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MOTION OF CHARGED PARTICLES FROM THE GEOMETRIC VIEW POINT

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Abstract. This is a review article on the motion of charged particles related to the author's study. The equation of motion of a charged particle is defined as a curve satisfying a certain differential equation of second order in a semi-Riemannian manifold furnished with a closed two-form. Charged particle is a generalization of geodesic. We shall oversee the geometric aspect of charged particles.

1. Introduction

Let F be a closed two-form and U a function on a connected semi-Riemannian manifold $(M, \langle\, ,\rangle)$, where $\langle\, ,\rangle$ is a semi-Riemannian metric on M. We denote by $\bigwedge^m(M)$ the space of m-forms on M. Denote by $\iota(X) : \bigwedge^m(M) \to \bigwedge^{m-1}(M)$ the interior product operator induced from a vector field X on M, and by $\mathcal{L} : T(M) \to T^*(M)$, the Legendre transformation from the tangent bundle $T(M)$ of M onto the cotangent bundle $T^*(M)$, which is defined by

$$\mathcal{L} : T(M) \to T^*(M), \quad u \mapsto \mathcal{L}(u), \quad \mathcal{L}(u)(v) = \langle u, v\rangle, \quad u, v \in T(M). \tag{1}$$

A curve $x(t)$ in M is called the *motion of a charged particle under electromagnetic field F and potential energy U*, if it satisfies the following second order differential equation

$$\nabla_{\dot{x}}\dot{x} = -\mathrm{grad}U - \mathcal{L}^{-1}(\iota(\dot{x})F) \tag{2}$$

where ∇ is the Levi-Civita connection of M. Here $\nabla_{\dot{x}}\dot{x}$ means the acceleration of the charged particle. Since $-\mathcal{L}^{-1}(\iota(\dot{x})F)$ is perpendicular to the direction $\dot{x}$ of the movement, $-\mathcal{L}^{-1}(\iota(\dot{x})F)$ means the Lorentz force. This equation originated in the theory of general relativity (see § 2 or [26]). When $F = 0$ and $U = 0$, then $x(t)$ is merely a geodesic. When M is a Kähler manifold with a complex structure J, then it is natural to take a scalar multiple of the Kähler form Ω defined by $\Omega(X,Y) = \langle X, JY\rangle$ as an electromagnetic field F. We call $\kappa\Omega$ the *Kähler electromagnetic field*, where κ is a constant. The author believes that the motion of charged particle under Kähler electromagnetic field oughts to reflect the Kähler structure of its spacetime M (see Cor. 10 in § 4). Returning to the general case, if $x(t)$ is the motion of a charged particle (2) under F and U, then the total energy

$$\frac{1}{2}\langle \dot{x}, \dot{x}\rangle + U(x(t)) \tag{3}$$

is a constant. If F has an *electromagnetic potential* A, that is $F = \mathrm{d}A$, then we define a functional E by

$$E(x) = \int_0^1 \left(\frac{1}{2}\langle \dot{x}, \dot{x}\rangle + \frac{1}{2}A(\dot{x}) - U(x(t))\right)\mathrm{d}t.$$

Here we set

$$(2\mathrm{d}A)(X,Y) = X(A(Y)) - Y(A(X)) - A([X,Y]).$$

The Euler-Lagrange equation for E describes the motion of a charged particle (2) under F and U. For instance, if M is a Hermitian symmetric space of non-compact type, since it is diffeomorphic to a Euclidean space, any electromagnetic field has an electromagnetic potential. On the other hand, for a Kähler electromagnetic field on a compact Kähler manifold, there does not exist an electromagnetic potential ([19, p. 132, 6]).

We denote by $\pi : T(M) \to M$ the natural projection. Based on (3), we define a function H on $T(M)$ as

$$H(u) = \frac{1}{2}\langle u, u\rangle + U(\pi(u)) \quad \text{for} \quad u \in T(M). \tag{4}$$

Here we mainly deal with charged particles in the case where M is a homogeneous space. In the beginning of each section is given an abstract. For almost all assertions, we shall omit their proofs. See the original papers.

2. Physical Background

In this section we explain the physical background of the motion of charged particle (2) according to [16].

We denote by $\rho = \rho(t, x_1, x_2, x_3)$ and $\boldsymbol{J} = \boldsymbol{J}(t, x_1, x_2, x_3)$ the charge and current density respectively. The equation of continuity is given by

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\boldsymbol{J} = 0. \tag{5}$$

The magnetic field $\boldsymbol{B} = (B_1, B_2, B_3)$ and the electric field $\boldsymbol{E} = (E_1, E_2, E_3)$ are time dependent vector fields on $\mathbb{R}^3$. Maxwell's equations are given by

$$\begin{aligned}
\mathrm{div}\boldsymbol{B} &= 0 \quad \text{(non-existence of magnetic monopoles)} && (6)\\
\frac{\partial \boldsymbol{B}}{\partial t} + \mathrm{rot}\ \boldsymbol{E} &= 0 \quad \text{(Faraday's law)} && (7)\\
\mathrm{div}\boldsymbol{E} &= \frac{\rho}{\epsilon_0} \quad \text{(Gauss' law)} && (8)\\
-\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{1}{\mu_0}\mathrm{rot}\ \boldsymbol{B} &= \boldsymbol{J} \quad \text{(Ampère-Maxwell's law).} && (9)
\end{aligned}$$

The speed of light c is given by $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$. Let (t, x_1, x_2, x_3) be the canonical coordinates of $\mathbb{R}^4$. We define a two-form on $\mathbb{R}^4$ by

$$F = \sum_{i=1}^{3} E_i \mathrm{d}x_i \wedge \mathrm{d}t + \mathfrak{S}_{1,2,3} B_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3$$

where we denote by $\mathfrak{S}_{1,2,3}$ the cyclic sum. Then we have

$$\mathrm{d}F = (\mathrm{div}\boldsymbol{B})\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathfrak{S}_{1,2,3}\left(\frac{\partial B_1}{\partial t} + (\mathrm{rot}\boldsymbol{E})_1\right)\mathrm{d}t \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3.$$

Hence the conditions (6) and (7) are equivalent to the condition $\mathrm{d}F = 0$. We define a Lorentz metric on $\langle\, ,\rangle$ on $\mathbb{R}^4$ by

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \delta_{ij}, \qquad \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = -c^2, \qquad \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x_j} \right\rangle = 0.$$

We denote by $\mathbb{R}^4_1 = (\mathbb{R}^4, \langle\, ,\rangle)$ the four dimensional Minkowski space-time. The Hodge star operator $* : \bigwedge^2(\mathbb{R}^4_1) \to \bigwedge^2(\mathbb{R}^4_1)$ is conformal invariant and satisfies $*^2 = -1$. We define the current density one-form $j \in \bigwedge^1(\mathbb{R}^4_1)$ by

$$j = \frac{1}{c^2 \epsilon_0} \sum_{i=1}^{3} J_i \mathrm{d}x_i - \frac{\rho}{\epsilon_0}\mathrm{d}t.$$

Since

$$\mathrm{d} * j = \frac{1}{c\epsilon_0}\left(\frac{\partial \rho}{\partial t} + \mathrm{div}\ \boldsymbol{J}\right)\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$$

the condition (5) is equivalent to $\delta j = 0$. Since

$$\begin{aligned}
*F &= \frac{1}{c}\mathfrak{S}_{1,2,3} E_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 - c\sum_{i=1}^{3} B_i \mathrm{d}x_i \wedge \mathrm{d}t\\
\delta F &= *\mathrm{d} * F = -(\mathrm{div}\ \boldsymbol{E})\mathrm{d}t - \frac{1}{c}\sum_{i=1}^{3}\left(\frac{1}{c}\frac{\partial E_i}{\partial t} - c(\mathrm{rot}\ \boldsymbol{B})_i\right)\mathrm{d}x_i
\end{aligned}$$

the conditions (8) and (9) are equivalent to the condition $\delta F = j$.
Let $x(\tau) = (t(\tau), x_1(\tau), x_2(\tau), x_3(\tau))$ be a curve in $\mathbb{R}^4_1$, where τ is called a proper time. The equation of motion for an electric charged particle with mass m and electric charge q is give by $m\nabla_{\dot{x}}\dot{x} = -q\mathcal{L}^{-1}(\iota(\dot{x})F)$, which is equivalent to

$$\begin{aligned} m\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} &= \frac{q}{c^2}\sum_{i=1}^{3} E_i \frac{\mathrm{d}x_i}{\mathrm{d}\tau} \\ m\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} &= q\left(E_1\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\mathrm{d}x_2}{\mathrm{d}\tau}B_3 - \frac{\mathrm{d}x_3}{\mathrm{d}\tau}B_2\right) \\ m\frac{\mathrm{d}^2 x_2}{\mathrm{d}\tau^2} &= q\left(E_2\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\mathrm{d}x_3}{\mathrm{d}\tau}B_1 - \frac{\mathrm{d}x_1}{\mathrm{d}\tau}B_3\right) \\ m\frac{\mathrm{d}^2 x_3}{\mathrm{d}\tau^2} &= q\left(E_3\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\mathrm{d}x_1}{\mathrm{d}\tau}B_2 - \frac{\mathrm{d}x_2}{\mathrm{d}\tau}B_1\right). \end{aligned}$$

