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ON THE UNCERTAINTY RELATIONS IN STOCHASTIC MECHANICS

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Abstract.

It is shown that the Bohm equations for the phase S and squared modulus ρ of the quantum mechanical wave function can be derived from the classical ensemble equations admiting an aditional momentum p_s of the form proportional to the osmotic velocity in the Nelson stochastic mechanics and using the variational principle with appropriate change of variables. The possibility to treat grad$S$ and p_s as two parts of the momentum of quantum ensemble particles is considered from the view point of uncertainty relations of Robertson - Schrödinger type on the examples of the stochastic image of quantum mechanical canonical coherent and squeezed states.

1. Introduction

The uncertainty (indeterminacy) principle in quantum physics, which quantitatively is expressed in the form of uncertainty relations (URs) [13, 14, 24, 25] is commonly regarded as the most radical departure from the classical physics.

However in the recent decades publications have appeared [5, 11, 12, 21, 23] in which inequalities are introduced in Nelson stochastic mechanics (SM) [19] and discussed as Heisenberg-type URs. The equations of motion in this mechanics coincide with the David Bohm equations [1] (the continuity equation and the modified Hamilton-Jacobi equation, the latter known also as Hamilton-Jacobi-Madelung (HJM) equation) for the phase S and squared modulus $|\psi|^2 \equiv \rho$ of the Schrödinger wave function ψ. Bohm equations for S and ρ have been later derived from 'the stochastic variational principles of control theory' by Guerra and Marra [9], and by Reginatto [23], using the 'principle of minimum Fisher information' [6].

Hall and Reginatto [12] introduced the so called 'exact UR' and showed that it 'leads from classical equations of motion to the Schrödinger equation' and to uncertainty inequality of the form of Heisenberg UR. They derived the continuity equation and the modified HJ equation from a variational principle, introducing into the Lagrangian of the HJ equation an additional momentum $\boldsymbol{p}_{\mathrm{N}}$ of the classical particle assuming that its first moment and its covariance with the 'classical' moment ∇S are universally vanishing. From some general consideration they 'derived' that the variance of this extra momentum should be proportional to the Fisher information of the coordinate probability density ρ. As a result their Lagrangian takes the form of Reginatto' Lagrangian, wherefrom the Bohm equations are derived [23] and the product of coordinate and $\boldsymbol{p}_{\mathrm{N}}$ variances equals constant for any ρ (which in fact is a minimization of the Cramer-Rao inequality). This equality and the related uncertainty principle are called 'exact' UR and 'exact' uncertainty principle. The system described by the so derived Bohm equations is interpreted in [12] as '*quantum ensemble*'. If $\nabla S/m$ and the variance of $\boldsymbol{p}_{\mathrm{N}}/m$ are identified with the current velocity and the mean squared osmotic velocity the formal connection to the Nelson SM is established [12].

However no particular underlying physical model was assumed for the fluctuations of the momentum $\boldsymbol{p}_{\mathrm{N}}$ - they were regarded as fundamentally nonanalyzable [12]. Having no model for $\boldsymbol{p}_{\mathrm{N}}$ one has to postulate infinitely many constraints in order to recover the statistical properties of quantum mechanical momentum $\hat{\boldsymbol{p}}$. The two constraints postulated in [12] (namely $\overline{\boldsymbol{p}_{\mathrm{N}}} = 0$ and $\overline{\nabla S \cdot \boldsymbol{p}_{\mathrm{N}}} = 0$) ensure the coinsidence only of the first two moments of $\nabla S + \boldsymbol{p}_{\mathrm{N}}$ and $\hat{\boldsymbol{p}}$.

It is our aim here to introduce a model for such additional momentum to account partially for the quantum fluctuations and to examin its properties and consistency. ('Partially', because no classical model, we believe, could provide the 'full' account). Another our aim is to briefly review the URs in the Nelson stochastic mechanics (SM) from the point of view of the more precise Robertson-Schrödinger (R-S) inequality. Unlike the Heisenberg UR the R-S UR in quantum mechanics involves all the three second moments of the two quantum observables $\hat{A}$ and $\hat{B}$, the variances $(\Delta\hat{A})^2$, $(\Delta\hat{B})^2$ and the squared covariance $(\Delta_{AB})^2$. If the covariance is vanishing then the R-S UR recovers the Heisenberg UR.

In the next section we briefly review the Heisenberg and Robertson-Schrödinger URs (R-S UR). In the third section we recall the main features of the Hall and Reginatto 'quantum ensemble' approach and Nelson SM. A model of additional momentum p_{s} is introduced, its potential S_{s} being interpreted as the intensity dependent part of the quantum wave function phase S. It is shown that Bohm equations for S and ρ can be derived from the Reginatto variational principle considering the probability density ρ and $S_{-} \equiv S - S_{\rm s}$ as new independent variables. In Section 4 the stochastic analogues of the R-S URs are reviewed and discussed in connection with the introduced auxilliary momentum model p_s and on the example of S and ρ corresponding to canonical coherent and squeezed states. In Section 5 the first and second moments of coordinate and the related momenta and URs are calculated on the example of stochastic images of canonical coherent and squeezed states (CS and SS) and compared with the corresponding moments and URs in quantum mechanics. Nelson SM images of canonical CS and SS have been discussed previously in several papers: of CS in [10,16–18,22] and of SS and CS - in [16–18,22] in the context of 'stochastic mechanics and control theory'.

2. Robertson-Schrödinger UR in Quantum Mechanics

The indeterminacy principle was introduced in 1927 by Heisenberg [13] who demonstrated the impossibility of simultaneous precise measurement of the canonical quantum observables $\hat{x}$ and $\hat{p}$ (the particles coordinate and momentum) by positing an approximate relation $\delta p\, \delta x \sim \hbar$, where $\hbar$ is the Plank constant.

