

**REMARK ON THE INTEGRALS OF MOTION ASSOCIATED WITH
LEVEL k REALIZATION OF THE ELLIPTIC ALGEBRA $U_{q,p}(\widehat{\mathfrak{sl}}_2)$**

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Abstract. We give one parameter deformation of level k free field realization of the screening current of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. By means of these free field realizations, we construct infinitely many commutative operators, which are called the nonlocal integrals of motion associated with the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ for level k . They are given as integrals involving a product of the screening current and elliptic theta functions. This paper give level k generalization of the nonlocal integrals of motion given in [1].

1. Introduction

One of the results in Bazhanov, Lukyanov and Zamolodchikov [4] is construction of field theoretical analogue of the commuting transfer matrix $\mathbf{T}(z)$, acting on the highest weight representation of the Virasoro algebra. Their commuting transfer matrix $\mathbf{T}(z)$ is the trace of the image of the universal R -matrix associated with the quantum affine symmetry $U_q(\widehat{\mathfrak{sl}}_2)$. This construction is very simple and the commutativity $[\mathbf{T}(z), \mathbf{T}(w)] = 0$ is direct consequence of the Yang-Baxter equation. They call the coefficients of the Taylor expansion of $\mathbf{T}(z)$ the nonlocal integrals of motion. The higher-rank generalization of [4] is considered in [5, 6]. The elliptic deformation of the nonlocal integrals of motion is considered in [1]. Bazhanov, Lukyanov and Zamolodchikov [4] constructed the continuous transfer matrix $\mathbf{T}(z)$ by taking the trace of the image of the universal R -matrix associated with $U_q(\widehat{\mathfrak{sl}}_2)$. However, it is not so easy to calculate the image of the elliptic version of the universal R -matrix, which is obtained by using the twister [10]. Hence the construction method of the elliptic version [1] should be completely different from those in [4]. Instead of considering the transfer matrix $\mathbf{T}(z)$, the authors [1] give the integral representation of the integrals of motion directly. The commutativity of the integrals of motion is not consequence of the Yang-Baxter equation. It is consequence of the commutative subalgebra of the Feigin-Odesskii

algebra [11]. The higher-rank generalization of [1] is considered in [2, 3]. This paper is a continuation of [1–3]. This paper give level k generalization of the nonlocal integrals of motion given in [1].

The organization of this paper is as following. In Section 2 we give one parameter “ s ” deformation of the level k free field realization of the screening current of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. In Section 3 we construct infinitely many commutative operators, which are called the nonlocal integrals of motion associated with the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ for level k . In Section 3 we state main theorem and give conjecture. In appendix we summarize the normal ordering of basic operators.

2. Elliptic Current

In this section we give one parameter “ s ” deformation of the level k free field realization of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. We fix complex numbers x, r, r^*, s , ($|x| < 1, \operatorname{Re}(r), \operatorname{Re}(r^*) > 0, s \neq 2$), and $k = r - r^* \neq 0, -2$. We use symbols

$$[n] = \frac{x^n - x^{-n}}{x - x^{-1}}, \quad [n]_+ = x^n + x^{-n}.$$

We set the parameters τ, τ^*

$$x = e^{-\pi\sqrt{-1}/r\tau} = e^{-\pi\sqrt{-1}/r^*\tau^*}. \quad (1)$$

Let us use parameterization $z = x^{2u}$. The symbol $[u]_r$ stands for the Jacobi elliptic theta function

$$[u]_r = x^{\frac{u^2}{r}-u} \Theta_{x^{2r}}(z), \quad [u]_{r^*} = x^{\frac{u^2}{r^*}-u} \Theta_{x^{2r^*}}(z) \quad (2)$$

where we have used

$$\Theta_p(z) = (z; p)_\infty (p/z; p)_\infty (p; p)_\infty, \quad (z; p)_\infty = \prod_{n=0}^{\infty} (1 - p^n z). \quad (3)$$

The theta function $[u]_r$ enjoys the quasi-periodicity property

$$[u + r]_r = -[u]_r, \quad [u + r\tau]_r = -e^{-\pi\sqrt{-1}\tau - \frac{2\pi\sqrt{-1}}{r}u} [u]_r. \quad (4)$$

2.1. Bosons

We set the bosons $\alpha_m^j, \tilde{\alpha}_m^j$ $j = 1, 2, m \in \mathbb{Z}_{\neq 0}$

$$[\alpha_m^j, \alpha_n^j] = -\frac{1}{m} \frac{[2m][rm]}{[km][(r-k)m]} \delta_{m+n,0}, \quad j = 1, 2$$

$$[\alpha_m^1, \alpha_n^2] = \frac{1}{m} \left(\frac{x^{(-r+k)m}([sm] - [(s-2)m])}{[(r-k)m]} + \frac{x^{km}([sm] + [(s-2)m])}{[km]} \right) \times \delta_{m+n,0} \quad (5)$$

$$[\tilde{\alpha}_m^j, \tilde{\alpha}_n^j] = -\frac{1}{m} \frac{[2m][r-k]m}{[km][rm]} \delta_{m+n,0}, \quad j = 1, 2$$

$$[\tilde{\alpha}_m^1, \tilde{\alpha}_n^2] = \frac{1}{m} \left(\frac{x^{rm}(-[sm] + [(s-2)m])}{[rm]} + \frac{x^{km}([sm] + [(s-2)m])}{[km]} \right) \times \delta_{m+n,0} \quad (6)$$

$$[\alpha_m^j, \tilde{\alpha}_n^j] = -\frac{1}{m} \frac{[2m]}{[km]} \delta_{m+n,0}, \quad j = 1, 2$$

$$[\alpha_m^1, \tilde{\alpha}_n^2] = \frac{1}{m} \frac{[sm] + [(s-2)m]}{[km]} \delta_{m+n,0} \quad (7)$$

$$[\tilde{\alpha}_m^1, \alpha_n^2] = \frac{1}{m} \frac{[sm] + [(s-2)m]}{[km]} \delta_{m+n,0}.$$