The Lorentz metric $\langle\, ,\rangle$ naturally induces a scaler product $\langle\, ,\rangle$ on $\bigwedge^k(\mathbb{R}^4_1)$. See [20] for the detail. For instance

$$\begin{aligned} &\langle \mathrm{d}x_i, \mathrm{d}x_j\rangle = \delta_{ij}, \qquad \langle \mathrm{d}t, \mathrm{d}t\rangle = -\frac{1}{c^2}, \qquad \langle \mathrm{d}t, \mathrm{d}x_i\rangle = 0 \\ &\langle \mathrm{d}x_i \wedge \mathrm{d}x_j, \mathrm{d}x_k \wedge \mathrm{d}x_l\rangle = \delta_{ik}\delta_{jl}, \qquad i \neq j, k \neq l \\ &\langle \mathrm{d}x_i \wedge \mathrm{d}t, \mathrm{d}x_j \wedge \mathrm{d}t\rangle = -\frac{1}{c^2}\delta_{ij}, \qquad \langle \mathrm{d}x_i \wedge \mathrm{d}t, \mathrm{d}x_j \wedge \mathrm{d}x_k\rangle = 0. \end{aligned}$$

Since

$$\langle F, *F\rangle = \frac{2}{c}\boldsymbol{E}\cdot\boldsymbol{B}, \qquad \langle F + *F, F + *F\rangle = -\langle F - *F, F - *F\rangle = \frac{4}{c}\boldsymbol{E}\cdot\boldsymbol{B}$$

the condition $\boldsymbol{E} \perp \boldsymbol{B}$ is equivalent to one of (hence all) the following conditions:

$$\langle F, *F\rangle = 0, \qquad \langle F - *F, F - *F\rangle = 0, \qquad \langle F + *F, F + *F\rangle = 0.$$

3. Hamiltonian Dynamics of a Charged Particle

In this section, we show that, according to [13], even if the electromagnetic field F does not have an electromagnetic potential, the motion of a charged particle (2) is a Hamiltonian system with H defined by (4) and a noncanonical symplectic structure on $T(M)$ (Theorem 3). We here mention some fundamental definitions concerning symplectic geometry. A *symplectic structure* on a manifold is a closed two-form which is nondegenerate at each point. A *symplectic manifold* is a manifold possessing a symplectic structure. A symplectic manifold is even-dimensional and orientable. A diffeomorphism on a symplectic manifold is called a *symplectic transformation* if it preserves the symplectic structure, though, in old literatures, a symplectic transformation was called a canonical transformation.

3.1. Hamiltonian Dynamics of a Geodesic

In this subsection, we review the Hamiltonian dynamics of a geodesic, which is defined by $\nabla_{\dot{x}}\dot{x} = 0$, in a semi-Riemannian manifold $(M, \langle\, ,\rangle)$, in order to contrast it with the Hamiltonian dynamics of a charged particle discussed in the next subsection. The results obtained here will be used in the next subsection. Define a function H on $T(M)$ by

$$H(u) = \frac{1}{2}\langle u, u\rangle \quad \text{for} \quad u \in T(M)$$

which corresponds to the kinetic energy. There exists a canonical symplectic structure ω^* on $T^*(M)$. We denote by ω the pull back of ω^* by the Legendre transformation $\mathcal{L} : T(M) \to T^*(M)$. Then ω is a symplectic structure on $T(M)$ (see (11) in the proof of Proposition 1 below). We denote by X_H the Hamiltonian vector field of the Hamiltonian H with respect to ω, that is, $\mathrm{d}H = \iota(X_H)\omega$. We denote by $\{\, ,\}$ the Poisson bracket on $C^\infty(T(M))$ with respect to ω, which is defined by

$$\{f, g\} = X_f(g) = \omega(X_g, X_f) \quad \text{for} \quad f, g \in C^\infty(T(M)).$$

Each orbit of the geodesic flow on $T(M)$ coincides with the integral curve of X_H ([10]). We define a mapping

$$P : \mathfrak{X}(M) \to (C^\infty(T(M)), \{\, ,\}), \qquad Y \mapsto P_Y$$

by $P_Y(u) = \langle u, Y\rangle$. The mapping P is, defined via the Legendre transformation $\mathcal{L}$ specified in (1)

$$P_Y = \mathcal{L} \circ Y.$$

Here the differential one-form $\mathcal{L} \circ Y$ being considered as a function on $T(M)$, whose restriction to each fibre of the bundle is linear. It is clear that P is injective. If Y is a Killing vector field, then P_Y is a conservative constant for geodesics (see [20, Lemma 9.26]). In other words

$$\{H, P_Y\} = 0 \tag{10}$$

for any Killing vector field Y.

Proposition 1 ([10], p. 222). $\{P_Y, P_Z\} = P_{[Y,Z]}$ *for all* $Y, Z \in \mathfrak{X}(M)$.

Proof: Let $(x^1, \cdots, x^n)$ be a local coordinate system in M. The components g_{ij} of $\langle\,,\rangle$ with respect to $(x^1, \cdots, x^n)$ are given by $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$. We denote by (g^{ij}) the inverse matrix of (g_{ij}). We introduce a local coordinate system $(x^1, \cdots, x^n, u^1, \cdots, u^n)$ in $T(M)$ by setting

$$u = \sum_{i=1}^{n} u^i(u) \frac{\partial}{\partial x^i}, \qquad u \in T(M).$$

The local expression for the canonical symplectic structure ω is then given by

$$\omega = \sum_{i,j,k} \frac{\partial g_{ij}}{\partial x^k} u^j \mathrm{d}x^i \wedge \mathrm{d}x^k + \sum_{i,j} g_{ij} \mathrm{d}x^i \wedge \mathrm{d}u^j = -\mathrm{d}\left(\sum g_{ij} u^j \mathrm{d}x^i\right). \qquad (11)$$

The vector fields Y and Z can be written as $Y = \sum Y^i \frac{\partial}{\partial x^i}, Z = \sum Z^i \frac{\partial}{\partial x^i}$, so

$$P_Z = \sum g_{ij} Z^i u^j, \quad \text{and} \quad P_{[Y,Z]} = \sum g_{jk} \left(Y^i \frac{\partial Z^j}{\partial x^i} - Z^i \frac{\partial Y^j}{\partial x^i} \right) u^k.$$

Since $\mathrm{d}P_Y = \iota(X_{P_Y})\omega$, we have

$$X_{P_Y} = \sum Y^i \frac{\partial}{\partial x^i} - \sum \left(Y^k \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial Y^k}{\partial x^i} g_{jk} \right) g^{il} u^j \frac{\partial}{\partial u^l}. \qquad (12)$$

Hence we obtain

$$\begin{aligned} \{P_Y, P_Z\} &= X_{P_Y}(P_Z) \\ &= \sum Y^i \frac{\partial (g_{jk} Z^j)}{\partial x^i} u^k - \sum \left(Y^k \frac{\partial g_{ij}}{\partial x_k} + \frac{\partial Y^k}{\partial x^i} g_{jk} \right) g^{il} u^j g_{pl} Z^p \\ &= \sum Y^i \frac{\partial (g_{jk} Z^j)}{\partial x^i} u^k - \sum \left(Y^j \frac{\partial g_{ik}}{\partial x_j} + \frac{\partial Y^j}{\partial x^i} g_{jk} \right) Z^i u^k \\ &= \sum g_{jk} \left(Y^i \frac{\partial Z^j}{\partial x^i} - Z^i \frac{\partial Y^j}{\partial x^i} \right) u^k = P_{[Y,Z]}. \end{aligned}$$

■

A diffeomorphism φ of M induces a transformation φ_* of $T(M)$. Thus a vector field Y of M induces vector fields of $T(M)$ in the following two ways: One is the Hamiltonian vector field X_{P_Y} of P_Y, and the other is $\frac{\mathrm{d}\varphi_{t*}(u)}{\mathrm{d}t}_{|t=0}$ $(u \in T(M))$, where φ_t is the one parameter transformation group of M generated by Y.

