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MEAN SQUARE FORMULA FOR THE DOUBLE ZETA-FUNCTION

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Abstract: We prove the mean square formula of the Euler–Zagier type double zeta-function $\zeta_2(s_1, s_2)$ and provide an improvement on the Ω results of Kiuchi, Tanigawa, and Zhai. We also calculate the double integral $\int_2^N \int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 dt_2$ under certain conditions.

Keywords: double zeta-functions, mean square formula, Riemann zeta-function.

1. Introduction

Let $s_j = \sigma_j + it_j$ $(j = 1, 2)$ be complex variables with σ_j, $t_j \in \mathbb{R}$, and let $\zeta(s)$ be the Riemann zeta-function, which is defined as $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}\, s > 1$. The double zeta-function of Euler–Zagier type is defined by

$$\zeta_2(s_1, s_2) = \sum_{1 \leqslant m < n} \frac{1}{m^{s_1} n^{s_2}}, \tag{1.1}$$

which is absolutely convergent for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. The proof of Atkinson's formula for the mean value theorem of the Riemann zeta-function $\zeta(s)$ (see F.V. Atkinson [2] or A. Ivić [6]) applies to the function (1.1). Some analytic properties of (1.1) have been obtained by S. Akiyama, S. Egami and Y. Tanigawa [1], H. Ishikawa and K. Matsumoto [5], I. Kiuchi and Y. Tanigawa [9], I. Kiuchi, Y. Tanigawa and W. Zhai [10], K. Matsumoto [11], [12], J. Q. Zhao [15], and others.

K. Matsumoto and H. Tsumura [13] were the first to study a new type of mean value formula for $\int_2^T |\zeta_2(s_1, s_2)|^2 dt_2$ with a fixed complex number s_1 and any number $T > 2$. They conjectured that when $\sigma_1 + \sigma_2 = \frac{3}{2}$, the form of the main term of the mean square formula would not be CT with a constant C, and that, most probably, some log-factor would appear. Recently, their results were

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considered by S. Ikeda, K. Matsuoka and Y. Nagata [4], who showed that

$$\int_2^T |\zeta_2(s_1,s_2)|^2\,dt_1 = \left(\sum_{m=1}^{\infty}\frac{1}{m^{2\sigma_1}}\left|\zeta(s_2)-\sum_{n=1}^{m}\frac{1}{n^{s_2}}\right|^2\right)T + \begin{cases} O\left(T^{4-2\sigma_1-2\sigma_2}\right) & \text{if } \frac{3}{2}<\sigma_1+\sigma_2<2, \\ O\left(\log^2 T\right) & \text{if } \sigma_1+\sigma_2=2. \end{cases} \tag{1.2}$$

Here, the coefficient of the main term on the right-hand side of (1.2) converges if $\sigma_1+\sigma_2>\frac{3}{2}$. They also deduced that the asymptotic formula

$$\int_2^T |\zeta_2(s_1,s_2)|^2\,dt_1 = \frac{1}{|s_2-1|^2}T\log T + O(T) \tag{1.3}$$

holds on the line $\sigma_1+\sigma_2=\frac{3}{2}$. This result implied that the conjecture of Matsumoto and Tsumura on the line $\sigma_1+\sigma_2=\frac{3}{2}$ was true.

Before the introduction of our theorems, let us recall the mean square formulas of the Riemann zeta-function $\zeta(\sigma+it)$ (see [2], [6], [7], [14]). It is well known that

$$\int_2^T |\zeta(\sigma+it)|^2\,dt = \zeta(2\sigma)T + O(1) \tag{1.4}$$

with $\sigma>1$, and

$$\int_2^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = T\log\frac{T}{2\pi} + (2\gamma-1)T + O\left(T^{\frac{1}{2}}\right), \tag{1.5}$$

where γ denotes the Euler constant. For $\frac{1}{2}<\sigma<1$, A. Ivić and K. Matsumoto [8] obtained that

$$\int_2^T |\zeta(\sigma+it)|^2\,dt = \zeta(2\sigma)T + A(\sigma)T^{2-2\sigma} + E_\sigma(T) \tag{1.6}$$