We set the bosons $\beta_m^j, \gamma_m^j, j = 1, 2, m \in \mathbb{Z}_{\neq 0}$

$$[\beta_m^j, \beta_n^j] = \frac{[2m][(k+2)m]}{m} \delta_{m+n,0}, \quad j = 1, 2$$

$$[\beta_m^1, \beta_n^2] = -\frac{[(k+2)m]([sm] + [(s-2)m])}{m} \delta_{m+n,0}$$

$$[\gamma_m^j, \gamma_n^j] = \frac{1}{m} \frac{[2m]}{[km]} \delta_{m+n,0}, \quad j = 1, 2 \quad (8)$$

$$[\gamma_m^1, \gamma_n^2] = -\frac{1}{m} \frac{[sm] + [(s-2)m]}{[km]} \delta_{m+n,0}.$$

We set the zero-mode operators P_0, Q_0, h, α and $h_0, h_1, h_2, \alpha_0, \alpha_1, \alpha_2$

$$[P_0, iQ_0] = 1, \quad [h, \alpha] = 2 \quad (9)$$

$$[h_0, \alpha_0] = [h_1, \alpha_2] = [h_2, \alpha_1] = (2-s), \quad [h_1, \alpha_1] = [h_2, \alpha_2] = 0.$$

We set the Fock space $\mathcal{F}_{K,L}, K, L \in \mathbb{Z}$

$$\mathcal{F}_{K,L} = \bigoplus_{n,n_0,n_1,n_2 \in \mathbb{Z}} \mathbb{C}[\alpha_{-m}^j, \tilde{\alpha}_{-m}^j, \beta_{-m}^j, \gamma_{-m}^j, (j = 1, 2; m \in \mathbb{N}_{\neq 0})] \otimes |K, L\rangle_{n,n_0,n_1,n_2} \quad (10)$$

where

$$|K, L\rangle_{n, n_0, n_1, n_2} = \exp \left(L \sqrt{\frac{2r}{r-k}} - K \sqrt{\frac{2(r-k)}{r}} \right) iQ \quad (11)$$

$$\otimes e^{n\alpha} \otimes e^{n_0\alpha_0} \otimes e^{n_1\alpha_1} \otimes e^{n_2\alpha_2}.$$

Upon specialization $s \rightarrow 2$ simplification occurs

$$\alpha_m^2 = -\alpha_m^1, \quad \tilde{\alpha}_m^1 = \frac{[(r-k)m]}{[rm]} \alpha_m^1, \quad \tilde{\alpha}_m^2 = -\frac{[(r-k)m]}{[rm]} \alpha_m^1 \quad (12)$$

$$\beta_m^2 = -\beta_m^1, \quad \gamma_m^2 = -\gamma_m^1, \quad h_0 = h_1 = h_2 = \alpha_0 = \alpha_1 = \alpha_2 = 0.$$

The bosons $\alpha_m^1, \beta_m^1, \gamma_m^1$ are the same bosons which were introduced to construct the elliptic current associated with the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the deformed Virasoro algebra $\text{Vir}_{q,t}$ [7–9]. In order to construct infinitely many commutative operators, we introduce one parameter s deformation of the bosons in [7–9]. This additional parameter s plays an important role in proof of the main theorem.

We introduce the operators $C_j(z), C_j^\dagger(z), j = 1, 2$, acting on the Fock space $\mathcal{F}_{J,K}$

$$C_1(z) = e^{-\sqrt{\frac{2r}{k(r-k)}} iQ_0} e^{-\sqrt{\frac{2r}{k(r-k)}} P_0 \log z} : \exp \left(- \sum_{m \neq 0} \alpha_m^1 z^{-m} \right) :$$

$$C_2(z) = e^{\sqrt{\frac{2r}{k(r-k)}} iQ_0} e^{\sqrt{\frac{2r}{k(r-k)}} P_0 \log z} : \exp \left(- \sum_{m \neq 0} \alpha_m^2 z^{-m} \right) :$$

$$C_1^\dagger(z) = e^{\sqrt{\frac{2(r-k)}{kr}} iQ_0} e^{\sqrt{\frac{2(r-k)}{kr}} P_0 \log z} : \exp \left(\sum_{m \neq 0} \tilde{\alpha}_m^1 z^{-m} \right) :$$

$$C_2^\dagger(z) = e^{-\sqrt{\frac{2(r-k)}{kr}} iQ_0} e^{-\sqrt{\frac{2(r-k)}{kr}} P_0 \log z} : \exp \left(\sum_{m \neq 0} \tilde{\alpha}_m^2 z^{-m} \right) : . \quad (13)$$