When Y is a Killing vector field, by (10) Noether's theorem tells us that the one-parameter transformation group of $T(M)$ generated by X_{P_Y} is a symplectic transformation which preserves H.

Proposition 2 ([13]). *Let φ_{t*} be the one-parameter transformation group of $T(M)$ induced from the one parameter transformation group φ_t of M generated by a Killing vector field Y. Then φ_{t*} coincides with the one-parameter transformation group generated by the Hamiltonian vector field of P_Y.*

Proof: Since Y is a Killing vector field

$$\begin{aligned}\sum_k Y^k \frac{\partial g_{ij}}{\partial x^k} &= Y\left(\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle\right) \\ &= \left\langle \left[Y, \frac{\partial}{\partial x^i}\right], \frac{\partial}{\partial x^j}\right\rangle + \left\langle \frac{\partial}{\partial x^i}, \left[Y, \frac{\partial}{\partial x^j}\right]\right\rangle \\ &= -\sum_k \left(\frac{\partial Y^k}{\partial x^i} g_{kj} + \frac{\partial Y^k}{\partial x^j} g_{ki}\right).\end{aligned}$$

Applying $\sum_i g^{il}$ to the equation above, we have

$$\frac{\partial Y^l}{\partial x^j} = -\sum \left(\frac{\partial Y^k}{\partial x^i} g_{kj} + Y^k \frac{\partial g_{ij}}{\partial x^k}\right) g^{il}.$$

Using (12), we obtain

$$X_{P_Y} = \sum Y^i \frac{\partial}{\partial x^i} + \sum \frac{\partial Y^l}{\partial x^j} u^j \frac{\partial}{\partial u^l} = \frac{\mathrm{d}\varphi_{t*}}{\mathrm{d}t}_{|t=0}.$$

■

3.2. Hamiltonian Dynamics of a Charged Particle

In this subsection, we study the Hamiltonian dynamics of the motion of a charged particle (2) in a connected semi-Riemannian manifold $(M, \langle\, ,\rangle)$. We define a function H on $T(M)$ by (4). For a closed two-form F, we define a closed two-form ω_F on $T(M)$ by

$$\omega_F = \omega - \pi^* F.$$

For each tangent vector $u \in T(M)$, we denote by x_u the motion of a charged particle (2) with the initial vector u. The electromagnetic flow $\Phi_t : T(M) \to T(M)$ is defined by $\Phi_t(u) = \dot{x}_u(t)$.

Theorem 3 ([13]).

1) *The closed two-form ω_F is a symplectic structure on $T(M)$.*

2) *We denote by* X_H^F *the Hamiltonian vector field of the Hamiltonian* H *with respect to* ω_F*. Each orbit of the electromagnetic flow on* $T(M)$ *coincides with the integral curve of* X_H^F*.*

Remark 4. This theorem is well-known when $F = 0$. The theorem is also well-known when $M = \mathbb{R}_1^4$ and $U = 0$ ([21], [10, § 20] and [18, § 4]).

Henceforth, we set $U = 0$. We define a tensor field ϕ of type $(1, 1)$ by

$$F(X, Y) = \langle X, \phi Y \rangle \quad \text{that is} \quad \phi X = -\mathcal{L}^{-1}(\iota(X)F) \tag{13}$$

which is skew-symmetric with respect to $\langle \, , \rangle$. We consider the motion of a charged particle

$$\nabla_{\dot{x}} \dot{x} = \phi \dot{x} \tag{14}$$

under electromagnetic field F. We define a Lie subalgebra $\mathcal{I}_\phi(M)$ in $\mathfrak{X}(M)$ by

$$\mathcal{I}_\phi(M) = \{X \in \mathfrak{X}(M) \; ; \; L_X \langle \, , \rangle = 0, L_X \phi = 0\}$$

where L_X is the Lie derivative with respect to the vector field X. The condition $L_X \langle \, , \rangle = 0$ means X is a Killing vector field. Using (13), we have

$$\mathcal{I}_\phi(M) = \{X \in \mathfrak{X}(M) \; ; \; L_X \langle \, , \rangle = 0, L_X F = 0\}$$

For $X \in \mathcal{I}_\phi(M)$, we have $\mathrm{d}(\iota(X)F) = 0$ (we refer to the proof of Theorem 9). The following proposition will be used in the proof of Theorem 9.

Proposition 5 ([13]). *Let* X *and* Y *be in* $\mathcal{I}_\phi(M)$*. Then*

$$\iota([X, Y])F = -\mathrm{d}(F(X, Y)).$$

Proof: Let Z be any vector field on M. Since F is closed, we have

$$\mathfrak{S}_{X,Y,Z} X(F(Y, Z)) - \mathfrak{S}_{X,Y,Z} F([X, Y], Z) = 0$$

which implies that

$$\begin{aligned} \mathrm{d}(F(X,Y)(Z) &= Z(F(X,Y)) \\ &= -X(F(Y,Z)) - Y(F(Z,X)) \\ &\quad + F([X,Y],Z) + F([Y,Z],X) + F([Z,X],Y) \\ &= -X(\langle Y, \phi Z \rangle) - Y(\langle Z, \phi X \rangle) \\ &\quad + \langle [X,Y], \phi Z \rangle + \langle [Y,Z], \phi X \rangle + \langle [Z,X], \phi Y \rangle. \end{aligned}$$

Since X and Y are Killing vector fields

$$\mathrm{d}(F(X,Y)(Z) = -\langle Y, [X, \phi Z]\rangle - \langle Z, [Y, \phi X]\rangle + \langle [Z, X], \phi Y\rangle.$$

Since X and Y are infinitesimal automorphisms of ϕ,

$$\begin{aligned}(\mathrm{d}(F(X,Y))(Z) &= -\langle Y, \phi[X, Z]\rangle - \langle Z, \phi[Y, X]\rangle + \langle [Z, X], \phi Y\rangle \\ &= F(Z, [X, Y]) = -\iota([X, Y])F(Z).\end{aligned}$$

Hence $\mathrm{d}(F(X,Y) = -\iota([X,Y])F$. ∎

Let Y be in $\mathcal{I}_\phi(M)$. Assume that there exists a function f_Y such that

$$\iota(Y)F = \mathrm{d}f_Y.$$

For instance, if M satisfies one of the conditions 1), 2) or 3) in Theorem 9, then such a function f_Y exists. We define a function P_Y^F on $T(M)$ by

$$P_Y^F(u) = \langle u, Y\rangle - f_Y(\pi(u)) = (P_Y - f_Y \circ \pi)(u) \quad \text{for all} \quad u \in T(M).$$

We denote by $\{\,,\}_F$ the Poisson bracket with respect to ω_F.

Proposition 6 ([13]). *Let Y be in $\mathcal{I}_\phi(M)$. Assume that there exists a function f_Y such that $\iota(Y)F = \mathrm{d}f_Y$. Then*

1) *The Hamiltonian vector field of P_Y^F with respect to ω_F coincides with the Hamiltonian vector field X_{P_Y} of P_Y with respect to ω.*

2) $\{H, P_Y^F\}_F = 0$, *where* $H(u) = \frac{1}{2}\langle u, u\rangle$.

