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A NONCOMMUTATIVE UNITON THEORY

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Abstract. Harmonic two-spheres in the unitary group may be constructed and described in terms of unitons. We present an analogue of this theory for those solutions of the noncommutative $\mathrm{U}(1)$ sigma-model that may be represented as finite-dimensional perturbations of zero-energy solutions. In particular, we establish that the energy of every such solution is an integer multiple of 8π, describe all solutions of small energy and give many explicit examples of non-Grassmannian solutions.

1. Introduction

This paper is an extended version of the author's talk at the XXV workshop on geometric methods in physics (Bialowieza, Poland, 02–08 July 2006). We start by briefly recalling some aspects of the theory of harmonic maps from the two-dimenisonal sphere to the unitary group. Then we consider a noncommutative analogue of this sigma model and state the main results of the uniton theory. As an application, we describe all solutions of small energy and give explicit examples (apparently absent in the existing literature) of non-Grassmannian solutions of any admissible energy. Complete proofs will be given elsewhere.

2. Harmonic Maps From $\mathbb{S}^2$ to $\mathrm{U}(n)$

Consider the energy functional

$$E(\varphi) = \frac{1}{2}\int_{\mathbb{C}} |\varphi^{-1}\mathrm{d}\varphi|^2 \,\mathrm{d}x\mathrm{d}y = 2\int_{\mathbb{C}} |\varphi^{-1}\overline{\partial}\varphi|^2 \,\mathrm{d}x\mathrm{d}y \tag{1}$$

on the set of all smooth maps $\varphi : \mathbb{C} \to \mathrm{U}(n)$. Here $\overline{\partial} = (\partial_x + \mathrm{i}\partial_y)/2$ is the derivative with respect to $\overline{z}$, and $|A|^2 = \mathrm{tr}(AA^*)$ for any matrix A. The critical points of $E(\varphi)$ are called *harmonic maps from* $\mathbb{S}^2$ *to* $\mathrm{U}(n)$. They coincide with finite-energy solutions of the Euler–Lagrange equations $\partial(\varphi^{-1}\overline{\partial}\varphi) + \overline{\partial}(\varphi^{-1}\partial\varphi) = 0$, where $\partial = (\partial_x - \mathrm{i}\partial_y)/2$ is the derivative with respect to z. The simplest examples of such solutions are given by holomorphic and anti-holomorphic maps from $\mathbb{S}^2 = \mathbb{CP}^1$ to the complex Grassmannians $\mathrm{Gr}_k(\mathbb{C}^n)$, which are embedded in the unitary group $\mathrm{U}(n)$ as totally geodesic submanifolds by $V \mapsto \varphi = I - 2Q$, where $Q : \mathbb{C}^n \to \mathbb{C}^n$ is the orthogonal projector with image V.

The set of all *Grassmannian solutions* (that is, those with values in some $\mathrm{Gr}_k(\mathbb{C}^n)$) is not exhausted by holomorphic and anti-holomorphic solutions (except for the $\mathbb{CP}^1$-model, where $n = 2$ and $k = 1$). However, every Grassmannian solution can be obtained from holomorphic and antiholomorphic ones by a finite sequence of geometric modifications, where each step consists in subtracting a holomorphic subbundle from the image of Q and adding a holomorphic subbundle of the image of $Q^\perp := I - Q$ (see [10]).

A construction of Uhlenbeck [7] gives a similar factorization for all (not necessarily Grassmannian) solutions. Namely, for every harmonic map $\varphi : \mathbb{S}^2 \to \mathrm{U}(n)$ there exists a sequence of harmonic maps $\varphi_0, \varphi_1, \dots, \varphi_m : \mathbb{S}^2 \to \mathrm{U}(n)$ such that φ_0 is a constant map (equivalently, a solution of energy 0), $\varphi_m = \varphi$, and each φ_{j+1} is obtained from φ_j by adding a φ_j-uniton. This means that $\varphi_{j+1}(\zeta) = \varphi_j(\zeta)(I - 2P_j(\zeta))$ for some family $\{P_j(\zeta)\,;\, \zeta \in \mathbb{C}\}$ of finite-dimensional orthogonal projectors satisfying the equations

$$PAP^\perp = 0, \qquad P^\perp AP + 2P^\perp \overline{\partial} P = 0 \tag{2}$$

where $P = P_j$ is the family of projectors and $A = \varphi_j^{-1}\overline{\partial}\varphi_j$ is the logarithmic $\overline{\partial}$-derivative of φ_j. If $A = 0$, then equations (2) reduce to $P^\perp \overline{\partial} P = 0$, which means that $P^\perp$ determines a holomorphic map from $\mathbb{CP}^1$ to the Grassmannian. As shown in [8], equations (2) imply the following formula for the jump of energy

$$E(\varphi_{j+1}) - E(\varphi_j) = 8\pi c_1(P_j)[\mathbb{S}^2] \tag{3}$$

where $c_1(P_j)$ is the first Chern class of the vector bundle over $\mathbb{S}^2$ whose fibre at $\zeta \in S^2$ is the image of $P_j(\zeta)$. Since the numbers $c_1(P_j)[\mathbb{S}^2]$ are always integer, it follows that the energy of any harmonic map from $\mathbb{S}^2$ to $\mathrm{U}(n)$ is an integer multiple of 8π. These results lead to a description of all harmonic maps from $\mathbb{S}^2$ to $\mathrm{U}(n)$ for $n \leqslant 4$ (see [10], [9]).