Here $: * :$ represents normal ordering. We set the operators $\tilde{\Psi}_{j,I}(z), \tilde{\Psi}_{j,II}(z), \tilde{\Psi}_{j,I}^\dagger(z), \tilde{\Psi}_{j,II}^\dagger(z), j = 1, 2$, acting on the Fock space $\mathcal{F}_{J,K}$

$$\tilde{\Psi}_{j,I}(z) = \exp \left(-(x - x^{-1}) \sum_{m>0} \frac{x^{\frac{km}{2}}}{[m]_+} \beta_m^j z^{-m} \right)$$

$$\times \exp \left(- \sum_{m>0} x^{-\frac{km}{2}} \gamma_{-m}^j z^m \right) \exp \left(- \sum_{m>0} x^{\frac{km}{2}} \frac{[(k+1)m]_+}{[m]_+} \gamma_m^j z^{-m} \right)$$

$$\begin{aligned}
\tilde{\Psi}_{j,II}(z) &= \exp\left((x-x^{-1})\sum_{m>0}\frac{x^{\frac{km}{2}}}{[m]_+}\beta_{-m}^j z^m\right) \\
&\quad \times \exp\left(-\sum_{m>0}x^{\frac{km}{2}}\frac{[(k+1)m]_+}{[m]_+}\gamma_{-m}^j z^m\right) \exp\left(-\sum_{m>0}x^{-\frac{km}{2}}\gamma_m^j z^{-m}\right) \\
\tilde{\Psi}_{j,I}^\dagger(z) &= \exp\left((x-x^{-1})\sum_{m>0}\frac{x^{-\frac{km}{2}}}{[m]_+}\beta_m^j z^{-m}\right) \\
&\quad \times \exp\left(\sum_{m>0}x^{\frac{km}{2}}\gamma_{-m}^j z^m\right) \exp\left(\sum_{m>0}x^{-\frac{km}{2}}\frac{[(k+1)m]_+}{[m]_+}\gamma_m^j z^{-m}\right) \\
\tilde{\Psi}_{j,II}^\dagger(z) &= \exp\left(-(x-x^{-1})\sum_{m>0}\frac{x^{-\frac{km}{2}}}{[m]_+}\beta_{-m}^j z^m\right) \\
&\quad \times \exp\left(\sum_{m>0}x^{-\frac{km}{2}}\frac{[(k+1)m]_+}{[m]_+}\gamma_{-m}^j z^m\right) \exp\left(\sum_{m>0}x^{\frac{km}{2}}\gamma_m^j z^{-m}\right).
\end{aligned} \tag{14}$$

We set the operators $\Psi_{j,I}(z)$, $\Psi_{j,II}(z)$, $\Psi_{j,I}^\dagger(z)$, $\Psi_{j,II}^\dagger(z)$, $j = 1, 2$, acting on the Fock space $\mathcal{F}_{J,K}$

$$\begin{aligned}
\Psi_{1,I}(z) &= \tilde{\Psi}_{1,I}(z)e^{\alpha+\alpha_0+\alpha_1}x^{\frac{h}{2}+h_0+h_1}z^{-\frac{h}{k}} \\
\Psi_{1,II}(z) &= \tilde{\Psi}_{1,II}(z)e^{\alpha+\alpha_0+\alpha_1}x^{-\frac{h}{2}+h_0-h_1}z^{-\frac{h}{k}} \\
\Psi_{2,I}(z) &= \tilde{\Psi}_{2,I}(z)e^{-\alpha-\alpha_0+\alpha_2}x^{-\frac{h}{2}+h_0+h_2}z^{\frac{h}{k}} \\
\Psi_{2,II}(z) &= \tilde{\Psi}_{2,II}(z)e^{-\alpha-\alpha_0+\alpha_2}x^{\frac{h}{2}+h_0-h_2}z^{\frac{h}{k}} \\
\Psi_{1,I}^\dagger(z) &= \tilde{\Psi}_{1,I}^\dagger(z)e^{-\alpha-\alpha_0+\alpha_1}x^{\frac{h}{2}-h_0-h_1}z^{\frac{h}{k}} \\
\Psi_{1,II}^\dagger(z) &= \tilde{\Psi}_{1,II}^\dagger(z)e^{-\alpha-\alpha_0+\alpha_1}x^{-\frac{h}{2}-h_0+h_1}z^{\frac{h}{k}} \\
\Psi_{2,I}^\dagger(z) &= \tilde{\Psi}_{2,I}^\dagger(z)e^{\alpha+\alpha_0+\alpha_2}x^{-\frac{h}{2}-h_0-h_2}z^{-\frac{h}{k}} \\
\Psi_{2,II}^\dagger(z) &= \tilde{\Psi}_{2,II}^\dagger(z)e^{\alpha+\alpha_0+\alpha_2}x^{\frac{h}{2}-h_0+h_2}z^{-\frac{h}{k}}.
\end{aligned} \tag{15}$$

Definition 1. We set the operators $E_j(z)$, $F_j(z)$, $j = 1, 2$, which can be regarded as one parameter deformation of the level k elliptic currents associated with the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [7, 9]

$$E_j(z) = C_j(z)\Psi_j(z), \quad F_j(z) = C_j^\dagger(z)\Psi_j^\dagger(z), \quad j = 1, 2 \tag{16}$$

where we have set

$$\begin{aligned}\Psi_j(z) &= \frac{1}{x-x^{-1}}(\Psi_{j,I}(z) - \Psi_{j,II}(z)) \\ \Psi_j^\dagger(z) &= \frac{-1}{x-x^{-1}}(\Psi_{j,I}^\dagger(z) - \Psi_{j,II}^\dagger(z)), \quad j = 1, 2.\end{aligned}\tag{17}$$

We have following as direct consequence of the normal orderings of the basic operators summarized in appendix.