Proof: 1) Using $\iota(Y)F = \mathrm{d}f_Y$ and (12), we have $\mathrm{d}(f_Y \circ \pi) = \iota(X_{P_Y})\pi^*F$. Thus

$$\mathrm{d}P_Y^F = \mathrm{d}P_Y - \mathrm{d}(f_Y \circ \pi) = \iota(X_{P_Y})\omega - \iota(X_{P_Y})\pi^*F = \iota(X_{P_Y})\omega_F.$$

2) $\{H, P_Y^F\}_F = -X_{P_Y}(H) = \{H, P_Y\} = 0$

where 1) guarantees the first equality, and the last follows from (10). ∎

Using Noether's theorem, Proposition 2 and Proposition 6, we obtain the following conclusion: The one-parameter transformation group of $T(M)$ which is induced from the one-parameter transformation group of M generated by $X \in \mathcal{I}_\phi(M)$ is a symplectic transformation that preserves H.

Assume that there exists a function f_Y such that $\mathrm{d}f_Y = \iota(Y)F$ for any vector field $Y \in \mathcal{I}_\phi(M)$. We examine the relation between $\{P_Y^F, P_Z^F\}_F$ and $P_{[Y,Z]}^F$ for

$Y, Z \in \mathcal{I}_\phi(M)$. In order to formulate this, we define an equivalence relation $\sim$ on $C^\infty(T(M))$ by

$$f_1 \sim f_2 \Leftrightarrow f_2 - f_1 = \text{a constant function} \quad f_1, f_2 \in C^\infty(T(M)).$$

We denote by $C^\infty(T(M))/\mathbb{R}$ the set of equivalence classes in $C^\infty(T(M))$. If we set

$$\{[f_1], [f_2]\}_F = [\{f_1, f_2\}_F] \quad \text{for} \quad f_1, f_2 \in C^\infty(T(M))$$

then the induced Poisson bracket $\{\,,\}_F$ on $C^\infty(T(M))/\mathbb{R}$ is well-defined, where we denote by $[f]$ the equivalence class of $f \in C^\infty(T(M))$. By Lemma 1, Proposition 5 and Proposition 6, we have the following

Proposition 7 ([13]). *Assume that there exists a function f_Y such that $\mathrm{d}f_Y = \iota(Y)F$ for any $Y \in \mathcal{I}_\phi(M)$. Then the mapping*

$$[P^F] : (\mathcal{I}_\phi(M), [\,,]) \to (C^\infty(T(M))/\mathbb{R}, \{\,,\}_F), \quad Y \mapsto [P^F_Y]$$

is a Lie homomorphism, that is

$$\{[P^F_Y], [P^F_Z]\}_F = [P^F_{[Y,Z]}] \qquad \textit{for} \qquad Y, Z \in \mathcal{I}_\phi(M).$$

4. Simpleness of the Motion of Charged Particles

In general, it is an interesting problem whether a given equation of motion has a periodic solution or not. In this section, we apply the conservation law obtained in the previous section to the simpleness of the motion of a charged particle according to [15]. Here a curve in a manifold is *simple* if it is a simply closed periodic curve, or if it does not intersect itself. Hence a curve is not simple if it has a self-intersection point but it is not simply closed.

Definition 8 ([13]). *Let $(M, \langle\,,\rangle)$ be a semi-Riemannian manifold and ϕ a tensor field of type $(1,1)$ on M which is skew-symmetric with respect to the semi-Riemannian metric $\langle\,,\rangle$. Such a manifold $(M, \langle\,,\rangle, \phi)$ is called G-homogeneous or simply homogeneous if a Lie transformation group G of isometries acts transitively and effectively on M, and ϕ is invariant under the action of G.*

Theorem 9 ([13], [16]). *Let $(M, \langle\,,\rangle, \phi)$ be a G-homogeneous semi-Riemannian manifold. Assume that the two-form Ω defined by $\Omega(X, Y) = \langle X, \phi Y\rangle$ is closed. If one of the following three conditions 1), 2) or 3) holds, then the motion of charged particle $\nabla_{\dot{x}}\dot{x} = \kappa\phi(\dot{x})$ is simple, where κ is a constant.*

1) *One dimensional de-Rham cohomology group $H^1(M)$ vanishes.*

2) $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, *where $\mathfrak{g}$ is a Lie algebra of G.*

3) $(M, \langle\ ,\ \rangle, \phi, \eta, \xi)$ *is an almost α-Sasakian manifold, where α is a nonzero constant.*

See § 5 for the definition of almost α-Sasakian manifold.

Proof: Let X be a Killing vector field which is an infinitesimal automorphism of ϕ. We show that $\iota(X)\Omega$ is closed. Using Cartan's relation and the assumption that Ω is closed we have

$$2\mathrm{d}(\iota(X)\Omega) = L_X\Omega - 3!\iota(X)\mathrm{d}\Omega = L_X\Omega.$$

Hence, for any vector field Y and Z, we have

$$\begin{aligned} 2(\mathrm{d}(\iota(X)\Omega))(Y, Z) &= (L_X\Omega)(Y, Z) \\ &= X(\Omega(Y, Z)) - \Omega([X, Y], Z) - \Omega(Y, [X, Z]) \\ &= X\langle Y, \phi Z\rangle - \langle [X, Y], \phi Z\rangle - \langle Y, \phi[X, Z]\rangle \\ &= \langle Y, [X, \phi Z]\rangle - \langle Y, \phi[X, Z]\rangle = 0 \end{aligned}$$

where the fourth equality comes from the fact $L_X\langle\ ,\ \rangle = 0$, and $L_X\phi = 0$ guarantees the last equality. Hence $\iota(X)\Omega$ is closed.

We show that there exists a function f_X such that $\iota(X)\Omega = \mathrm{d}f_X$ if M satisfies one of the conditions 1), 2) or 3) in Theorem 9.

1) Since $\mathrm{d}(\iota(X)\Omega) = 0$ and $H^1(M) = \{0\}$, there exists a function f_X such that $\iota(X)\Omega = \mathrm{d}f_X$.

2) Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, there exists a function f_X such that $\iota(X)\Omega = \mathrm{d}f_X$ by Proposition 5.

3) If we put $f_X = -\frac{1}{2\alpha}\eta(X)$, then $\iota(X)\Omega = \mathrm{d}f_X$ by Proposition 20.

Let $x(t)$ be the motion of charged particle. Since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\langle \dot{x}(t), X_{x(t)}\rangle - \kappa f_X(x(t))) &= \langle \nabla_{\dot{x}}\dot{x}, X\rangle + \langle \dot{x}, \nabla_{\dot{x}}X\rangle - \kappa(\mathrm{d}f_X)(\dot{x}) \\ &= \kappa\langle \phi(\dot{x}), X\rangle - \kappa\Omega(X, \dot{x}) = 0 \end{aligned}$$

$\langle \dot{x}(t), X_{x(t)} \rangle - \kappa f_X(x(t))$ is a constant independent of t. Assume that $x(0) = x(1)$. Since

$$\langle \dot{x}(0), X_{x(0)} \rangle - \kappa f_X(x(0)) = \langle \dot{x}(1), X_{x(0)} \rangle - \kappa f_X(x(0))$$

we have

$$\langle \dot{x}(0) - \dot{x}(1), X_{x(0)} \rangle = 0.$$

Since $(M, \langle\, , \rangle, \phi)$ is homogeneous

$$T_{x(0)}(M) = \mathrm{span}\{X_{x(0)}\ ;\ X - \text{Killing},\ L_X\phi = 0\}.$$

Since $\langle\, , \rangle$ is nondegenerate, we have $\dot{x}(0) = \dot{x}(1)$. Since $\nabla_{\dot{x}}\dot{x} = \kappa\phi(\dot{x})$ is an ordinary differential equation of second order, we have $x(t+1) = x(t)$. Hence $x(t)$ is a a simply closed periodic curve. ■

Corollary 10 ([16]). *A homogeneous Kähler manifold M does not contain a totally geodesic Kähler immersed complex torus if M satisfies one of the conditions* 1) *or* 2) *in* Theorem 9.