3. Noncommutativity

Let H be a separable Hilbert space with an orthonormal basis $\{e_0, e_1, \dots\}$, and let a be the standard annihilation operator on H with domain $\mathrm{dom}(a) = \{x \in H;$

$\sum_{j=0}^{\infty}(j+1)|x_j|^2 < \infty\}$, and $a(e_j) = \sqrt{j}e_{j-1}$ for $j = 0, 1, 2, \dots$. Using the Weyl calculus of pseudodifferential operators (see, for example, [3], Section 18.5), we can transfer the U(1)-version of the harmonic map problem (1) to a family $\mathbb{R}^2_\theta$ ($\theta \geqslant 0$) of noncommutative planes obtained from the usual plane $\mathbb{C} = \mathbb{R}^2_0$ by the so-called *Moyal deformation* (or Moyal–Weyl quantization, see [2]). This results in the following picture ([4], [1]). The maps $\varphi : \mathbb{C} \to \mathrm{U}(n)$ are replaced by unitary operators $\Phi \in \mathrm{U}(H)$, the derivatives $\overline{\partial}$ and ∂ are replaced by commutators with $(2\theta)^{-1/2}a$ and $-(2\theta)^{-1/2}a^*$ respectively, and the integral over $\mathbb{C}$ is replaced by $2\pi\theta\mathrm{Tr}$, where $\mathrm{Tr}X$ is the trace of any trace-class operator X on H. The energy functional (1) is replaced by the θ-independent functional

$$E(\Phi) := 2\pi\|[a, \Phi]\|^2_{\mathrm{HS}} \tag{4}$$

where $\|A\|^2_{\mathrm{HS}} := \mathrm{Tr}(AA^*)$ is the squared Hilbert–Schmidt norm. We conisder the energy functional (4) on the space $\mathrm{U}(H) \cap \mathcal{D}$, where $\mathcal{D}$ consists of all continuous linear operators $B : H \to H$ of the form $B = \lambda I + K$ with $\lambda \in \mathbb{C}$ and the image of K being a finite-dimensional vector subspace of $\mathrm{dom}(a^2)$. This enables us to obtain non-trivial analogues of the results of Section 2 without studying delicate domain questions for various operators.

4. Grassmannian Solutions and BPS Solutions

Extremals of (4) are solutions $\Phi \in \mathrm{U}(H) \cap \mathcal{D}$ of the Euler-Lagrange equations $\partial_+(\Phi^{-1}\partial_-\Phi) + \partial_-(\Phi^{-1}\partial_+\Phi) = 0$, where we put $\partial_- X := [a, X]$ and $\partial_+ X := -[a^*, X]$ for all $X \in \mathcal{D}$. A solution Φ is said to be *Grassmannian* if $\Phi = I - 2P$ for some orthoprojector $P \in \mathcal{D}$. The simplest examples of such solutions appear when $P^\perp a P = 0$ or, equivalently, $a(\mathrm{im}P) \subset \mathrm{im}P$. The corresponding operators $\Phi = I - 2P$ are called *BPS-solutions* (in honour of Bogomolny, Prasad and Sommerfeld, see [4]). These are the noncommutative analogues of antiholomorphic maps from $\mathbb{S}^2$ to the Grassmannians.

The energy of every BPS solution $\Phi = I - 2P$ equals $8\pi\dim\mathrm{im}\,P$. For every integer $k \geq 0$, the set of all BPS solutions of energy $8\pi k$ is naturally identified with $\mathbb{C}^k$ (see, for example, [1]) by writing $\mathrm{im}\,P = \ker\Pi(a)$, where $\Pi(z) = z^k + c_1 z^{k-1} + \cdots + c_k$ is any monic polynomial of degree k with complex coefficients. This descrption can also be restated in terms of the so-called (non-normalized) coherent states for the harmonic oscillator [5], which are defined by

$$R_\lambda := \sum_{j=0}^{\infty} \frac{\lambda^j}{\sqrt{j!}} e_j \quad \text{and} \quad R^k_\lambda := (a^* - \overline{\lambda} I)^k R_\lambda \tag{5}$$

where $\lambda \in \mathbb{C}$ and $k = 0, 1, 2, \ldots$. Namely, a generic BPS solution $I - 2P$ of energy $8\pi k$ is given by $\operatorname{im} P = \langle R_{\lambda_1}, \ldots, R_{\lambda_k} \rangle$ for any distinct numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. We may also let some of these numbers coincide. If precisely j elements of the set $\{\lambda_1, \ldots, \lambda_k\}$ are equal to λ, then the corresponding group of j basis vectors must be replaced by $\langle R_\lambda, R_\lambda^1, \ldots, R_\lambda^{j-1} \rangle$.