Theorem 1. *The elliptic currents $E_j(z)$, $j = 1, 2$, satisfy the following commutation relations*

$$\begin{aligned}[u_1 - u_2]_{r-k}[u_1 - u_2 - 1]_{r-k}E_j(z_1)E_j(z_2) \\ = [u_2 - u_1]_{r-k}[u_2 - u_1 - 1]_{r-k}E_j(z_2)E_j(z_1), \quad j = 1, 2 \\ \left[u_1 - u_2 + \frac{s}{2}\right]_{r-k} \left[u_1 - u_2 - \frac{s}{2} + 1\right]_{r-k} E_1(z_1)E_2(z_2) \\ = \left[u_2 - u_1 + \frac{s}{2}\right]_{r-k} \left[u_2 - u_1 - \frac{s}{2} + 1\right]_{r-k} E_2(z_2)E_1(z_1).\end{aligned}\tag{18}$$

The elliptic currents $F_j(z)$, $j = 1, 2$, satisfy the following commutation relations

$$\begin{aligned}[u_1 - u_2]_r[u_1 - u_2 + 1]_rF_j(z_1)F_j(z_2) \\ = [u_2 - u_1]_r[u_2 - u_1 + 1]_rF_j(z_2)F_j(z_1), \quad j = 1, 2 \\ \left[u_1 - u_2 - \frac{s}{2}\right]_r \left[u_1 - u_2 + \frac{s}{2} - 1\right]_r F_1(z_1)F_2(z_2) \\ = \left[u_2 - u_1 - \frac{s}{2}\right]_r \left[u_2 - u_1 + \frac{s}{2} - 1\right]_r F_2(z_2)F_1(z_1).\end{aligned}\tag{19}$$

The currents $E_j(z)$ and $F_j(z)$ satisfy

$$\begin{aligned}[E_j(z_1), F_j(z_2)] = \frac{x^{(-1)^j(s-2)}}{x-x^{-1}} \left(: C_j(z_1)C_j^\dagger(z_2)\Psi_{j,I}(z_1)\Psi_{j,I}^\dagger(z_2) : \delta\left(\frac{x^k z_2}{z_1}\right) \right. \\ \left. - : C_j(z_1)C_j^\dagger(z_2)\Psi_{j,II}(z_1)\Psi_{j,II}^\dagger(z_2) : \delta\left(\frac{x^{-k} z_2}{z_1}\right) \right), \quad j = 1, 2.\end{aligned}\tag{20}$$

Here we have used the delta-function $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$.

Upon specialization $s = 2$ the currents $E_1(z)$, $F_1(z)$ degenerate to elliptic currents in [9]. We set $E_j^{DV}(z) = E_j(z)|_{s=2}$, $F_j^{DV}(z) = F_j(z)|_{s=2}$, $j = 1, 2$.

3. Integrals of Motion

In this section we construct infinitely many commutative operators \mathcal{G}_m^* , \mathcal{G}_m , $m \in \mathbb{N}$, which we call the nonlocal integrals of motion for level k .

3.1. Nonlocal Integrals of Motion

Let us set the theta function $\vartheta_\alpha^*(u)$, $\vartheta_\alpha(u)$, $\alpha \in \mathbb{C}$, by

$$\begin{aligned} \vartheta^*(u+1) &= \vartheta^*(u) \\ \vartheta^*(u+r^*\tau^*) &= \exp \left[-2\pi\sqrt{-1}\tau^* - \frac{2\pi\sqrt{-1}}{r^*} \left(2u - \sqrt{\frac{2rr^*}{k}}P_0 - \frac{r^*}{k}h \right) \right] \vartheta^*(u) \\ \vartheta(u+1) &= \vartheta(u) \\ \vartheta(u+r\tau) &= \exp \left[-2\pi\sqrt{-1}\tau - \frac{2\pi\sqrt{-1}}{r} \left(2u - \sqrt{\frac{2rr^*}{k}}P_0 - \frac{r}{k}h \right) \right] \vartheta(u). \end{aligned} \quad (21)$$

Let us use the parameterization $z_j^{(t)} = x^{2u_j^{(t)}}$, $t = 1, 2$, $j = 1, 2, \dots, m$.

Definition 2. We define the operator \mathcal{G}_m^* for the regime $\operatorname{Re}(r) > k$ and $0 < \operatorname{Re}(s) < 2$ by

$$\begin{aligned} \mathcal{G}_m^* &= \int \cdots \int_{C^*} \prod_{j=1}^m \frac{dz_j^{(1)}}{z_j^{(1)}} \prod_{j=1}^m \frac{dz_j^{(2)}}{z_j^{(2)}} E_1(z_1^{(1)}) E_1(z_2^{(1)}) \cdots E_1(z_m^{(1)}) \\ &\quad \times E_2(z_1^{(2)}) E_2(z_2^{(2)}) \cdots E_2(z_m^{(2)}) \\ &\times \frac{\prod_{t=1,2} \prod_{1 \leq i < j \leq m} [u_i^{(t)} - u_j^{(t)}]_{r-k} [u_j^{(t)} - u_i^{(t)} + 1]_{r-k}}{\prod_{1 \leq i, j \leq m} [u_i^{(1)} - u_j^{(2)} - \frac{s}{2}]_{r-k} [u_j^{(2)} - u_i^{(1)} - \frac{s}{2} + 1]_{r-k}} \vartheta^* \left(\sum_{j=1}^m (u_j^{(2)} - u_j^{(1)}) \right) \end{aligned} \quad (22)$$