Proof: Let $T = \mathbb{C}^n/\Gamma$ be a complex torus of complex dimension n, where $\Gamma = \sum_{j=1}^{2n} \mathbb{R}a_j$ is a lattice of $\mathbb{C}^n$. It is sufficient to prove that there exists a charged particle which is not simple in T. Denote by $\pi : \mathbb{C}^n \to T$ the natural projection. Let p and q be points in $\mathbb{C}^n$ such that $p \neq q$ and $\pi(p) = \pi(q)$. Let $\tilde{x}(t)$ be the motion of a charged particle in $\mathbb{C}^n$ through p and q under a Kähler electromagnetic field, which is a circle in the usual sense. If we put $x(t) = \pi(\tilde{x}(t))$, then $x(t)$ is a motion of a charged particle in T which is not simply closed. ■

In a similar way to the proof of Theorem 9, we can prove the following theorem of Kobayashi ([20, p. 321]), when M is a homogeneous Riemannian manifold.

Theorem 11 ([16]). *Every geodesic in a homogeneous semi-Riemannian manifold is a simple curve.*

5. Sasakian Manifold

The following definitions are to be found in [7].

Definition 12. Let $(M, \langle\, , \rangle)$ be an odd dimensional Riemannian manifold with the Riemannian metric $\langle\, , \rangle$. An *almost contact metric structure* on M is defined by a tensor field ϕ of type $(1,1)$, a vector field ξ and a one-form η on M such that

$$\phi^2 = -1 + \eta \otimes \xi \tag{15}$$

$$\phi(\xi) = 0 \tag{16}$$

$$\eta(\phi X) = 0 \tag{17}$$

$$\eta(\xi) = 1 \tag{18}$$

$$\langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y) \tag{19}$$

$$\eta(X) = \langle X, \xi \rangle. \tag{20}$$

A manifold equipped with an almost contact metric structure is called an *almost contact metric manifold*.

Example 13. Let M be an oriented real hypersurface in a Hermitian manifold $(\bar{M}, \langle\, , \rangle, J)$. We denote by ν a unit normal vector field of M. Set

$$\xi = -J\nu, \qquad \eta(X) = \langle X, \xi \rangle, \qquad \phi(X) = (JX)^T. \tag{21}$$

Then $(M, \langle\, , \rangle, \phi, \eta, \xi)$ is an almost contact metric manifold.

Definition 14. We define a two-form Ω on an almost contact metric manifold $(M, \langle\, , \rangle, \phi, \eta, \xi)$ by $\Omega(X, Y) = \langle X, \phi Y \rangle$.

We will study the motion of a charged particle in an almost contact metric manifold $(M, \langle\, , \rangle, \phi, \eta, \xi)$ defined by

$$\nabla_{\dot{x}} \dot{x} = \kappa \phi(\dot{x}).$$

Since $\phi(\dot{x})$ is perpendicular to both $\dot{x}$ and ξ, the vector field ξ and the force $\kappa\phi(\dot{x})$ mean the *magnetic field* and the *Lorentz force*, respectively, in magnetic theory. Hence in this paper we call ϕ the *Lorentz tensor*, after Lorentz.

Definition 15. Let $(M, \langle\, , \rangle, \phi, \eta, \xi)$ be an almost contact metric manifold. We define a tensor $[\phi, \phi]$ of type $(1,2)$ by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

which is called the *Nijenhuis torsion* of ϕ.

Since the torsion tensor of the Levi-Civita connection vanishes, we can write

$$\begin{aligned} [\phi, \phi](X, Y) &= (\nabla_{\phi X}\phi)(Y) - (\nabla_{\phi Y}\phi)(X) \\ &\quad + \phi((\nabla_Y \phi)(X) - (\nabla_X \phi)(Y)). \end{aligned} \tag{22}$$

Definition 16. An almost contact metric manifold $(M, \langle\ ,\ \rangle, \phi, \eta, \xi)$ is said to be *normal* if $[\phi, \phi] + 2d\eta \otimes \xi = 0$, where

$$2\mathrm{d}\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]).$$

Proposition 17 (Blair [5], p. 50, Proposition). *If an almost contact metric manifold $(M, \langle\ ,\ \rangle, \phi, \eta, \xi)$ is normal then*

$$L_\xi \eta = 0, \qquad L_\xi \phi = 0, \qquad (L_{\phi X}\eta)(Y) = (L_{\phi Y}\eta)(X).$$

Definition 18. An almost contact metric manifold $(M, \langle\ ,\ \rangle, \phi, \eta, \xi)$ is said to be an *almost α-Sasakian manifold* if

$$\mathrm{d}\eta(X, Y) = \alpha\langle X, \phi Y\rangle \tag{23}$$

where α is a function. In this paper we deal with M only if α is a constant.

Hence η means a (scalar multiple of) *magnetic potential* for magnetic field ξ when α is a non-zero constant.

Proposition 19. *Let $(M, \langle\ ,\ \rangle, \phi, \eta, \xi)$ be an almost α-Sasakian manifold. Then*

1) *The integral curves of ξ are geodesics.*

2) $L_\xi \eta = 0$.

The following proposition was used in the proof of Theorem 9.

Proposition 20. *Let $(M, \langle\ ,\ \rangle, \phi, \eta, \xi)$ be an almost α-Sasakian manifold, where α is a non-zero constant. If X is a Killing vector field which is an infinitesimal automorphism of ϕ, then*

$$\iota(X)\Omega = -\frac{1}{2\alpha}\mathrm{d}(\eta(X)).$$

Definition 21. An almost α-Sasakian manifold $(M, \langle\ ,\ \rangle, \phi, \eta, \xi)$ is called *α-Sasakian* if M is normal and one-Sasakian manifold is simply called *Sasakian manifold*.

Proposition 22. *Let $(M, \langle\ ,\ \rangle, \phi, \eta, \xi)$ be an α-Sasakian manifold, where α is a non-zero constant. Then*

1) *ξ is a Killing vector field.*

2) $\nabla_Y \xi = -\alpha\phi Y$.

The following proposition is a theorem of Blair when $\alpha = 1$ (see [5, p. 73]).

Proposition 23. *Let $(M, \langle\, ,\rangle, \phi, \eta, \xi)$ be an almost α-Sasakian manifold, where α is a constant.*

1) *If $(M, \langle\, ,\rangle, \phi, \eta, \xi)$ satisfies*

$$(\nabla_X \phi)(Y) = \alpha(\langle X, Y\rangle\xi - \eta(Y)X)$$

then it is an α-Sasakian manifold.

2) *Conversely assume that $(M, \langle\, ,\rangle, \phi, \eta, \xi)$ is an α-Sasakian manifold, where α is a non-zero constant. Then*

$$(\nabla_X \phi)(Y) = \alpha(\langle X, Y\rangle\xi - \eta(Y)X). \tag{24}$$

Proposition 24. *Let $(M, \langle\, ,\rangle, \phi, \eta, \xi)$ be an oriented real hypersurface in a Kähler manifold $(\bar{M}, \langle\, ,\rangle, J)$, where (ϕ, η, ξ) is defined by* (21)*. Assume that the manifold M is totally umbilic, that is, there exists a constant α such that $B(X, Y) = -\alpha\langle X, Y\rangle\nu$, where B is the second fundamental form of M. Then $(M, \langle\, ,\rangle, \eta, \phi, \xi)$ is an α-Sasakian manifold.*

At the end of this section we focus our attention on the motion of a charged particle under Lorentz force in an odd-dimensional sphere $\mathbb{S}^{2n+1}$ of unit radius. We denote by J the complex structure of $\mathbb{C}^{n+1}$. If we set

$$\xi_x = -Jx, \qquad \phi X = (JX)^T, \qquad \eta(X) = \langle X, \xi\rangle$$

then $\mathbb{S}^{2n+1}$ is a Sasakian manifold by Proposition 24 since it is a totally umbilic real hypersurface in the complex Euclidean space $\mathbb{C}^{n+1}$.

Theorem 25 ([16]). *Let $x(t)$ be the motion of a charged particle $\nabla_{\dot{x}}\dot{x} = \kappa\phi(\dot{x})$ under Lorentz force in the odd-dimensional sphere $\mathbb{S}^{2n+1}$ of unit radius. Assume that $x(0) = e_1$ and that*

$$\dot{x}(0) = \mathrm{i}v_1 e_1 + \sum_{j=2}^{n+1} v_j e_j, \qquad v_1 \in \mathbb{R}, \qquad v_2, \cdots, v_{n+1} \in \mathbb{C}^*.$$

Then $x(t)$ is given by

$$x(t) = \exp\left(\frac{\mathrm{i}}{2}\kappa t\right)\left\{\left(\cos\omega t + \frac{\mathrm{i}}{\omega}(v_1 - \frac{\kappa}{2})\sin\omega t\right)e_1 + \frac{\sin\omega t}{\omega}\sum_{j=2}^{n+1} v_j e_j\right\}$$

where

$$\omega = \sqrt{\frac{1}{4}\kappa^2 - \kappa v_1 + v^2} > 0, \qquad v = ||\dot{x}(0)||, \qquad \mathrm{i} = \sqrt{-1}.$$

The motion is periodic if and only if κ/ω is rational.