Heisenberg considered this inequality as the "direct descriptive interpretation" of the canonical commutation relation between the operators of the coordinate and momentum: $[\hat{x}, \hat{p}] = {\rm i}\hbar$, $[\hat{x}, \hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x}$. A rigorous proof of the Heisenberg relation was soon published by Kennard and Weyl [14] who established the inequality

$$(\Delta\hat{p})^2(\Delta\hat{x})^2 \;\geq\; \hbar/4 \tag{1}$$

where $(\Delta\hat{p})^2$ and $(\Delta\hat{x})^2$ are the variances (dispersions) of $\hat{p}$ and $\hat{x}$, defined by Weyl for every quantum state $|\psi\rangle$ via the formula $(\Delta\hat{p})^2 := \langle\psi|(\hat{p} - \langle\psi|\hat{p}|\psi\rangle)^2|\psi\rangle$, and similarly $(\Delta\hat{x})^2$ is defined. In correspondence with the classical probability theory the standard deviation $\Delta\hat{A}$ is considered as a measure for the uncertainty (indeterminacy) of the quantum observable $\hat{A}$ in the corresponding state $|\psi\rangle$. The inequality (1) became known as the *Heisenberg* UR.

The extension of (1) to the case of two arbitrary quantum observables (Hermitian operators $\hat{A}$ and $\hat{B}$) was made by Robertson and Schrödinger [24, 25], who established more precise inequality, that involves all the three second moments of the two observables,

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 - (\Delta_{AB})^2 \;\geq\; \frac{1}{4}\left|\langle[\hat{A}, \hat{B}]\rangle\right|^2 \tag{2}$$

where Δ_{AB} is the covariance (which in mathematical literature is denoted usually as ${\rm Cov}(AB)$) of $\hat{A}$ and $\hat{B}$, $\;\Delta_{AB} := (1/2)\langle\hat{A}\hat{B} + \hat{B}\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$.

In the case of coordinate and momentum observables relation (2) takes the shorter form

$$(\Delta\hat{x})^2(\Delta\hat{p})^2 - (\Delta_{xp})^2 \;\geq\; \hbar^2/4. \tag{3}$$

The inequality (2) is referred either as Schrödinger or Robertson-Schrödinger UR (R-S UR). In states with vanishing covariance the R-S UR (3) recovers the Heisenberg's one, equation (1). The minimization of (1), i.e., the equality in (1), means the equality in (3), the inverse being not true. Thus the R-S UR provides a more stringent limitation (from below) to the product of two variances. Besides the R-S UR is more symmetric than the Heisenberg UR: the equality in it is invariant under nondegenerate linear transformations of the two observables (in the case of x and p R-S UR is invariant under linear canonical transformations) [26]. Despite these advantages the relation (3) and/or (2) are lacking in almost all quantum mechanics text books. The interest in R-S relation has been renewed in the last three decades [2, 4, 27] (50 years after its discovery) in connection with the description and experimental realization of the *squeezed states* of the electromagnetic radiation (see the 'squeezed review' [4, 26]).

3. 'Quantum Ensemble' and Stochastic Mechanics

The quantum-classical relations are subject of a host of publications, which started from the early days of quantum mechanics. Aiming to provide an alternative interpretation of quantum mechanics in terms of 'hidden variables' David Bohm [1] noted that the phase $S = \hbar \arg\psi$ and the squared modulus $|\psi|^2 \equiv \rho$ of the quantum-mechanical particle wave function ψ obeys a system of classical-type equations, namely the probability conservation equation and a modified Hamilton-Jacobi equation

$$\frac{\partial\rho}{\partial t} + \frac{1}{m}\mathrm{div}\,(\rho\nabla S) = 0, \qquad \frac{\partial S}{\partial t} + \frac{1}{2m}\,(\nabla S)^2 + V(\boldsymbol{x},t) + V_{\mathrm{q}} = 0 \tag{4}$$

where V is the external particle potential, and V_{q} (the so called 'quantum potential' [1]) is given by

$$V_{\mathrm{q}} = \frac{\hbar^2}{8m}\left(\frac{(\nabla\rho)^2}{\rho^2} - 2\frac{\nabla^2\rho}{\rho}\right).$$

Pursuing the classical interpretation and derivation of the Schrödinger equation Nelson [19] had derived equations for the velocity fields in the forward and backward Fokker-Planck equations of a diffusion process and, noting that the 'osmotic' velocity $\boldsymbol{u}$ is a gradient ($\boldsymbol{u} = D\nabla\ln\rho$, ρ being the probability density of the process, D - the diffusion coefficient) and supposing that the current velocity $\boldsymbol{v}$ is also a gradient, $\boldsymbol{v} = (1/m)\nabla S$, he had established that with $D = \hbar/2m$ the probability density ρ and the current velocity potential S satisfy the Bohm equations (4), i.e., $\psi := \sqrt{\rho}\exp(\mathrm{i}S/\hbar)$ obey the Schrödinger equation. This theory is known as Nelson SM.

Reginatto [23] noted that the Bohm equations (and thereby the Schrödinger equation) can be obtained from the variational principle and the principle of minimum Fisher information [6] applied to the 'classical ensemble of particles'. In this derivation Reginatto started from the classical Hamilton-Jacobi (HJ) equation (we consider the case of $n = 1$, and external potential V)

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\nabla S\right)^2 + V(\boldsymbol{x}, t) = 0. \tag{5}$$

Supposing that the coordinates are subject to fluctuations described by the probability density ρ he had postulated the validity of the continuity equation of the same form as in (4)

$$\frac{\partial \rho}{\partial t} + \frac{1}{m}\mathrm{div}\left(\rho\nabla S\right) = 0 \tag{6}$$

and noted that it can be derived from the functional

$$\Phi_A = \int \rho\left(\partial_t S + \frac{1}{2m}\nabla S \cdot \nabla S + V\right) \mathrm{d}^3x\, \mathrm{d}t \tag{7}$$

as extremal equation with respect to variation of the classical action S. (As noted in [23] the variation with respect to ρ trivially results into HJ equation (5)). Physical system which motion is described by the equations (5) and (6) is called *classical ensemble* of particles [12, 23].