where the integral contour C^* encircles $z_j^{(t)} = 0$, $t = 1, 2$, $j = 1, 2, \dots, m$, in such a way that

$$|z_j^{(t)}| = 1, \quad t = 1, 2, \quad j = 1, 2, \dots, m.$$

We define the operator \mathcal{G}_m for the regime $\operatorname{Re}(r) > 0$ and $0 < \operatorname{Re}(s) < 2$ by

$$\mathcal{G}_m = \int \cdots \int_C \prod_{j=1}^m \frac{dz_j^{(1)}}{z_j^{(1)}} \prod_{j=1}^m \frac{dz_j^{(2)}}{z_j^{(2)}} F_1(z_1^{(1)}) F_1(z_2^{(1)}) \cdots F_1(z_m^{(1)})$$

$$\begin{aligned}
& \times F_2(z_1^{(2)})F_2(z_2^{(2)}) \cdots F_2(z_m^{(2)}) \quad (23) \\
& \times \frac{\prod_{t=1,2} \prod_{1 \leq i < j \leq m} [u_i^{(t)} - u_j^{(t)}]_r [u_j^{(t)} - u_i^{(t)} - 1]_r}{\prod_{1 \leq i, j \leq m} [u_i^{(1)} - u_j^{(2)} + \frac{s}{2}]_r [u_j^{(2)} - u_i^{(1)} + \frac{s}{2} - 1]_r} \vartheta \left(\sum_{j=1}^m (u_j^{(1)} - u_j^{(2)}) \right)
\end{aligned}$$

where the integral contour C^* encircles $z_j^{(t)} = 0$, $t = 1, 2$, $j = 1, 2, \dots, m$, in such a way that

$$|z_j^{(t)}| = 1, \quad t = 1, 2, \quad j = 1, 2, \dots, m.$$

We call the operators \mathcal{G}_m^* and \mathcal{G}_m the nonlocal integrals of motion for level k .

The definition of the operators \mathcal{G}_m^* , \mathcal{G}_m for generic $s \in \mathbb{C}$, $s \neq 2$, should be understood as analytic continuation. In the limit $s \rightarrow 2$, the contour C^* , C pinch at $z_j^{(t)} = z_i^{(t')}$. Hence the definition of \mathcal{G}_m^* , \mathcal{G}_m do not hold for $s = 2$. We give modified definition of \mathcal{G}_m^* , \mathcal{G}_m for $s = 2$, below. We note that parameter $s \neq 2$ plays an important role in the proof of Theorem 2.

Definition 3. We define the operator \mathcal{G}_m^{DV*} for the regime $\text{Re}(r) > k$ and $s = 2$ by

$$\begin{aligned}
\mathcal{G}_m^{DV*} &= \int \cdots \int_{C_{\text{Arg}}^*} \prod_{j=1}^m \frac{dz_j^{(1)}}{z_j^{(1)}} \prod_{j=1}^m \frac{dz_j^{(2)}}{z_j^{(2)}} E_1^{DV}(z_1^{(1)}) \cdots E_1^{DV}(z_m^{(1)}) \\
& \quad \times E_2^{DV}(z_1^{(2)}) \cdots E_2^{DV}(z_m^{(2)}) \quad (24) \\
& \times \frac{\prod_{t=1,2} \prod_{1 \leq i < j \leq m} [u_i^{(t)} - u_j^{(t)}]_{r-k} [u_j^{(t)} - u_i^{(t)} + 1]_{r-k}}{\prod_{1 \leq i, j \leq m} [u_i^{(1)} - u_j^{(2)} - 1]_{r-k} [u_j^{(2)} - u_i^{(1)}]_{r-k}} \vartheta^* \left(\sum_{j=1}^m (u_j^{(2)} - u_j^{(1)}) \right)
\end{aligned}$$

where the integral contour C_{Arg}^* encircles $z_j^{(t)} = 0$, $t = 1, 2$, $j = 1, 2, \dots, m$, in such a way that

$$|x^2 z_m^{(2)}|, |x^{2r^*} z_m^{(2)}| < |z_1^{(1)}| < |z_1^{(2)}| < |z_2^{(1)}| < |z_2^{(2)}| < \cdots < |z_m^{(1)}| < |z_m^{(2)}|.$$