6. Sasaki-Kähler Submersion

The image of any horizontal geodesic under a Riemannian submersion is a geodesic [20]. However, in general the image of a geodesic under a Riemannian submersion is not a geodesic. In this section, according to [17], we define a Sasaki-Kähler submersion from a Sasakian manifold onto a Kähler manifold, and show that the image of the motion of a charged particle is the motion of a charged particle. In particular, the image of a geodesic is the motion of a charged particle under a Sasaki-Kähler submersion. A Sasaki-Kähler submersion is a kind of Riemannian submersion [3].

6.1. Charged Particles and Okumura Geodesics

Let $(M, \langle\,,\rangle, \phi, \eta, \xi)$ be a Sasakian manifold. For a constant $r \in \mathbb{R}$, we define a tensor field A of type $(1, 2)$ by

$$A(X)Y = \mathrm{d}\eta(X, Y)\xi + r\eta(X)\phi Y + \eta(Y)\phi X.$$

Then $A(X)$ is skew-symmetric with respect to g. The Okumura linear connection $\tilde{\nabla}$ is defined by $\tilde{\nabla}_X Y = \nabla_X Y + A(X)Y$, which satisfies $\tilde{\nabla}\langle\,,\rangle = 0$ and $\tilde{\nabla}\xi = 0$ (see [23]). We have

$$\tilde{\nabla}_X X = \nabla_X X + (r + 1)\eta(X)\phi X. \tag{25}$$

A curve $x(t)$ in M is called the motion of a charged particle if $\nabla_{\dot{x}}\dot{x} = \kappa\phi(\dot{x})$ for a constant κ. The constant κ is the charge-to-mass ratio for $x(t)$.

Proposition 26 ([17]).

1) *If $x(t)$ is an Okumura geodesic, that is $\tilde{\nabla}_{\dot{x}}\dot{x} = 0$, then $\eta(\dot{x}(t))$ is a constant.*

2) *If $x(t)$ is the motion of a charged particle, then $\eta(\dot{x}(t))$ is a constant.*

Proposition 26 and (25) immediately imply the following:

Proposition 27 ([17]).

1) *Let $x(t)$ be an Okumura geodesic. Set $c = \eta(\dot{x}(t))$, then $x(t)$ is the motion of a charged particle of the charge-to-mass ratio $\kappa = -(r+1)c$.*

2) *Let $x(t)$ be the motion of a charged particle. Set $c = \eta(\dot{x}(t))$.*

   2.1) *When $c \neq 0$, then $x(t)$ is an Okumura geodesic for $r = -(\frac{\kappa}{c} + 1)$.*

   2.2) *When $c = 0$, then $\tilde{\nabla}_{\dot{x}}\dot{x} = \kappa\phi(\dot{x})$.*

Corollary 28 ([17]). *A curve $x(t)$ is a geodesic with respect to the Levi-Civita connection if and only if*

1) *$x(t)$ is an Okumura geodesic for $r = -1$ when $\eta(\dot{x}) \neq 0$*

2) *$x(t)$ is an Okumura geodesic for any r when $\eta(\dot{x}) = 0$.*

6.2. Sasaki-Kähler Submersion and Charged Particles

Definition 29 ([17]). *Let $\pi : \bar{M} \to M$ be a Riemannian submersion from a Sasakian manifold $(\bar{M}, \langle\, ,\rangle, \phi, \eta, \xi)$ of dimension $2n+1$ onto a Kähler manifold $(M, \langle\, ,\rangle, J)$ of real dimension $2n$. We call π a Sasaki-Kähler submersion if*

1) *$\pi^{-1}(y)$ $(y \in M)$ is the image of an integral curve of ξ*

1) *$\mathrm{d}\pi\phi X = J\mathrm{d}\pi X$ for any horizonal vector X.*

Here horizontal vector means $\eta(X) = 0$.

For instance, we can construct a Sasaki-Kähler submersion from any Hermitian symmetric space M.

Theorem 30 ([17]). *Let $\pi : \bar{M} \to M$ be a Sasaki-Kähler submersion. Assume that $x(t) \in \bar{M}$ is the motion of a charged particle of the charge-to-mass ratio κ. Define a constant c by $c = \eta(\dot{x})$. Then $y(t) = \pi(x(t))$ is the motion of a charged particle of the charge-to-mass ratio $\kappa + 2c$, that is $\nabla_{\dot{y}}\dot{y} = (\kappa + 2c)J\dot{y}$, where ∇ is the Levi-Civita connection of M. In particular, if $x(t)$ is a geodesic, then $y(t)$ is the motion of a charged particle of the charge-to-mass ratio $2c$.*

Proof: Since $||\dot{x}||$ is a constant, $\dot{x}(t) = 0$ for some t if and only if $\dot{x}(t) = 0$ for any t. In this case, $x(t)$ is a single point. Hence we may assume $\dot{x}(t) \neq 0$ for any t. If $\dot{x}(t)$ is proportional to ξ for some t, then $x(t)$ is an integral curve of ξ. In this case, $y(t)$ is a single point. Hence we may assume that $\dot{x}$ is not proportional to ξ for any t. In other words, we may assume $\dot{y}(t) \neq 0$ for any t. Hence there exists a (local) vector field X of M such that $X = \dot{y}$. If we denote by $\bar{X}$ the horizontal lift of X, then we have $\dot{x} = \bar{X} + \eta(\dot{x})\xi = \bar{X} + c\xi$. Since $x(t)$ is the motion of a charged particle, we get

$$\kappa\phi\bar{X} = \kappa\phi\dot{x} = \bar{\nabla}_{\dot{x}}\dot{x} = \bar{\nabla}_{\bar{X}+c\xi}(\bar{X} + c\xi) = \bar{\nabla}_{\bar{X}}\bar{X} + c(-2\phi\bar{X} + [\xi, \bar{X}])$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$. Since ξ and 0 are π-related, and $\bar{X}$ and X are π-related, we have $\pi[\xi, \bar{X}] = [\pi\xi, \pi\bar{X}] = 0$. Hence $[\xi, \bar{X}]$ is vertical. Since ξ is a Killing vector field and $\bar{X}$ is perpendicular to ξ, we have $\eta([\xi, \bar{X}]) = \langle \xi, [\xi, \bar{X}]\rangle = \xi(\langle \xi, \bar{X}\rangle) = 0$. Hence $[\xi, \bar{X}] = 0$, which implies that $\kappa\phi\bar{X} = \bar{\nabla}_{\bar{X}}\bar{X} - 2c\phi\bar{X}$. Using [20, p. 212, Lemma 45, (3)], we obtain $\nabla_{\dot{y}}\dot{y} = \nabla_X X = \mathrm{d}\pi(\bar{\nabla}_{\bar{X}}\bar{X}) = (\kappa + 2c)\pi\phi\bar{X} = (\kappa + 2c)J\dot{y}$. ■

7. Charged Particles in Special Homogeneous Spaces

7.1. Charged Particles in Special Homogeneous Spaces

In this subsection we shall construct a Riemannian homogeneous space M with an invariant $(1, 1)$-tensor I and consider the motion of charged particles under electromagnetic field κI according to [14].