To obtain the second of the Bohm equations (4) the principle of minimal Fisher information was applied by adding to Φ_A the term [23]

$$\Phi'_A = \lambda \int I_{\mathrm{F}}(\rho)\mathrm{d}t, \qquad I_{\mathrm{F}} = \int \frac{1}{\rho}\nabla\rho \cdot \nabla\rho\, \mathrm{d}^3x, \tag{8}$$

where I_{F} is the Fisher information of the probability density $\rho(\boldsymbol{x}, t)$, and the multiplier λ is put equal to $\hbar^2/8m$. Thus the Bohm equations (4) are derived from the action functional

$$\begin{aligned} \Phi_B &= \Phi_A + \Phi'_A \\ &= \int \left[\rho\left(\partial_t S + \frac{1}{2m}\nabla S \cdot \nabla S + V\right) + \frac{\hbar^2}{8m}\frac{1}{\rho}\nabla\rho\cdot\nabla\rho\right] \mathrm{d}^3x\mathrm{d}t \end{aligned} \tag{9}$$

by independent variation of ρ and S. In fact Bohm equations have been derived previously from the same action functional (9) by Guerra and Moratto [9] but with no reference to Fisher information.

By different argumentation the same action functional (9) has been derived and used later in [12], where the term $2m\lambda I_{\mathrm{F}}$ is interpreting as a variance $(\sigma_{\boldsymbol{p}_{\mathrm{N}}})^2$ of an additional momentum $\boldsymbol{p}_{\mathrm{N}}$, subjected to the constraints (of vanishing first moment and vanishing covariance with ∇S)

$$\langle \boldsymbol{p}_{\mathrm{N}} \rangle = 0, \qquad \langle \boldsymbol{p}_{\mathrm{N}} \cdot \nabla S \rangle = 0. \tag{10}$$

This variance obeys the inequality (in the one-dimensional case)

$$(\sigma_x)^2 (\sigma_{p_{\mathrm{N}}})^2 \geq \hbar^2/4 \tag{11}$$

which directly stems from well known Cramer-Rao inequality $(\sigma_x)^2 I_{\mathrm{F}} \geq 1$, where $(\sigma_x)^2$ is the variance (the squared uncertainty) of x. The authors of [12] consider the total momentum of the particle p as a sum of $\partial_x S$ and p_{N}. Then, in view of (10) and (11), one gets

$$(\sigma_x)^2 (\sigma_p)^2 \geq (\sigma_x)^2 (\sigma_{p_{\mathrm{N}}})^2 \geq \hbar^2/4. \tag{12}$$

The authors argue that this is a derivation of Heisenberg UR.

However no particular underlying physical model was assumed for the fluctuations of the momentum $\boldsymbol{p}_{\mathrm{N}}$ - they were regarded as 'fundamentally nonanalyzable'. Despite the proclaimed 'nonanalizability' of $\boldsymbol{p}_{\mathrm{N}}$ the authors of [12] succeeded to find that its variance should be proportional to the Fisher information of ρ, resorting in this way to the Reginatto functional (9), wherefrom they derive the Bohm equations for ρ and S. The so derived equations (4) are referred to as equations of motion of *quantum ensemble* [12, 23].

We note that in fact the second constraint in (10) simply reduces the functional (6) in [12] to Reginatto functional (9), which ensures the variational derivation of the Bohm equations for ρ and S. Retaining the idea of introducing an additional stochastic momentum however there is an alternative way to derive variationally the Bohm equations, namely the change the independent variables. In this way one may expect to introduce a model of an *analizable* additional stochastic momentum $\boldsymbol{p}_{\mathrm{s}}$ to account *partially* for the quantum momentum fluctuations.

We consider the total momentum $\boldsymbol{p}$ of the particle as a sum of two parts

$$\boldsymbol{p} = \boldsymbol{p}_{\mathrm{c}} + \boldsymbol{p}_{\mathrm{s}} \tag{13}$$

supposing that the first one stems from the deterministic classical motion and the second one is induced by the coordinate randomization. We suppose further that both $\boldsymbol{p}_{\mathrm{c}}$ and $\boldsymbol{p}_{\mathrm{s}}$ are gradients of corresponding potentials (actions)

$$\boldsymbol{p}_{\mathrm{c}} = \nabla S, \qquad \boldsymbol{p}_{\mathrm{s}} = \nabla S_{\mathrm{s}} \tag{14}$$

where the momentum potential S originates from the classical HJ equation, and the potential S_{s} - from the coordinate stochasticity. In the absence of stochasticity S is the classical particle action that obey the HJ equation. We make the natural anzatz that the potential S_{s} depends on $\boldsymbol{x}$ and t via the coordinate probability distribution $\rho(\boldsymbol{x}, t)$ only: $S_{\mathrm{s}} = S_{\mathrm{s}}(\rho(\boldsymbol{x}, t))$.

Supposing that coordinate stochasticity induces new momentum part it is then natural to expect that the latter in turn will affect the particle action S. The simplest way to take into account this 'feed back' is to suppose that part of S becomes ρ-dependent. We suppose that this part is proportional to S_{s}, and denote the difference by S_{-}. So that we put

$$S = S_{-} + S_{\mathrm{s}}(\rho) \tag{15}$$

and treat S_{-} and ρ as *independent fields*. Next we put $S = S_{-} + S_{\mathrm{s}}(\rho)$ into the Reginatto action functional (9) and apply the variational principle to the resulting functional

$$\begin{aligned} \Phi_{B'} \;=\; & \int \rho \left(\partial_t (S_{-} + S_{\mathrm{s}}) + \frac{1}{2m} \nabla (S_{-} + S_{\mathrm{s}}) \cdot \nabla (S_{-} + S_{\mathrm{s}}) \right) \mathrm{d}^3 x \, \mathrm{d}t \\ & + \int \rho \left(V + \frac{\hbar^2}{8m} \frac{1}{\rho^2} \nabla \rho \cdot \nabla \rho \right) \mathrm{d}^3 x \, \mathrm{d}t \end{aligned} \tag{16}$$

treating δS_{-} and $\delta \rho$ as independent variations, vanishing at the end points. The resulting equations of the extremals read

$$\frac{\partial \rho}{\partial t} + \frac{1}{m} \mathrm{div} \left(\rho \nabla S_{-} + \frac{\partial S_{\mathrm{s}}}{\partial \rho} \nabla \rho \right) = 0 \tag{17}$$

$$\begin{aligned} \frac{\partial S_{-}}{\partial t} + \frac{1}{2m} \left(\nabla (S_{-} + S_{\mathrm{s}}) \right)^2 - \frac{1}{m} \frac{\partial S_{\mathrm{s}}}{\partial \rho} \left(\nabla \rho \cdot \nabla S_{-} + \rho \nabla^2 S_{-} \right) \\ - \frac{1}{m} \frac{\partial S_{\mathrm{s}}}{\partial \rho} \left(\left(\rho \nabla^2 \rho + (\nabla \rho)^2 \right) \frac{\partial S_{\mathrm{s}}}{\partial \rho} + \rho (\nabla \rho)^2 \frac{\partial^2 S_{\mathrm{s}}}{\partial \rho^2} \right) = 0. \end{aligned} \tag{18}$$

Putting here $S_{-} = S - S_{\mathrm{s}}$ and using again the continuity equation we obtain the Bohm equations (4) for ρ and S, as desired. Taking into account that $\sqrt{\rho}\exp(\mathrm{i}S/\hbar)$

satisfies the Schrödinger equation we see that the $\boldsymbol{p}_{\rm s}$-potential $S_{\rm s}$ has the meaning of *ρ-dependent part* of the wave function phase.

