb{R}^n$-coordinates. The origin corresponds to the critical point $[e_0]$. Also, during the process, adiabaticity (action invariance, i.e., probability conservation) is clearly destroyed.

We stress the fact that our geometric picture should be seen as a (possibly useful) description, not as a "realistic" explanation. On the other hand, various mechanisms of dissipation have been invoked in the physical literature (see [49] for a thorough recent discussion) in connection with the collapse of the wave function. Among these, the idea of relaxing unitarity whilst keeping linearity is also present. Geometric invariant theory possibly makes this mathematically natural.

We may also depict the following "hydrodynamical" picture of the "collapse of the wave function" that is performing an energy measurement on a quantum system causes a perturbation of the Schrödinger fluid, forcing the quantum state to reach to a minimal (indeed, zero) pressure, i.e., an eigenstate.

The geometrical and hydrodynamical set-up may be useful in "visualising" the Quantum Zeno Effect (see e.g. [49], p.110, 3.3.1). Continual measurement "freezes" the motion: the rate of decay of a pure state (as a function of t) goes as $(\Delta H)^2 t^2$, the "space" (squared) travelled by the state under the Schrödinger motion (Lagrangian portrait), and related in turn to the fluid pressure. Upon repeating the measurement N times within the time interval t one finds $(\Delta H)^2 \frac{t^2}{N}$, tending to zero as N goes to infinity.

4.6. Brunnian Links, Projective Geometry and Measurement

In this Subsection we wish to point out the emergence of a possibly interesting geometrical pattern in discussing measurements made upon particular entangled states. We first resume the discussion begun in Subsection 4.3.

The eigevalues of the helicity operator $\mathcal{H}$ are $\pm n, \pm(n-2), \ldots, \pm(n-2[\frac{n}{2}])$, with (non normalized) eigenvectors given (up to phase) below, starting from $\mathcal{H}|0\rangle = |0\rangle$, $\mathcal{H}|1\rangle = -|1\rangle$

$$\begin{aligned} \phi_n &= |0...0\rangle \\ \phi_{n-2} &= |1...0\rangle + |01...0\rangle + ... + |0...1\rangle \\ &... \\ \phi_{-n} &= |1...1\rangle. \end{aligned}$$

The (non normalized) state (vector) $\phi_n + \phi_{-n} = |0...0\rangle + |1...1\rangle$ is a generalized GHZ-state (see e.g. [52]) and a measurement of the helicity carried out upon any particle yields a completely disentangled state as outcome (either $|0...0\rangle$ or

$|1...1\rangle$). According to the suggestion of Aravind, this arrangement (state *plus* measurement!) can be depicted by a Brunnian link (a link such that removing any of its components yields a trivial link. In the case $n = 3$ we find the celebrated *Borromean rings*.

We now wish to show that similar remarks apply to the states ϕ_{n-2} etc. confining ourselves to the first one. The following statement is easily proved, and we refer to any book in graph theory for the basic terminology. For the knot theory involved, we limit ourselves to mention [102] and [64].

Theorem 22 ([18]). i) *All potential successive measurements of the state*

$$\phi_{n-2} := f_1 + f_2 + ... + f_n$$

give rise to an oriented graph *which can be geometrically portrayed as follows*: *its* nodes *are the vertices of the simplex* $(f_1, f_2, ...f_n)$ *in* $\mathbb{C}^n$ *where*

$$f_j := |00...1...0\rangle$$

(1 *at the* j*th position*)*, together with the barycentres of its various subsimplices and in total, they are amount to* $2^n - 1$.