Let G be a connected Lie group and K a compact subgroup of G. We consider the coset manifold $M = G/K$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Since K is compact, there exists an $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}. \tag{26}$$

We denote by π the natural projection from G onto M, and by $o = \pi(e)$, the origin of M. Then we can identify $\mathfrak{m}$ with $T_o(M)$ through π_*. We assume that there exist such $\mathrm{Ad}(K)$-invariant subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of $\mathfrak{m}$ which span m, i.e.,

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \tag{27}$$

and such that

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k} \oplus \mathfrak{m}_2, \qquad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}, \qquad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1. \tag{28}$$

For X in $\mathfrak{g}$, we denote by X_i the $\mathfrak{m}_i$-component of X. Moreover we assume that there exist a nonzero constant $c \in \mathbb{R}$ and $\mathrm{Ad}(K)$-invariant inner product $\langle\, ,\rangle$ in $\mathfrak{m}$ such that

$$\mathfrak{m}_1 \perp \mathfrak{m}_2, \quad \langle [X,Y]_2, Z\rangle + c\langle X, [Z,Y]\rangle = 0, \quad X,Y \in \mathfrak{m}_1, Z \in \mathfrak{m}_2. \tag{29}$$

If we extend the inner product $\langle\, ,\rangle$ to a G-invariant Riemannian metric $\langle\, ,\rangle$ on M, then M is a Riemannian homogeneous space and G acts on M isometrically. We denote by $\mathfrak{c}$ the center of $\mathfrak{k}$. For W in $\mathfrak{c}$, we define an endomorphism I of $\mathfrak{m}$ by

$$I : \mathfrak{m} \to \mathfrak{m}; X_1 + X_2 \mapsto [W, X_1] + \frac{1}{c}[W, X_2], \quad X_1 \in \mathfrak{m}_1, X_2 \in \mathfrak{m}_2. \tag{30}$$

Since $\mathrm{Ad}(k)I = I\mathrm{Ad}(k)$ for any k in K, we can extend I to a G-invariant $(1,1)$-tensor I on M. We then have

$$\langle IX, Y\rangle + \langle X, IY\rangle = 0 \quad \text{for} \quad X, Y \in \mathfrak{X}(M).$$

Let κ be a constant. A curve $x(t)$ is called the *motion of a charged particle under electromagnetic field* κI, if it satisfies the following differential equation

$$\nabla_{\dot{x}}\dot{x} = \kappa I\dot{x}. \tag{31}$$

When $\kappa = 0$, then $x(t)$ is a geodesic.

Theorem 31 ([14]). *Let $M = (G/K, \langle\, ,\rangle)$ be a Riemannian homogeneous space with a G-invariant skew-symmetric $(1,1)$-tensor I satisfying the conditions* (26), (27), (28), (29) *and* (30). *Let $x(t)$ be the motion of a charged particle defined by* (31) *under electromagnetic field κI with initial conditions $x(0) = o$ and $\dot{x}(0) = X_1 + X_2$ $(X_1 \in \mathfrak{m}_1, X_2 \in \mathfrak{m}_2)$. Then $x(t)$ is given by*

$$x(t) = \pi\left(\exp t(X_1 + cX_2 + \kappa W)\exp t(1-c)\left(X_2 + \frac{\kappa}{c}W\right)\right).$$

If $x(t)$ intersects itself, then it is simply closed.

Remark 32. In the case when $\kappa = 0$, this is a theorem of Dohira [9].

Example 33 (geodesics in compact four-symmetric spaces). *Let G be a compact connected Lie group and θ an automorphism of G of order four. We also denote by θ the differential of θ. We define a closed subgroup K of G by $K = \{g \in G\ ;\ \theta(g) = g\}$ and a subspace $\mathfrak{m}$ in the Lie algebra $\mathfrak{g}$ of G by*

$$\mathfrak{m} = \{X \in \mathfrak{g}\ ;\ (\theta^3 + \theta^2 + \theta + 1)(X) = 0\}.$$

We define subspaces $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$ *in* $\mathfrak{m}$ *by*

$$\begin{aligned}\mathfrak{m}_1 &= \{X \in \mathfrak{m}\ ;\ \theta^2(X) = -X\} = \{X \in \mathfrak{g}\ ;\ \theta^2(X) = -X\}\\ \mathfrak{m}_2 &= \{X \in \mathfrak{m}\ ;\ \theta^2(X) = X\} = \{X \in \mathfrak{g}\ ;\ \theta(X) = -X\}.\end{aligned}$$

Let $F : G/K \to G$ *be a Cartan embedding, which is defined by*

$$F : G/K \to G, \qquad gK \mapsto g\theta(g^{-1}).$$

Take an $\mathrm{Ad}(G)$ *and* θ *invariant inner product* $(\,,)$ *on* $\mathfrak{g}$. *Then* F *and* $(\,,)$ *induce a* G*-invariant Riemannian metric* $\langle\,,\rangle$ *on* G/K. *Since* $F_*X = X - \theta X, \quad X \in \mathfrak{m}$, *we have*

$$\langle X, Y\rangle = (X - \theta X, Y - \theta Y) \quad \textit{for} \quad X, Y \in \mathfrak{m}.$$

If we set $c = 2$*, then the conditions* (26), (27), (28) *and* (29) *are satisfied. Hence a curve* $x(t)$ *in* $(G/K, \langle\,,\rangle)$ *is a geodesic such that* $x(0) = o$ *and* $\dot{x}(0) = X_1 + X_2 (X_i \in \mathfrak{m}_i)$ *if and only if*

$$x(t) = \pi\left(\exp t(X_1 + 2X_2)\exp(-tX_2)\right).$$

7.2. Charged Particles in Hermitian Symmetric Spaces

In this subsection we shall apply Theorem 31 to the motion of charged particles in Hermitian symmetric spaces according to [14] and [16]. Every motion of a charged particle in a Hermitian symmetric space under Kähler electromagnetic field is simple. Let $(G, K, \theta, \langle\,,\rangle, J)$ be an almost effective Hermitian symmetric pair. Then the coset manifold $M = G/K$ is a Hermitian symmetric space. Conversely, every Hermitian symmetric space is obtained in this way. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

be the canonical decomposition of the Lie algebra $\mathfrak{g}$ of G. We denote by $\mathfrak{c}$ the center of $\mathfrak{k}$. There exists an element J_o in $\mathfrak{c}$ such that $J = \mathrm{ad}(J_o)$ is a complex structure on $\mathfrak{m}$. Setting $\mathfrak{m}_2 = \{0\}$ and $W = J_o$ in Theorem 31, we redemonstrate the following.

Corollary 34 (Adachi-Maeda-Udagawa [2]). *Let* $M = G/K$ *be a Hermitian symmetric space . Let* $x(t)$ *be the motion of a charged particle defined by* $\nabla_{\dot{x}}\dot{x} = \kappa J\dot{x}$ *under the electromagnetic field* κJ *with initial conditions* $x(0) = o$ *and* $\dot{x}(0) = X \in \mathfrak{m}$. *Then* $x(t)$ *is given by*

$$x(t) = \pi(\exp t(\kappa J_o + X)). \tag{32}$$

Corollary 35 ([14]). *Let $x(t)$ be the motion of a charged particle in a Hermitian symmetric space. Then its velocity vector $\dot{x}(t)$ can then be extended to a Killing vector field which is an infinitesimal automorphism of J.*

We here mention some fundamental properties of the motion of charged particles under a Kähler electromagnetic field. Let $x(t)$ be the motion of a charged particle under a Kähler electromagnetic field $\kappa\Omega$ in a Kähler manifold $(M, \langle\, ,\rangle, J)$. If g is a holomorphic isometry of M, then $gx(t)$ is also the motion of a charged particle under $\kappa\Omega$. Two motions $x_1(t)$ and $x_2(t)$ are called *congruent* if there exists a holomorphic isometry g with $x_2 = g \circ x_1$.

Let M be a Hermitian symmetric space of compact type, with fixed rank r. Let $\delta(> 0)$ be the maximum of the sectional curvatures of M. We denote by $\mathbb{S}^2(1/\sqrt{\delta})$ the two-dimensional sphere of radius $1/\sqrt{\delta}$. Then there exists a totally geodesic Kähler embedding

$$\iota : (\mathbb{S}^2(1/\sqrt{\delta}))^r = \mathbb{S}^2(1/\sqrt{\delta}) \times \cdots \times \mathbb{S}^2(1/\sqrt{\delta}) \to M$$

which is called a Hermann map (see [11], [24, § 3] and [25, § 3] for details). Let trajectory $x(t)$ describes the motion of charged particle in M. Since M is homogeneous, replacing $x(t)$ with a congruent class there of if necessary, we may assume that $x(0) \in (\mathbb{S}^2(1/\sqrt{\delta}))^r$. Since $\mathrm{rank}((\mathbb{S}^2(1/\sqrt{\delta}))^r) = r$, we may assume that

$$\dot{x}(0) \in T_o(\mathbb{S}^2(1/\sqrt{\delta}))^r \quad (o = x(0)).$$

Because $(\mathbb{S}^2(1/\sqrt{\delta}))^r$ is a totally geodesic complex submanifold in M, we have $x(\mathbb{R}) \subset (\mathbb{S}^2(1/\sqrt{\delta}))^r$. Since the motions of the charged particles in $\mathbb{S}^2$ are small circles, there exists an r-dimensional flat torus T in $(\mathbb{S}^2(1/\sqrt{\delta}))^r$ such that $x(\mathbb{R}) \subset T$ and such that $x(t)$ is a geodesic in T. Hence we obtain the following.