We note that this result is valid for any differential function $S_{\rm s}(\rho)$ (but not if $S_{\rm s}$ depends on $\nabla\rho$). One can use this freadom to subject $S_{\rm s}$ to some desired constraints. One natural constraint is the vanishing average of $\boldsymbol{p}_{\rm s}$, $\overline{\boldsymbol{p}_{\rm s}} = 0$. This can be satisfied with $S_{\rm s} = \lambda\ln(l^3\rho)$, where l is a length parameter (so that $l^3\rho$ be dimensionless), i.e.,

$$\boldsymbol{p}_{\rm s} = \frac{\lambda}{\rho}\nabla\rho. \tag{19}$$

This ensures the coincidence of total momentum average $\overline{\boldsymbol{p}} := \int\rho(\boldsymbol{x},t)\boldsymbol{p}\,{\rm d}^3x$, with the average $\langle\hat{\boldsymbol{p}}\rangle$ of quantum momentum $\hat{\boldsymbol{p}}$ in the corresponding state $\psi = \sqrt{\rho}\exp({\rm i}S/\hbar)$ (using the known equality $\langle\hat{\boldsymbol{p}}\rangle = \overline{\boldsymbol{p}_{\rm c}}$ [17]). Furthemore we fix the parameter λ as $\hbar/2$, i.e., we use $\boldsymbol{p}_{\rm s} = \hbar(\nabla\rho)/2\rho$. To shorten the notation herefrom we consider the one-dimensional motion only.

Formally the quantities $\boldsymbol{p}_{\rm s}/m$, $\nabla S/m$ and $\boldsymbol{p} = (\boldsymbol{p}_{\rm s} + \nabla S)/m$ coincide with the osmotic, current and forward velocities u, v, v_+ in the Nelson SM, where many of their properties are thoroughly examined (see for example [16–18, 22] and references there in). Our interest here is focused on the properties related to the possibility $p_{\rm s}$ to describe (at least partially) nonclassical fluctuations of the quantum-mechanical momentum $\hat{p}$. In this aim we compare statistical properties of $p_{\rm s}$ with those of 'nonclassical part' $\hat{p}_{\rm nc}$ of $\hat{p}$ [11], $\hat{p}_{\rm nc} := \hat{p} - p_{\rm c}$, $p_{\rm c} = \partial S/\partial x$. Hall [11] found the first two moments of $p_{\rm nc}$ as

$$\langle\hat{p}_{\rm nc}\rangle = 0, \qquad \langle\hat{p}_{\rm nc}^2\rangle = \langle\hat{p}^2\rangle - \langle p_{\rm c}^2\rangle = (\Delta\hat{p})^2 - (\Delta p_{\rm c})^2 \tag{20}$$

$$(\delta x)^2(\Delta\hat{p}_{\rm nc})^2 = \frac{\hbar^2}{4} \tag{21}$$

where δx is the Fisher length, $(\delta x)^2 = 1/I_{\rm F}$, the equality (21) being refered as 'exact' UR. The above three properties are shared by $p_{\rm s}$ too (established in terms of the osmotic momentum mu earlier, see e.g. [5, 17] and references their in):

$$\langle p_{\rm s}\rangle = 0, \qquad \langle p_{\rm s}^2\rangle = \langle\hat{p}^2\rangle - \langle p_{\rm c}^2\rangle = (\Delta\hat{p})^2 - (\sigma_{p_{\rm c}})^2 \tag{22}$$

$$(\delta x)^2(\sigma_{p_s})^2 = \frac{\hbar^2}{4} \tag{23}$$

where $\sigma_{p_{\rm s}}$, $\sigma_{p_{\rm c}}$ are variances of $p_{\rm s}$, $p_{\rm c}$: $(\sigma_{p_{\rm c}})^2 := \overline{p_{\rm c}^2} - \overline{p_{\rm c}}^{\,2}$. Further common properties of $\hat{p}_{\rm nc}$ and $p_{\rm s}$ could be revieled on examples of some specific states only.

We however have to note here an important difference in the properties of $\hat{p}_{\rm nc}$ and $p_{\rm s}$: the linear correlation between $p_{\rm c}$ and $p_{\rm s}$ (i.e., the covariance $C_{p_{\rm s}p_{\rm c}}$ may not vanish, while the covariance $\Delta_{p_{\rm nc}p_{\rm c}}$ of $\hat{p}_{\rm nc}$ and $p_{\rm c}$ vanishes in all states [11]. $C_{p_{\rm s}p_{\rm c}}$ may vanish in some specific families of states only, e.g. in ρ corresponding to the cannonical CS. (It is not vanishing e.g. in squeezed states - see Section 5). When $C_{p_{\rm s}p_{\rm c}} \neq 0$ the total second moment $p \equiv p_{\rm c} + p_{\rm s}$ is not equal to that of $\hat{p}$. With nonvanishing $p_{\rm s}$-$p_{\rm c}$ covariance the Hall and Reginatto scheme of derivation of "quantum ensemble" equations (i.e., Bohm equations (4)), is not applicable. Therefore *if ensemble interpretation* is applied to our scheme (with $p_{\rm s}$) of derivation of Schrödinger equation, the resulting nonclassical ensemble could be called "*semi-quantum*" or, more briefly $p_{\rm s}$-ensemble. And if one interpretes $p_{\rm c}/m$ and $p_{\rm s}/m$ as current and osmotic velocities respectively then Nelson SM scheme is applicable.