The $n + 1 = (n - 1) + 2$ *points* $[f_1], [f_2], ..., [f_n], [\phi_{n-2}]$ *provide a projective frame for the complex* $(n - 1)$*-dimensional projective space corresponding to* $< f_1, f_2, ..., f_n >$*, with* $[\phi_{n-2}]$ *being the unit point. Furthermore, upon passing to* $\mathbb{F}_2$*-coefficients* ($\mathbb{F}_2$ *being the Galois field with two elements*)*, one gets the projective space* $\mathbb{P}(\mathbb{F}_2^n)$. *Its* arrows *connect a barycentre with a basis vertex and with the* (*sub*)*face opposite to it.*

ii) *The successive measurement of the state* ϕ_{n-2} *with respect to a fixed particle* (*or, better, position*) *give rise to a* binary tree $(B_{n-2}, B_{n-3}, ...B_0, B_0')$. *The* leaves B_i *can be depicted as Brunnian* (*or Borromean*) *links of decreasing complexity. The last two leaves are* (*two-component*) *Hopf links.*

Comment. We briefly discuss the case $n = 3$. Upon measuring helicity in the state $[f_1 + f_2 + f_3]$, if say, we measure 1 at the first position, then we get $[f_1]$, which is completely disentangled, so the leaf B_1 is represented by the Borromean rings. Upon measuring 0, we find $[f_2 + f_3]$, and the state is partially entangled. A successive measurement (of the second particle) produces a disentangled state in both cases, so the corresponding leaves B_0 and B_0' can be both represented by a Hopf link (discarding a disjoint circle given by the first measured particle). Geometric interpretation leads to the well-known (projective) *Fano plane*.

4.7. Second Quantization and Bohr-Sommerfeld Quantization

In this section we discuss another implication of complete integrability, following [17]. Recall that in [105] and [9] it is observed that (geometric) quantization of a quantum mechanical system looked upon classically yields its second quantization. We comment on this as follows: having realized a (finite dimensional) quantum mechanical system as a classically completely integrable system (with the Riemannian structure giving the extra "quantum" ingredient) formally resembling a collection of classical harmonic oscillators (with a norm constraint) - this is clear from Section 3, but see also, e.g. [60] - we may wish to quantize it, for instance, via Bohr-Sommerfeld quantization (ignoring the Maslov correction for the moment, see e.g. [127]). We proceed as follows: first recall the formula for the classical Hamiltonian h (for $\| v \| = 1$ and $\lambda_0 = 0$)

$$h([v]) = \langle v|Hv\rangle = \sum_{j=1}^{n} \lambda_j I_j.$$

Next, Bohr-Sommerfeld quantization requires in our case that

$$I_j = n_j \in \mathbf{N}, \quad j = 1, 2, \ldots, n$$

giving rise to the (non negative) energy levels

$$H(\{n_j\}) = \sum_{j=1}^{n} \lambda_j n_j.$$

Taking into account the bounds $0 \leq I_j \leq 1$, $j = 1, 2, \ldots, n$, this is possible if and only if $n_j = 0$ for all j's or $n_j = \delta_{jk}$ for some k. That is we exactly recover the eigenstates and energy level of the initial system (the vertices of the moment map polytope).

However, upon removing the above constraints we get precisely the (bosonic) second quantization prescription (with the n_j's becoming occupation numbers). Taking Maslov's correction into due account would yield the zero point energy contribution which is discarded in the infinite dimensional situation. Hence, we summarize the preceding discussion by saying that *second quantization can be interpreted as a kind of Bohr-Sommerfeld quantization of a quantum mechanical system looked upon classically*.

Moreover one can, by resorting e.g. to [47], realize the (bosonic) second quantization scheme geometrically upon considering direct sums of tensor products of the hyperplane section bundle $\mathcal{O}(1)$ on $\mathbb{P}(V)$ (whose holomorphic sections yield an

$(n+1)$-dimensional complex vector space Γ, cf. [53] and Section 3) and defining the symmetric Fock space $\mathcal{F}$ as

$$\mathcal{F} := \oplus_{k=0}^{\infty} \Gamma^k$$

(symmetric tensor product understood, and obviously taking $\Gamma^0 = \mathbb{C}$).

5. Concluding Remarks

In this paper we surveyed geometric quantum mechanics, together with some of its applications. We stressed the fact that geometric quantization, provides, in our opinion, a necessary tool for understanding GQM and ultimately quantum mechanics itself. We wish again to point out the recent papers [4, 32, 50, 51] wherein many quantum mechanical problems are independently addressed in the same geometric spirit of the present work.

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