Theorem 36 ([16]). *Let M be a Hermitian symmetric space of compact type, whose rank is equal to r. For any motion $x(t)$ of a charged particle under a Kähler electromagnetic field in M, there exists an r-dimensional flat torus T in M such that $x(t)$ is a geodesic in T.*

Remark 37. When M is of rank one, the above theorem shows that every motion of a charged particle is simply closed. This fact is well known. When M is a complex Grassmann manifold, then the above theorem corresponds to a theorem of Adachi, Maeda and Udagawa [2, Theorem 2.2]. When $r \geq 2$, the above theorem shows that there exist both a simply closed motion and an open motion of charged particles of any given κ. This fact was mentioned in [2, Corollary 2.1].

In a similar way we get the following.

Theorem 38 ([16]). *Let M be a Hermitian symmetric space of non-compact type, whose rank is equal to r. Let $-\delta(<0)$ be the minimum of the sectional curvatures of M. We denote by $H^2(-\delta)$ the two-dimensional real hyperbolic space of constant curvature $-\delta$. For any motion $x(t)$ of a charged particle under a Kähler electromagnetic field in M, there exists a totally geodesic complex submanifold*

$$(H^2(-\delta))^r = H^2(-\delta) \times \cdots \times H^2(-\delta) \subset M$$

such that $x(t)$ is the motion of charged particle in $(H^2(-\delta))^r$.

Remark 39. The motion of charged particles in $H^2(-\delta)$ was studied by Comtet [8] and Sunada [22]. The motion of charged particles in $(H^2(-\delta))^r$ was studied by Adachi [1].

7.3. Charged Particles in Kähler C-spaces

In this subsection we shall apply Theorem 31 to the motion of charged particles in Kähler C-spaces with certain conditions according to [14]. By a C-space we mean a compact simply connected complex homogeneous space, and by a Kähler C-space, a C-space M which admits a Kähler metric such that a group of holomorphic isometries acts transitively on M. Every motion of a charged particle in a Kähler C-space under Kähler electromagnetic field is simple.

We shall construct Kähler C-spaces according to [4, Ch. 8]. Let G be a compact connected semisimple Lie group and W in its Lie algebra $\mathfrak{g}$. We define a closed subgroup K of G by

$$K = \{g \in G \,;\, \mathrm{Ad}(g)W = W\}.$$

Then K is connected, and coset manifold $M = G/K$ is compact and simply connected, which is called a generalized flag manifold. We can identify the tangent space $T_o(M)$ at the origin o with $\mathfrak{m} = \mathrm{im}\,\mathrm{ad}(W)$. In order to define a G-invariant complex structure J on M, take a maximal torus T of G such that W is in its Lie algebra $\mathfrak{t}$. Take a biinvariant Riemannian metric $(\,,)$ on G. We denote by Δ the set of nonzero roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. Take a lexicographic ordering on $\mathfrak{t}$ such that $(W,\alpha) \geq 0$ for any positive root α. We denote by Δ^+ the set of positive roots. We have the following direct sum decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta^+} (\mathbb{R}F_\alpha \oplus \mathbb{R}G_\alpha)$$

where for each $H \in \mathfrak{t}$, $[H, F_\alpha] = (\alpha, H)G_\alpha, [H, G_\alpha] = -(\alpha, H)F_\alpha$. Set

$$\Delta_W = \{\alpha \in \Delta \ ; \ (\alpha, W) = 0\}, \qquad \Delta_W^+ = \Delta_W \cap \Delta^+$$

then we have

$$\mathfrak{k} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta_W^+} (\mathbb{R}F_\alpha \oplus \mathbb{R}G_\alpha), \qquad \mathfrak{m} = \sum_{\alpha \in \Delta^+ - \Delta_W^+} (\mathbb{R}F_\alpha \oplus \mathbb{R}G_\alpha).$$

We define a complex structure J on $\mathfrak{m}$ by

$$JF_\alpha = G_\alpha, \qquad JG_\alpha = -F_\alpha \qquad \text{for} \qquad \alpha \in \Delta^+ - \Delta_W^+.$$

Since $\mathrm{Ad}(k)J = J\mathrm{Ad}(k)$ for any k in K, we can extend J to a G-invariant almost complex structure on M. This almost complex structure J is integrable. We assume that G is simple. We denote by $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ the set of simple roots, and by $\alpha_0 = \sum m_j \alpha_j$, the highest root.

If we set

$$\Pi_W = \{\alpha_j \in \Pi \ ; \ (\alpha_j, W) > 0\} = \{\alpha_{i_1}, \cdots, \alpha_{i_s}\}$$

then it is known that the second betti number $b_2(M)$ of M is given by $b_2(M) = s = \#(\Pi_W)$ ([6]). We assume that $b_2(M) = 1$, that is, $\Pi_W = \{\alpha_i\}$. For a natural number n, set

$$\Delta^+(\alpha_i; n) = \{\alpha = \sum n_j \alpha_j \in \Delta^+ \ ; \ n_i = n\}, \quad \mathfrak{m}_n = \sum_{\alpha \in \Delta^+(\alpha_i; n)} (\mathbb{R}F_\alpha \oplus \mathbb{R}G_\alpha)$$

then we have

$$\Delta^+ - \Delta_W^+ = \Delta^+(\alpha_i) = \bigcup_{n \geq 1} \Delta^+(\alpha_i; n), \qquad \mathfrak{m} = \sum_{n \geq 1} \mathfrak{m}_n.$$

We set also $\mathfrak{m}_0 = \mathfrak{k}$ for simplicity. Then for $n, m \geq 0$ we have $[\mathfrak{m}_n, \mathfrak{m}_m] \subset \mathfrak{m}_{n+m} + \mathfrak{m}_{|n-m|}$. If we normalize W so that $(W, \alpha_i) = 1$, then we have $nJ = \mathrm{ad}(W)$ on $\mathfrak{m}_n$. We define a G-invariant Kähler metric $\langle \, , \rangle$ on M by

$$\langle X_n, X_m \rangle = n\delta_{nm}(X_n, X_m) \qquad \text{for} \qquad X_n \in \mathfrak{m}_n, \qquad X_m \in \mathfrak{m}_m.$$

We assume that $m_i = 2$. If we set $c = 2$, then conditions (26), (27), (28), (29) and (30) are satisfied. Hence we have the following corollary by Theorem 31.

Corollary 40 ([14]). *Let $M = (G/K, J)$ be a Kähler C-space with $b_2(M) = 1$. We assume that G is a compact connected simple Lie group. Further, we assume* *that there exists a simple root* α_i *such that* $\Pi_W = \{\alpha_i\}$ *and that* $m_i = 2$*, where* $\alpha_0 = \sum_j m_j\alpha_j$ *is the highest root. Let* $x(t)$ *be a motion of charged particle defined by* $\nabla_{\dot{x}}\dot{x} = \kappa J\dot{x}$ *under the electromagnetic field* κJ *with initial conditions* $x(0) = o$ *and* $\dot{x}(0) = X_1 + X_2$ $(X_1 \in \mathfrak{m}_1, X_2 \in \mathfrak{m}_2)$*. Then* $x(t)$ *is given by*

$$x(t) = \pi\left(\exp t(X_1 + 2X_2 + \kappa W)\exp\left(-t\left(X_2 + \frac{\kappa}{2}W\right)\right)\right)$$

where W *is in the center of the Lie algebra* $\mathfrak{k}$ *of* K*.*

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