5. Main Results

The following theorem is a noncommutative analogue of (2) and (3). The role of the Chern class in (3) is now played by (7).

Theorem 1. *Suppose that a unitary operator $\Phi \in \mathrm{U}(H) \cap \mathcal{D}$ and an orthoprojector $P \in \mathcal{D}$ satisfy the equations*

$$P\Phi^{-1}\partial_-\Phi P^\perp = 0, \qquad P^\perp\Phi^{-1}\partial_-\Phi P + 2P^\perp\partial_- P = 0. \tag{6}$$

Put $\Psi := \Phi(I - 2P)$. Then $E(\Psi) = E(\Phi) + 8\pi Q(P)$, where

$$Q(P) = \begin{cases} \dim \operatorname{im} P & \text{if} \quad \dim \operatorname{im} P < \infty \\ -\dim \ker P & \text{if} \quad \dim \ker P < \infty. \end{cases} \tag{7}$$

Moreover, if Φ is a solution, then Ψ is also a solution.

Let $\Phi \in \mathrm{U}(H) \cap \mathcal{D}$ be a solution. An orthoprojector $P \in \mathcal{D}$ is called a *uniton for Φ* (or a *Φ-uniton*) if it satisfies (6). Then we also say that the new solution $\Psi = \Phi(I - 2P)$ *is obtained from Φ by adding the Φ-uniton P*. A basic observation (whose commutative analogue plays an important role in [8], [9]) states that every solution $\Phi \in \mathrm{U}(H) \cap \mathcal{D}$ of positive energy possesses a canonical uniton P with $Q(P) < 0$ (namely, $\ker P = \operatorname{im}(\Phi^{-1}\partial_-\Phi)$). Combining this with Theorem 1, we get the following results on the uniton factorization and the quantization of energy.

Theorem 2. *For every solution $\Phi \in \mathrm{U}(H) \cap \mathcal{D}$ there is a sequence of solutions $\Phi_0, \Phi_1, \ldots, \Phi_m$ such that $\Phi_0 = \mathrm{e}^{\mathrm{i}\theta} I$ for some $\theta \in \mathbb{R}$, $\Phi_m = \Phi$, and each Φ_{j+1} is obtained from Φ_j by adding a Φ_j-uniton P_j with $Q(P_j) > 0$ $(j = 0, 1, \ldots, m-1)$.*

Theorem 3. *The energy of any solution $\Phi \in \mathrm{U}(H) \cap \mathcal{D}$ is an integer multiple of 8π.*

6. Applications

Solutions $\Phi_1, \Phi_2 \in \mathrm{U}(H) \cap \mathcal{D}$ are said to be *equivalent* if $\Phi_2 = \mathrm{e}^{\mathrm{i}\theta}\Phi_1$ for some $\theta \in \mathbb{R}$. Combining the following theorem with the results of Section 4, we see that the sets $\mathcal{M}_k$ of equivalence classes of all solutions of energy $8\pi k$ ($k = 1, 2, 3, 4$) may naturally be presented as

$$\mathcal{M}_1 = \mathbb{C}^1, \qquad \mathcal{M}_2 = \mathbb{C}^2, \qquad \mathcal{M}_3 = \mathbb{C}^3 \cup A, \qquad \mathcal{M}_4 = \mathbb{C}^4 \cup B \tag{8}$$

where $A = \mathbb{CP}^1 \times \mathbb{C}^1$ intersects $\mathbb{C}^3$ along $\{\infty\} \times \mathbb{C}^1$ and $B = \mathbb{CP}^1 \times \mathbb{C}^2$ intersects $\mathbb{C}^4$ along $\{\infty\} \times \mathbb{C}^2$. The sets $\mathcal{M}_k$ are real-analytic subvarieties in the unitary group $\mathrm{U}(H)$. Decompositions (8) lead to their regular stratifications in the sense of [6].