4. R-S Type URs for Stochastic System

Inequalities of the type of Robertson-Schrödinger UR (R-S URs) can be naturally and easily constructed for classical stochastic systems using the semi-definiteness of the covariance matrix (the matrix of dispersions [8]) of two random quantities. Gnedenko [8] proved that all principal minors of the matrix of dispersions of any n random quantities are nonnegative. For $n = 2$ this means that the product of the two variances is greater or equal to their squared covariance. Thus for any two random observables ξ, η the following inequality is valid

$$\sigma_\xi^2\, \sigma_\eta^2 \geq C_{\xi\eta}^2 \tag{24}$$

where σ_ξ^2 is the variances of ξ, $\sigma_\xi^2 = \overline{\xi^2} - \overline{\xi}^{\,2}$, and $C_{\xi\eta}$ is the covariance, $C_{\xi\eta} = \overline{\xi\eta} - \overline{\xi}\,\overline{\eta}$. Here $\overline{\xi}$ is the mean value of ξ. If the random quantity ξ admits a probability density $\rho(\xi, t)$ one has $\overline{\xi} = \int \rho(\xi, t)\xi\, {\rm d}\xi$. The inequality (24) is minimized iff ξ and η are linearly dependent [8]. For brevity the stochastic quantity and its values are denoted with the same letter.

We see that the inequality (24) is of almost the same form as the R-S UR (2) in quantum mechanics, the mean commutator of the two observables being missing only. Therefore the inequalities of the form (24) in stochastic mechanics and in any probability theory could be naturally called the R-S type URs. For given two quantities ξ, η such inequality should briefly be referred to as ξ-η UR.

Next we construct and discuss the R-S type URs for the coordinate and momentums of the stochastic particle. In the 'semi-quantum ensemble' interpretation we have to treat $p_{\rm s} + p_{\rm c} \equiv p$ as total particle momentum and compare the x-p UR with $\hat{x}$-$\hat{p}$ UR in quantum mechanics. Similarly URs between any other pair of the set

$(x,\, p_{\rm c},\, p_{\rm s},\, p)$ is to be compared with UR of the corresponding quantum pair from $(\hat{x},\, \hat{p}_{\rm c},\, \hat{p}_{\rm nc},\, \hat{p})$. In the stochastic mechanics interpretation the set $(x,\, p_{\rm c},\, p_{\rm s},\, p)$ coinsides with $(x,\, mv,\, mu,\, mv_+)$, where $v,\, u,\, v_+$ are current, osmotic and forward velocities.

In stochastic mechanics x-mu UR (the osmotic UR) was established in [21] and [5] in the 'Heisenberg form' $(\sigma_x)^2(\sigma_{mu})^2 \geq \hbar^2/4$, which we rewrite as

$$(\sigma_x)^2(\sigma_{p_{\rm s}})^2 \geq \hbar^2/4. \tag{25}$$

In [16,17] the osmotic inequality was extended to the processes with non constant diffusion coefficient $\nu(x,t)$ in the form $(\sigma_x)^2(\sigma_u)^2 \geq \overline{\nu}^{\,2}$. Comparing (25) with (24) we see that the squared x-$p_{\rm s}$ covariance is universally constant and equaled to $\hbar^2/4$. The covariance itself is

$$C_{xp_{\rm s}} = -\hbar/2. \tag{26}$$

In Heisenberg UR in quantum mechanics the universal term $\hbar^2/4$ comes from the nonvanishing commutator of coordinate and momentum operators. We now see that in SM and in 'quantum ensemble' aproach this term comes from the x-$p_{\rm s}$ covariance. The constancy of the covariance $C_{xp_{\rm s}}$ is, in fact, due to the vanishing first moment of our $p_{\rm s}$. Due to this property the variance of $p_{\rm s}$ is proportional to the Fisher information (as required in [12] for the variance of their 'nonanalyzable' $\boldsymbol{p}_{\rm N}$), and the x-$p_{\rm s}$ UR (25) coincides with the known Cramer-Rao inequality $\sigma_x^2\, I_{\rm F} \geq 1$. For Gaussian $\rho(x,t)$ one has $I_{\rm F}(\rho) = 1/\sigma_x^2$ [20]. Therefore for Gauss distribution the UR (25) is minimized along with the Cramer-Rao inequality. The UR (25) is to be compared with the $\hat{x}$-$\hat{p}_{\rm nc}$ UR $(\Delta\hat{x})^2(\Delta\hat{p}_{\rm nc})^2 \geq \hbar^2/4$ [11] and with the chain inequalities (12).

Unlike $C_{xp_{\rm s}}$ the covariances of other pairs of the set $\{x, p_{\rm c}, p_{\rm s}, p\}$, though having to obey the R-S type URs (24), do not take universally fixed values. In the next section we shall discuss this on the (one-dimensional) examples, comparing the calculated moments with the corresponding ones in quantum mechanics.

5. Examples: Coherent States and Squeezed States

In these section we calculate the first and second moments of x, $p_{\rm s}$, $p_{\rm c}$ and $p = p_{\rm s}{+}p_{\rm c}$ and the related R-S types of URs in 'stochastic states' $\rho(x,t)$ corresponding to the Glauber CS and canonical SS in quantum mechanics, and compare them with the related quantum moments. Nelson stochastic mechanics (SM) images of CS and SS have been discussed previously in several papers: of CS in [10–18] and of SS and CS - in [16–18, 22] in the context of 'stochastic mechanics and control theory'. Here we write these images and the related moments and URs in more standard quantum optical and quantum mechanical parameters (see e.g. [2, 4, 26–28]).

a) **Glauber coherent states.** Glauber CS [7] are defined as eigenstates $|\alpha\rangle$ of the boson annihilation operator $\hat{a}$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \hat{a} = \frac{1}{\sqrt{2\hbar}}\left(\hat{x}\sqrt{m\omega} + \frac{\mathrm{i}}{\sqrt{m\omega}}\hat{p}\right) \tag{27}$$

where $\hat{x}$ and $\hat{p}$ are coordinate and momentum operators, and m and ω are parameters of dimension of mass and frequency correspondingly. For the harmonic oscillator m is the mass of the particle, and ω is the oscillator frequency. These CS have been introduced by Glauber in 1963 [7] and are known as the *most classical* quantum states. In $|\alpha\rangle$ the first and second moments of $\hat{x}$ and $\hat{p}$ read

$$\langle\alpha|\hat{x}|\alpha\rangle = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\alpha, \qquad \langle\alpha|\hat{p}|\alpha\rangle = \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\alpha \tag{28}$$

$$(\Delta\hat{x})^2 = \frac{l^2}{2}, \qquad (\Delta\hat{p})^2 = \frac{\hbar^2}{2l^2}, \qquad \Delta_{xp} = 0. \tag{29}$$

where $l^2 = \hbar/m\omega$ (the length parameter). We see that the moments minimize R-S UR (3) on the lowest possible level (which is the equality in the Heisenberg UR): $(\Delta\hat{x})^2(\Delta\hat{p})^2 = \hbar/4$.