We define the operator \mathcal{G}_m^{DV} for the regime $\text{Re}(r) > 0$ and $s = 2$ by

$$\mathcal{G}_m^{DV} = \int \cdots \int_{C_{\text{Arg}}} \prod_{j=1}^m \frac{dz_j^{(1)}}{z_j^{(1)}} \prod_{j=1}^m \frac{dz_j^{(2)}}{z_j^{(2)}} F_1^{DV}(z_1^{(1)}) \cdots F_1^{DV}(z_m^{(1)})$$

$$\begin{aligned}
 & \times F_2^{DV}(z_1^{(2)}) \cdots F_2^{DV}(z_m^{(2)}) \\
 & \times \frac{\prod_{t=1,2} \prod_{1 \leq i < j \leq m} [u_i^{(t)} - u_j^{(t)}]_r [u_j^{(t)} - u_i^{(t)} - 1]_r}{\prod_{1 \leq i, j \leq m} [u_i^{(1)} - u_j^{(2)} + 1]_r [u_j^{(2)} - u_i^{(1)}]_r} \vartheta \left(\sum_{j=1}^m (u_j^{(1)} - u_j^{(2)}) \right)
 \end{aligned} \tag{25}$$

where the integral contour C_{Arg} encircles $z_j^{(t)} = 0$, $t = 1, 2$, $j = 1, 2, \dots, m$, in such a way that

$$|x^2 z_m^{(2)}|, |x^{2r} z_m^{(2)}| < |z_1^{(1)}| < |z_1^{(2)}| < |z_2^{(1)}| < |z_2^{(2)}| < \cdots < |z_m^{(1)}| < |z_m^{(2)}|.$$

3.2. Main Result

The following is the main theorem of this paper.

Theorem 2. *For the regime $s \neq 2$ and $\text{Re}(r) > k$ we have*

$$[\mathcal{G}_m^*, \mathcal{G}_n^*] = 0, \quad m, n \in \mathbb{N}. \tag{26}$$

For the regime $s \neq 2$ and $\text{Re}(r) > 0$ we have

$$[\mathcal{G}_m, \mathcal{G}_n] = 0, \quad m, n \in \mathbb{N}. \tag{27}$$

Let us sketch the proof of Theorem 2. The proof is given as the same manner as level $k = 1$ case in [1, 3]. By symmetrization of the screenings $E_j(z)$ the commutation relation $[\mathcal{G}_m^*, \mathcal{G}_n^*] = 0$ is reduced to the following sufficient condition of the theta functions, which is shown by induction as the same manner as [1, 3]. We note that this symmetrization procedure holds only for $s \neq 2$

$$\begin{aligned}
 & \sum_{K \cup K^c = L \cup L^c = \{1, 2, \dots, n+m\}} \sum_{|K|=|L|=n, |K^c|=|L^c|=m} \vartheta^* \left(\sum_{j \in K^c} u_j^{(2)} - \sum_{j \in L^c} u_j^{(1)} \right) \vartheta^* \left(\sum_{j \in K} u_j^{(2)} - \sum_{j \in L} u_j^{(1)} \right) \\
 & \times \prod_{\substack{i \in K^c \\ p \in K^c}} \prod_{\substack{j \in K^c \\ q \in K^c}} \frac{[u_j^{(2)} - u_p^{(1)} - \frac{s}{2}]_{r-k} [u_i^{(1)} - u_q^{(2)} - \frac{s}{2}]_{r-k}}{[u_i^{(1)} - u_p^{(1)}]_{r-k} [u_j^{(2)} - u_q^{(2)}]_{r-k}} \\
 & \times \prod_{\substack{i \in K^c \\ p \in K^c}} \prod_{\substack{j \in K^c \\ q \in K^c}} \frac{[u_p^{(1)} - u_j^{(2)} - \frac{s}{2} + 1]_{r-k} [u_q^{(2)} - u_i^{(1)} - \frac{s}{2} + 1]_{r-k}}{[u_p^{(1)} - u_i^{(1)} + 1]_{r-k} [u_q^{(2)} - u_j^{(2)} + 1]_{r-k}}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{K \cup K^c = L \cup L^c = \{1, 2, \dots, n+m\} \\ |K| = |L| = n, |K^c| = |L^c| = m}} \sum \vartheta^* \left(\sum_{j \in K^c} u_j^{(2)} - \sum_{j \in L^c} u_j^{(1)} \right) \vartheta^* \left(\sum_{j \in K} u_j^{(2)} - \sum_{j \in L} u_j^{(1)} \right) \\
&\times \prod_{\substack{i \in K^c \\ p \in K^c}} \prod_{\substack{j \in K^c \\ q \in K^c}} \frac{[u_q^{(2)} - u_i^{(1)} - \frac{s}{2}]_{r-k} [u_p^{(2)} - u_j^{(1)} - \frac{s}{2}]_{r-k}}{[u_p^{(1)} - u_i^{(1)}]_{r-k} [u_q^{(2)} - u_j^{(2)}]_{r-k}} \\
&\times \prod_{\substack{i \in K^c \\ p \in K^c}} \prod_{\substack{j \in K^c \\ q \in K^c}} \frac{[u_i^{(1)} - u_q^{(2)} - \frac{s}{2} + 1]_{r-k} [u_j^{(2)} - u_p^{(1)} - \frac{s}{2} + 1]_{r-k}}{[u_i^{(1)} - u_p^{(1)} + 1]_{r-k} [u_q^{(2)} - u_j^{(2)} + 1]_{r-k}}.
\end{aligned} \tag{28}$$

Naively, when we take the limit $s \rightarrow 2$, it seems that we have $[\mathcal{G}_m^{DV*}, \mathcal{G}_n^{DV*}] = 0$. However, very precisely, in order to take the limit $s \rightarrow 2$, we have to consider special treatment which we call “renormalized” limit in [1]. Here we state only conjecture on the operator \mathcal{G}_m^{DV*} . Theorem 2 give a supporting argument of the following conjecture.