Theorem 4. *Every solution of energy $8\pi.1$ or $8\pi.2$ is equivalent to a BPS solution. Every solution of energy $8\pi.3$ is equivalent either to a BPS solution or to $(I - 2P_0)(I - 2P_1)$, where $\operatorname{im} P_0 = \langle R_\alpha \rangle$ and $\operatorname{im} P_1 = \langle R^1_\alpha, R_\alpha + cR^2_\alpha \rangle$ for some $\alpha \in \mathbb{C}$, $c \in \mathbb{CP}^1$. Every solution of energy $8\pi.4$ is equivalent either to a BPS solution or to $(I - 2P_0)(I - 2P_1)$, where $\operatorname{im} P_0 = \langle R_\alpha \rangle$ and $\operatorname{im} P_1$ is either $\langle R^1_\alpha, R^2_\alpha, R_\alpha + cR^3_\alpha \rangle$ for some $\alpha \in \mathbb{C}$, $c \in \mathbb{CP}^1$ or $\langle R^1_\alpha, R_\alpha + cR^2_\alpha, S_\alpha R_\beta \rangle$ for some $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{CP}^1$. Here $S_\alpha x$ stands for the orthogonal projection of x onto $\langle R_\alpha, R^1_\alpha \rangle$, and $x + cy$ with $c = \infty$ is understood as y.*

Simple examples of non-BPS solutions of energy $8\pi.3$ (with $\alpha = 0$ in the notation of Theorem 4) are given by the following matrices in the basis $\{e_0, e_1, \dots\}$ (with I_∞ being the identity matrix of infinite rank):

$$\Phi = \begin{pmatrix} k & 0 & -\bar{l} & 0 \\ 0 & -1 & 0 & 0 \\ l & 0 & k & 0 \\ 0 & 0 & 0 & I_\infty \end{pmatrix}, \qquad \text{where} \quad k := \frac{1 - |c|^2}{1 + |c|^2}, \quad l := -\frac{2c}{1 + |c|^2}. \tag{9}$$

These solutions are non-Grassmannian for $c \in \mathbb{C} \setminus \{0\}$. If $c = 0$, we get a Grassmannian non-BPS solution $I - 2P$ with $\operatorname{im} P = \langle e_1 \rangle$. If $c = \infty$, we get a BPS solution $I - 2P$ with $\operatorname{im} P = \langle e_0, e_1, e_2 \rangle$. Hence non-Grassmannian solutions form a "complex film" that interpolates between Grassmannian ones. (This phenomenon holds in the commutative case as well, see [10].) General non-BPS solutions of energy $8\pi.3$ are obtained from (9) by the transformation $\Phi \mapsto U_\alpha \Phi U^*_\alpha$, where U_α is the unique unitary operator on H that sends each coherent state $R_\lambda/|R_\lambda|$ to $R_{\alpha+\lambda}/|R_{\alpha+\lambda}|$ (see the Appendix in [5] and notice that the commutative analogue of this transformation replaces $\varphi(z)$ by $\varphi(z - \alpha)$ in the notation of Section 2).

To give examples of non-Grassmannian solutions of any energy $8\pi k$ ($k \geqslant 3$), we take any positive integer j with $2j + 1 \leqslant k$ and consider the BPS solution $\Phi_0 = I - 2P_0$ with $\operatorname{im} P_0 = \langle e_0, e_1, \ldots, e_j \rangle$. For every $c \in \mathbb{C}$ we define a $(k - j)$-dimensional orthoprojector $P(c)$ in H by

$$\operatorname{im}P(c) = \langle e_j, \ldots, e_{k-j-1}, ce_0 + e_{k-j}, ce_1 + \frac{e_{k-j+1}}{\sqrt{k-j+1}},$$
$$\frac{ce_2}{\sqrt{2}} + \frac{e_{k-j+2}}{\sqrt{(k-j+1)(k-j+2)}}, \ldots, \frac{ce_{j-1}}{\sqrt{(j-1)!}} + \sqrt{\frac{(k-j)!}{k!}} e_k \rangle.$$

Then $P(c)$ is a Φ_0-uniton. The corresponding operators $\Phi(c) = (I - 2P_0)(I - 2P(c))$ are non-Grassmannian solutions of energy $8\pi k$ that interpolate between the BPS solution $\Phi(0) = I - 2P$ with $\operatorname{im} P = \langle e_0, e_1, \ldots, e_k \rangle$ and the Grassmannian non-BPS solution $\Phi(\infty) = I - 2Q$ with $\operatorname{im} Q = \langle e_j, \ldots, e_{k-j-1} \rangle$. These examples naturally appear in the problem of realizing the non-Grassmannian zero modes of the Hessian of the energy functional (4) at the BPS solution $\Phi(0)$ by tangent vectors to the moduli space of solutions (see [1], Section 4.3). Namely, it is very plausible that the tangent vectors to these families of solutions at $\Phi(0)$ span the whole tangent cone to the non-BPS part of the moduli space $\mathcal{M}_k$ at $\Phi(0)$. For $k \leqslant 4$, this conjecture is confirmed by Theorem 4.

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