To perform the comparison with the moments in 'semiclassical ensemble' and in SM we need the time-dependent CS, i.e., eigenstates of $\hat{a}$ that obey the Schrödinger equation. The first requirement can be met if the CS wave function depends, up to a x-independent phase factor, on t through the eigenvalue α: $\psi_\alpha(x,t) = \exp(\mathrm{i}\varphi(t))\psi_{\alpha(t)}(x)$, $\hat{a}\psi_\alpha(x,t) = \alpha(t)\psi_\alpha(x,t)$. Such *stable* CS $\psi_\alpha(x,t)$ exist for the stationary harmonic oscillator Hamiltonian, $\hat{H} = -(\hbar^2/2m)\partial_{xx} + (m\omega^2/2)x^2$ with $\alpha(t) = \alpha\exp(-\mathrm{i}\omega t)$, $\varphi(t) = -\omega t/2$ and

$$\psi_{\alpha(t)}(x) = \left(\frac{1}{\pi l^2}\right)^{\frac{1}{4}}\exp\!\left[-\frac{1}{2}\left(\frac{x}{l} - \alpha(t)\sqrt{2}\right)^2 + \frac{1}{2}\left(\alpha^2(t) - |\alpha(t)|^2\right)\right] \tag{30}$$

where $l = \sqrt{\hbar/m\omega}$ (the length parameter). For the stable CS $\psi_{\alpha(t)}(x)$ the first and the second moments of $\hat{x}$ and $\hat{p}$ are given by the same formulas (28), (29) but with the time-dependent eigenvalue $\alpha(t)$.

Next we put $|\psi_{\alpha(t)}(x)|^2 = \rho_{cs}(x,t)$ and calculate the stochastic moments of x and $p_{\rm s}$. Formula in (30) readily shows that (we put $\alpha_1 = \mathrm{Re}\,\alpha,\ \ \alpha_2 = \mathrm{Im}\,\alpha$)

$$\overline{x} = l\sqrt{2}\alpha_2(t) \quad \text{and} \quad \sigma_x^2 = \frac{l^2}{2} \tag{31}$$

and provides $p_{\rm s} = -\hbar\,(x - \overline{x})\,/l^2$, wherefrom

$$\overline{p}_{\rm s} = 0, \quad \text{and} \quad \sigma_{p_{\rm s}}^2 = \frac{\hbar^2}{2l^2}\cdot \tag{32}$$

We see that x-$p_{\rm s}$ UR (25) is minimized [16]: $\sigma_x^2\sigma_{p_{\rm s}}^2 = \hbar^2/4$.

To find the first and second moments of $p_{\rm c}$ and the 'total momentum' $p = p_{\rm c} + p_{\rm s}$ we need the action $S(x,t)$,

$$S(x,t) = \frac{\hbar}{l}\alpha_2(t)\sqrt{2}\,x - \hbar\alpha_1(t)\alpha_2(t) - \hbar\frac{\omega t}{2}\cdot \tag{33}$$

Then we get $p_{\rm c} = \hbar\sqrt{2}\alpha_2(t)$ and

$$\overline{p} = \overline{p_{\rm c}} = \int p_{\rm c}\,\rho_{cs}(x,t)\,\mathrm{d}x = \frac{\hbar}{l}\sqrt{2}\alpha_2(t) \tag{34}$$

verifying the known coincidence of $\overline{x}$ and $\overline{p}$ with quantum means $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ [5, 16].

Next we calculate the second moments of $p_{\rm c}$ and p and the related covariances. The covariance $C_{xp_{\rm s}}$, as noted in the previous section, is universally equal to $-\hbar/2$. The correlation between $p_{\rm c}$ and $p_{\rm s}$ in $\rho_{\rm cs}$ turned out to be vanishing:

$$C_{p_{\rm c}p_{\rm s}} = \overline{p_{\rm c}p_{\rm s}} = 0. \tag{35}$$

Thus the required in [12] properties (10) of the 'nonanalyzable' momentum $\boldsymbol{p}_{\rm N}$ are satisfied by $p_{\rm s}$ in $\rho_{\rm cs}$ and the first two moments of $p_{\rm s}$ and $\hat{p}_{\rm nc}$ and that of p and $\hat{p}$ do coincide. The third and higher moments of $p_{\rm s}$ and $\hat{p}_{\rm nc}$, however are found to coinside at all times in the ground state only. For example $\langle p_{\rm s}^3\rangle = 0$, while $\langle\hat{p}_{\rm nc}^3\rangle = \langle\hat{p}\rangle^3 + \langle\hat{p}\rangle/2 - \langle\hat{p}\rangle^2$, $\langle\hat{p}\rangle = \sqrt{2}\mathrm{Im}\,\alpha(t)$.

For the rest two variances and covariances in $\rho_{\rm cs}$ we get

$$\begin{aligned} C_{xp_{\rm c}} &= 0, \qquad C_{xp} = C_{xp_{\rm c}} + C_{xp_{\rm s}} = -\hbar/2 \\ \sigma_{p_{\rm c}} &= 0, \qquad \sigma_p^2 = \sigma_{p_{\rm s}}^2 = \frac{\hbar^2}{2l^2}\cdot \end{aligned} \tag{36}$$

Note the vanishing variance of the momentum $p_{\rm c} := \partial S/\partial x$ in $\rho_{\rm cs}$. As we shall see below this is again a particular property of $\rho_{\rm cs}$.