Conjecture 3. *For the regime $s = 2$ and $\text{Re}(r) > k$ we have*

$$[\mathcal{G}_m^{DV*}, \mathcal{G}_n^{DV*}] = 0, \quad m, n \in \mathbb{N}. \tag{29}$$

For the regime $s = 2$ and $\text{Re}(r) > 0$ we have

$$[\mathcal{G}_m^{DV}, \mathcal{G}_n^{DV}] = 0, \quad m, n \in \mathbb{N}. \tag{30}$$

In this paper we gave one parameter “ s ” deformation of level k free field realization of the screening current of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. By means of these free field realizations, we constructed infinitely many commutative operators, which we call the nonlocal integrals of motion associated with the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ for arbitrary level $k \neq 0, -2$. They are given as integrals involving a product of the screening current and Jacobi elliptic theta functions. The construction of the local integrals of motion \mathcal{I}_m for arbitrary level k is open problem. The construction of the local integrals of motion \mathcal{I}_m for level 1 only is summarized in [1–3].

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Appendix

Here we summarize the normal orderings of the basic operators (in all cases the subscript $j = 1, 2$).

$$\begin{aligned}
 C_j(z_1)C_j(z_2) &= :: z_1^{\frac{2}{r^*} + \frac{2}{k}} \frac{(x^{-2+2k} z_2/z_1; x^{2r^*})_\infty (x^{-2} z_2/z_1; x^{2k})_\infty}{(x^{2+2k} z_2/z_1; x^{2r^*})_\infty (x^2 z_2/z_1; x^{2k})_\infty} \\
 C_1(z_1)C_2(z_2) &= :: z_1^{-\frac{2}{r^*} - \frac{2}{k}} \frac{(x^s z_2/z_1; x^{2r^*})_\infty (x^{2-s} z_2/z_1; x^{2r^*})_\infty}{(x^{-s} z_2/z_1; x^{2r^*})_\infty (x^{s-2} z_2/z_1; x^{2r^*})_\infty} \\
 &\quad \times \frac{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{-s+2k} z_2/z_1; x^{2k})_\infty (x^{2-s+2k} z_2/z_1; x^{2k})_\infty} \\
 C_2(z_1)C_1(z_2) &= :: z_1^{-\frac{2}{r^*} - \frac{2}{k}} \frac{(x^{s+2r^*} z_2/z_1; x^{2r^*})_\infty (x^{2-s+2r^*} z_2/z_1; x^{2r^*})_\infty}{(x^{-s+2r^*} z_2/z_1; x^{2r^*})_\infty (x^{s-2+2r^*} z_2/z_1; x^{2r^*})_\infty} \\
 &\quad \times \frac{(x^s z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}{(x^{-s} z_2/z_1; x^{2k})_\infty (x^{2-s} z_2/z_1; x^{2k})_\infty} \\
 C_j^\dagger(z_1)C_j^\dagger(z_2) &= :: z_1^{-\frac{2}{r} + \frac{2}{k}} \frac{(x^{-2+2k} z_1/z_2; x^{2k})_\infty (x^{2+2r} z_1/z_2; x^{2r})_\infty}{(x^{2+2k} z_2/z_1; x^{2k})_\infty (x^{-2+2r} z_2/z_1; x^{2r})_\infty} \\
 C_1^\dagger(z_1)C_2^\dagger(z_2) &= :: z_1^{\frac{2}{r} - \frac{2}{k}} \frac{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{-s+2k} z_2/z_1; x^{2k})_\infty (x^{2-s+2k} z_2/z_1; x^{2k})_\infty} \\
 &\quad \times \frac{(x^{-s+2r} z_2/z_1; x^{2r})_\infty (x^{s-2+2r} z_2/z_1; x^{2r})_\infty}{(x^{s+2r} z_2/z_1; x^{2r})_\infty (x^{2-s+2r} z_2/z_1; x^{2r})_\infty} \\
 C_2^\dagger(z_1)C_1^\dagger(z_2) &= :: z_1^{\frac{2}{r} - \frac{2}{k}} \frac{(x^s z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}{(x^{-s} z_2/z_1; x^{2k})_\infty (x^{2-s} z_2/z_1; x^{2k})_\infty} \\
 &\quad \times \frac{(x^{-s} z_2/z_1; x^{2r})_\infty (x^{s-2} z_2/z_1; x^{2r})_\infty}{(x^s z_2/z_1; x^{2r})_\infty (x^{2-s} z_2/z_1; x^{2r})_\infty} \\
 C_j(z_1)C_j^\dagger(z_2) &= :: z_1^{-\frac{2}{k}} \frac{(x^{2+k} z_2/z_1; x^{2k})_\infty}{(x^{-2+k} z_2/z_1; x^{2k})_\infty} \\
 C_j^\dagger(z_1)C_j(z_2) &= :: z_1^{-\frac{2}{k}} \frac{(x^{2+k} z_2/z_1; x^{2k})_\infty}{(x^{-2+k} z_2/z_1; x^{2k})_\infty}
 \end{aligned}$$