Now one can easily check that the R-S type URs (11) for all the coordinate-momentum pairs x-$p_{\rm c}$, x-$p_{\rm s}$ and x-p are minimized in $\rho_{\rm cs}$. In particular the chain inequalities [5, 21]

$$(\Delta\hat{x})^2(\Delta\hat{p})^2 \geq \sigma_x^2\sigma_{p_{\rm s}}^2 \geq \hbar^2/4 \tag{37}$$

are also minimized. These minimizations follow from the fact that in $\rho_{\rm cs}$ all quantities x, $p_{\rm s}$, $p_{\rm c}$ are linearly dependent [8]. We note that in terms of the SM velocities $u = p_{\rm s}/m$, $v = {\rm grad}S/m$ all the above CS-related moments and URs (25), (37) were considered previously [5, 21] [16, 18].

In quantum mechanics CS $|\alpha\rangle$ are regarded as the 'most classical' states. They can be uniquely determined as states minimizing the inequality [26]

$$(\Delta\tilde{\hat{x}})^2 + (\Delta\tilde{\hat{p}})^2 \geq 1 \tag{38}$$

where $\tilde{x}$ and $\tilde{\hat{p}}$ are dimensionless coordinate and momentum. One can see that in $\rho_{\rm cs}$ the sum $\sigma_{\tilde{x}}^2 + \sigma_{\tilde{p}}^2$ also equals unity. However in other states the inequality $\sigma_{\tilde{x}}^2 + \sigma_{\tilde{p}}^2 \geq 1$ may be violated, as we shall see on the example of squeezed states.

b) **Squeezed States**. Squeezed states (SS) are defined as quantum states in which the variance (uncertainty) of coordinate or the variance of the momentum is less than its value in the ground state of the oscillator. The SS are known as *nonclassical states* since they exhibit many nonclassical properties. The famous example of SS are the eigenstates of the linear combination of Bose creation and annihilation operators $\tilde{u}\hat{a} + \tilde{v}\hat{a}^\dagger \equiv \hat{A}$ [28], which we rewrite in terms of $\hat{x}$ and $\hat{p}$ as $\mu\hat{x}/l + {\rm i}\nu l\hat{p}/\hbar$ ($\mu = (\tilde{u}+\tilde{v})/\sqrt{2},\ \nu = (\tilde{u}-\tilde{v})/\sqrt{2}$),

$$(\mu\hat{x}/l + {\rm i}\nu l\hat{p}/\hbar)|\alpha;\mu,\nu\rangle = \alpha|\alpha;\mu,\nu\rangle \tag{39}$$

where α is a complex number, l is the length parameter, and

$$|u|^2 - |v|^2 = 2{\rm Re}(\mu^*\nu) = 1 \tag{40}$$

which are to ensure $[\hat{A}, \hat{A}^\dagger] = 1$. It was noted in [27] that SS $|\alpha;\mu,\nu\rangle$ are states that minimize the R-S UR and coincide with the 'correlated CS' of ref. [2]. That is why they are also called generalized intelligent states or R-S intelligent states [27].

In the coordinate representation the SS wave functions take the form of exponential of a quadratic. These states are time-stable for any quadratic in $\hat{x}$ and $\hat{p}$ Hamiltonian, in particular for the harmonic oscillator with constant or time-dependent frequency $\omega(t)$. The normalized time-dependent wave function of an initial SS

$|\alpha;\mu_0,\nu_0\rangle$ reads

$$\begin{aligned}\psi_{\alpha\mu_0\nu_0}(x,t) &= \left(l\nu(t)\sqrt{2\pi}\right)^{-\frac{1}{2}}\exp\left[-\frac{\mu(t)}{2l^2\nu(t)}\left(x-\frac{l}{\mu(t)}\alpha\right)^2\right]\\ &\quad\times\exp\left[-\frac{1}{2}\left(|\alpha|^2-\frac{\mu^*(t)}{\mu(t)}\alpha^2\right)\right]\end{aligned}\tag{41}$$

where α is constant, and $\mu(t)$, $\nu(t)$ ($\mu(0)=\mu_0$, $\nu(0)=\nu_0$) satisfy certain first order equations, which can be reduced to the classical harmonic oscillator equation $\ddot{\varepsilon}+\omega(t)^2\varepsilon=0$ through the substitutions $\mu(t)=-\mathrm{i}\dot{\varepsilon}/\sqrt{2\omega_0}$, $\nu(t)=\varepsilon\sqrt{\omega_0/2}$, ω_0 being constant of inverse time dimension [3,26]. With such $\mu(t)$, $\nu(t)$ the operator $\hat{A}$ is a dynamical invariant of the nonstationary oscillator, i.e., $d\hat{A}/dt=0$. The family of stable SS includes the family of CS as a subset: If $\varepsilon(0)=1/\sqrt{\omega_0}$ and $\dot{\varepsilon}(0)=\mathrm{i}\sqrt{\omega_0}$ (that is $\mu_0\equiv\mu(0)=1/\sqrt{2}$, $\nu_0\equiv\nu(0)=1/\sqrt{2}$) then the wave function (41) represents the time-evolution of an initial Glauber CS $|\alpha\rangle$. In fact, in terms of ε, $\dot{\varepsilon}$ the wave functions (41) have been constructed and discussed earlier in [15] as time evolved CS for quadratic systems.

The first and the second moments of $\hat{x}$ and $\hat{p}$ in SS (41) read [3,26]

$$\langle\hat{x}\rangle=2l\mathrm{Re}(\alpha(t)\nu^*(t)),\qquad\langle\hat{p}\rangle=2\frac{\hbar}{l}\mathrm{Im}(\alpha(t)\mu^*(t))\tag{42}$$

$$(\Delta\hat{x})^2=l^2|\nu(t)|^2,\ \ (\Delta\hat{p})^2=\frac{\hbar^2}{l^2}|\mu(t)|^2,\ \ \Delta_{xp}=\hbar\mathrm{Im}(\mu^*(t)\nu(t))\tag{43}$$

the second moments saturating the R-S UR (3).

To calculate the stochastic moments in $\rho_{ss}=|\psi_{\alpha\mu_0\nu_0}(x,t)|^2$ we have to find the momentum potentials S and $S_\mathrm{s}=(\hbar/2)\ln\rho_{ss}$ (furthermore we skip the argument t of $\alpha(t)$, $\mu(t)$ and $\nu(t)$)

$$S(x,t)=-\frac{\hbar}{2l^2}\mathrm{Im}\left(\frac{\mu}{\nu}\right)x^2+\frac{\hbar}{l}\mathrm{Im}\left(\frac{\alpha}{\nu}\right)x+g_1(t)\tag{44}$$

$$\rho_{ss}(x,t)=\frac{1}{|\nu|l\sqrt{2\pi}}\exp\left[-\frac{1}{2l^2|\nu|^2}\left[x-2l\mathrm{Re}(\alpha\nu^*)\right]^2+g_2(t)\right]\tag{45}$$

where the terms $g_1(t)$, $g_2(t)$ are x-independent,

$$g_1(t)\ =\ -\frac{1}{2}\mathrm{Im}\left(\frac{\alpha^2}{\mu\nu}\right)-\frac{1}{2}\arg(\nu)+\frac{1}{2}\mathrm{Im}\left(\frac{\alpha^2\mu^*}{\mu}\right)$$

$$g_2(t) \;=\; \frac{2}{|\nu|^2}\mathrm{Re}^2(\alpha\nu^*) + \mathrm{Re}\left(\alpha^2/\mu\nu\right) + \mathrm{Re}\left(\mu^*\alpha^2/\mu\right) - |\alpha|^2 \,.$$

The first moments of x and p in ρ_{ss} coincide with the quantum means $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, equations (42). For the second moments of x, p_c, p_s, $p = p_\mathrm{s} + p_\mathrm{c}$ we find

$$\begin{aligned} &C_{xp_\mathrm{c}} = -\hbar\mathrm{Im}(\mu\nu^*), \qquad C_{p_\mathrm{s}p_\mathrm{c}} = \frac{\hbar^2}{2l^2|\nu|^2}\mathrm{Im}(\mu\nu^*) \\ &C_{xp} = -\hbar\left(\frac{1}{2} + \mathrm{Im}(\mu\nu^*)\right) \end{aligned} \tag{46}$$

$$\begin{aligned} &\sigma_x^2 = l^2|\nu|^2, \qquad \sigma_p^2 = \frac{\hbar^2}{l^2|\nu|^2}\left(\frac{1}{2} + \mathrm{Im}(\mu\nu^*)\right)^2 \\ &\sigma_{p_\mathrm{c}}^2 = \frac{\hbar^2}{l^2}|\mu|^2\sin^2(\delta\varphi), \qquad \sigma_{p_\mathrm{s}}^2 = \frac{\hbar^2}{4l^2|\nu|^2} \end{aligned} \tag{47}$$

where $\delta\varphi = \arg\mu - \arg\nu$. From (46) and (47) it follows that the R-S URs for *all pairs* of observables x, p_c, p_s, p *are minimized* in ρ_{ss},

$$\sigma_x^2\sigma_{p_\mathrm{c}}^2 = C_{xp_\mathrm{c}}^2 = \hbar^2\mathrm{Im}^2(\mu\nu^*),\;\; \sigma_x^2\sigma_p^2 = C_{xp}^2 = \hbar^2\left(\frac{1}{2} + \mathrm{Im}(\mu\nu^*)\right)^2 \tag{48}$$

$$\sigma_x^2\sigma_{p_\mathrm{s}}^2 = C_{xp_\mathrm{s}}^2 = \frac{\hbar^2}{4}, \qquad \sigma_{p_\mathrm{c}}^2\sigma_{p_\mathrm{s}}^2 = C_{p_\mathrm{c}p_\mathrm{s}}^2 = \frac{\hbar^4}{4l^4|\nu|^4}\mathrm{Im}^2(\mu\nu^*), \tag{49}$$

as expected due to the linearity of p_c, p_s, p in terms of x.

In ρ_{ss} however, unlike the case of ρ_{cs}, the dimensionless variances of x and momentum p (or p_s) are no more equal and none of the stochastic momentum uncertainties coincides identically with the quantum uncertainty $\Delta\hat{p}$. These second moment's differences could be interpreted as due to the *'nonclassicality'* of the SS. The calculations shows that the variance of 'semi-quantum ensembe' moment $p = p_\mathrm{s} + p_\mathrm{c}$ can be greater or less than $(\Delta\hat{p})^2$. The ratio

$$r_p = \left[(\Delta\hat{p})^2 - \sigma_p^2\right]/(\Delta\hat{p})^2$$

could be used to described the deviation of momentum fluctuations in p_s-ensemble state ρ_ψ from quantum fluctuations in ψ. For SS $\psi_{\alpha\mu\nu}$ it takes the form

$$r_p = -\frac{\mathrm{Im}(\mu\nu^*)}{|\mu\nu|^2} \tag{50}$$

and its value is oscillating between ± 1. It shows that the two variances coinside in states with μ and ν phase difference equal to $2n\pi$. Due to the nonvanishing covariance $C_{p_\mathrm{c}\,p_\mathrm{s}}$ the variance of p may vanish for certain values of μ, ν. In such states the variances σ_x^2 and σ_p^2 could not preserve the inequalities (1) and (38).

Conclusion

It has been shown that the Bohm equations, which are equivalent to the Schrödinger equations, Hamilton-Jacobi equation admiting an additional particle momentum $p_{\rm s}$ of the form of stochastic mechanics osmotic momentum and using the change of variables. The variational functional is similar that of Reginatto and Hall [11, 23] which incorporates Frieden [6] information. The fluctuations of $p_{\rm s}$, classical momentum $\partial S/\partial x \equiv p_{\rm c}$ and the 'total particle momentum' $p = p_{\rm s} + p_{\rm c}$ and the uncertainty relations (URs) are examined and compared corresponding quantum ones on the example of canonical coherent and squeezed states variances) of p and quantum $\hat{p}$ and the related coincide, while in SS they reveal differences. The latter are due to the nonvanishing deviation of variance of p from that of $\hat{p}$ bounded between ± 1. Thus in the ensemble interpretaion, our '$p_{\rm s}$-ensemble' can only approximately and partially reproduce statistical properties of 'quantum ensemble'. The correspondence with the Nelson stochastic mechanics is obtained via the identification of $p_{\rm s}/m$, $p_{\rm c}/m$ and p/m with the osmotic, current and forward particle velocities.

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