$$\begin{aligned}
\tilde{\Psi}_{1,I}(z_1)\tilde{\Psi}_{2,I}(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{2,I}(z_1)\tilde{\Psi}_{1,I}(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{1,II}(z_1)\tilde{\Psi}_{2,II}(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{2,II}(z_1)\tilde{\Psi}_{1,II}(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{1,I}^\dagger(z_1)\tilde{\Psi}_{2,I}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{2,I}^\dagger(z_1)\tilde{\Psi}_{1,I}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{1,II}^\dagger(z_1)\tilde{\Psi}_{2,II}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{2,II}^\dagger(z_1)\tilde{\Psi}_{1,II}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{1,I}(z_1)\tilde{\Psi}_{2,II}(z_2) &= :: \frac{(x^{-s+2k}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2+2k}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{2,II}(z_1)\tilde{\Psi}_{1,I}(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s}z_2/z_1; x^{2k})_\infty}{(x^s z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{1,II}(z_1)\tilde{\Psi}_{2,I}(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s}z_2/z_1; x^{2k})_\infty}{(x^s z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{2,I}(z_1)\tilde{\Psi}_{1,II}(z_2) &= :: \frac{(x^{-s+2k}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2+2k}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{1,I}^\dagger(z_1)\tilde{\Psi}_{2,II}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s}z_2/z_1; x^{2k})_\infty}{(x^s z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{2,II}^\dagger(z_1)\tilde{\Psi}_{1,I}^\dagger(z_2) &= :: \frac{(x^{-s+2k}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2+2k}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{1,II}^\dagger(z_1)\tilde{\Psi}_{2,I}^\dagger(z_2) &= :: \frac{(x^{-s+2k}z_2/z_1; x^{2k})_\infty (x^{2-s+2k}z_2/z_1; x^{2k})_\infty}{(x^{s+2k}z_2/z_1; x^{2k})_\infty (x^{s-2+2k}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{2,I}^\dagger(z_1)\tilde{\Psi}_{1,II}^\dagger(z_2) &= :: \frac{(x^{-s}z_2/z_1; x^{2k})_\infty (x^{2-s}z_2/z_1; x^{2k})_\infty}{(x^s z_2/z_1; x^{2k})_\infty (x^{s-2}z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,I}(z_1)\tilde{\Psi}_{j,I}(z_2) &= :: (1 - z_2/z_1) \frac{(x^{2+2k}z_2/z_1; x^{2k})_\infty}{(x^{-2}z_2/z_1; x^{2k})_\infty}
\end{aligned}$$

$$\begin{aligned}
\tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) &= :: (1 - z_2/z_1) \frac{(x^{2+2k} z_2/z_1; x^{2k})_\infty}{(x^{-2} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,I}(z_1)\tilde{\Psi}_{j,II}(z_2) &= :: \frac{(x^{2+2k} z_2/z_1; x^{2k})_\infty}{(x^{-2+2k} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,I}(z_2) &= :: \frac{(x^2 z_2/z_1; x^{2k})_\infty}{(x^{-2} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,I}^\dagger(z_1)\tilde{\Psi}_{j,I}^\dagger(z_2) &= :: (1 - z_2/z_1) \frac{(x^{2+2k} z_2/z_1; x^{2k})_\infty}{(x^{-2} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,II}^\dagger(z_1)\tilde{\Psi}_{j,II}^\dagger(z_2) &= :: (1 - z_2/z_1) \frac{(x^{2+2k} z_2/z_1; x^{2k})_\infty}{(x^{-2} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,I}^\dagger(z_1)\tilde{\Psi}_{j,II}^\dagger(z_2) &= :: \frac{(x^2 z_2/z_1; x^{2k})_\infty}{(x^{-2} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,II}^\dagger(z_1)\tilde{\Psi}_{j,I}^\dagger(z_2) &= :: \frac{(x^{2+2k} z_2/z_1; x^{2k})_\infty}{(x^{-2+2k} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,I}(z_1)\tilde{\Psi}_{j,I}^\dagger(z_2) &= :: \frac{1}{(1 - x^k z_2/z_1)} \frac{(x^{k-2} z_2/z_1; x^{2k})_\infty}{(x^{3k+2} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,I}^\dagger(z_1)\tilde{\Psi}_{j,I}(z_2) &= :: \frac{1}{(1 - x^{-k} z_2/z_1)} \frac{(x^{-k-2} z_2/z_1; x^{2k})_\infty}{(x^{k+2} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,I}(z_1)\tilde{\Psi}_{j,II}^\dagger(z_2) &= :: \frac{(x^{-2+k} z_2/z_1; x^{2k})_\infty}{(x^{2+k} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,II}^\dagger(z_1)\tilde{\Psi}_{j,I}(z_2) &= :: \frac{(x^{-2+k} z_2/z_1; x^{2k})_\infty}{(x^{2+k} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,I}^\dagger(z_2) &= :: \frac{(x^{-2+k} z_2/z_1; x^{2k})_\infty}{(x^{2+k} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,I}^\dagger(z_1)\tilde{\Psi}_{j,II}(z_2) &= :: \frac{(x^{-2+k} z_2/z_1; x^{2k})_\infty}{(x^{2+k} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}^\dagger(z_2) &= :: \frac{1}{(1 - x^k z_2/z_1)} \frac{(x^{-k-2} z_2/z_1; x^{2k})_\infty}{(x^{k+2} z_2/z_1; x^{2k})_\infty} \\
\tilde{\Psi}_{j,II}^\dagger(z_1)\tilde{\Psi}_{j,II}(z_2) &= :: \frac{1}{(1 - x^k z_2/z_1)} \frac{(x^{k-2} z_2/z_1; x^{2k})_\infty}{(x^{3k+2} z_2/z_1; x^{2k})_\infty}.
\end{aligned}$$

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