with $A(\sigma)=(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}$, where the error term of (1.6) was estimated as $E_\sigma(T)=O\left(T^{\frac{2}{3}(1-\sigma)}\log^{2/9}T\right)$. The mean square formula of $\zeta(1+it)$ was obtained by R. Balasubramanian, A. Ivić, and K. Ramachandra [3], who showed that

$$\int_2^T |\zeta(1+it)|^2\,dt = \zeta(2)T - \pi\log T + R(T) \tag{1.7}$$

with $R(T)=O\left((\log T)^{2/3}(\log\log T)^{1/3}\right)$, and the estimate

$$\int_2^T R(t)dt = O(T). \tag{1.8}$$

The main purpose of this paper is to prove the mean square formula for the double zeta-function $\zeta_2(s_1, s_2)$ within the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < \frac{3}{2}$. Ikeda, Matsuoka, and Nagata made use of the mean value theorems for Dirichlet polynomials and suitable approximations to the Euler–Maclaurin summation formula to obtain the formulas (1.2) and (1.3). We use the formulas (1.4)–(1.8) for the mean square of $|\zeta(\sigma + it)|$ and (2.8)–(2.10) in Lemma 3 below, as well as Lemma 1 below, which was derived from a weak form of the approximate formula of Kiuchi, Tanigawa, and Zhai [10] for $\zeta_2(s_1, s_2)$ to obtain the following formula.

Theorem 1. *Suppose that* $2 \leqslant t_1 \leqslant T$, $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $\sigma_1 + \sigma_2 = 1$. *Then, for any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = \frac{\zeta(2)}{4\pi |s_2 - 1|^2} \, T^2 + O\left(t_2^{-\frac{1}{2}} (\log t_2) T^{\frac{3}{2}}\right) \tag{1.9}$$

with $2 \leqslant t_2 \leqslant \frac{T^{\frac{1}{3}}}{\log T}$.

Remark 1. Inserting $t_2 = T^{\frac{1}{3}} (\log T)^{-1}$ into (1.9), the right-hand side of the formula (1.9) is estimated by $O(T^{\frac{4}{3}} \log^{\frac{3}{2}} T)$, but if we can take $t_2 = T$, we can estimate that $O(T \log T)$. The main term of this theorem is not $O(T \log^A T)$ $(A > 0)$, but T^2, since the analytic behaviour of the double zeta-function $\zeta_2(s_1, s_2)$ depends on both s_1 and s_2.

As an application of (1.9), we consider the evaluation of the double integral

$$\int_2^N \int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 dt_2$$

and then we deduce the following.

Corollary 1. *Let* $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $\sigma_1 + \sigma_2 = 1$. *Within the region* $2 \leqslant N \leqslant \frac{T^{\frac{1}{3}}}{\log T}$, *we obtain*

$$\lim_{T \to \infty} \frac{1}{T^2} \int_2^N \int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 dt_2 = \frac{\pi}{24\sigma_1} \left(\frac{\pi}{2} - \mathrm{Tan}^{-1} \frac{2}{\sigma_1} \right) + O\left(\frac{1}{N}\right).$$

Hence, this observation may be regarded as an average order of magnitude for the double integral $\int_2^N \int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 dt_2$, which is

$$\frac{\pi}{24\sigma_1} \left(\frac{\pi}{2} - \mathrm{Tan}^{-1} \frac{2}{\sigma_1} \right)$$

for $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $\sigma_1 + \sigma_2 = 1$.

Theorem 2. *Suppose that* $2 \leqslant t_1 \leqslant T,\ 0 < \sigma_1 < 1,\ 0 < \sigma_2 < 1,\ \frac{1}{2} \leqslant \sigma_1 + \sigma_2 < \frac{3}{2}$ *and* $\sigma_1 + \sigma_2 \neq 1$. *Then, for any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + O\left(t_2^{-\frac{1}{2}} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) \tag{1.10}$$

with $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$ *and* $2 \leqslant t_2 \leqslant T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$,

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) \tag{1.11}$$

with $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$ *and* $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$, *and*

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = \frac{\zeta(3)}{12\pi^2 |s_2-1|^2} T^3 + \begin{cases} O\left(T^2\right) & if \quad \sqrt{\log T} \leqslant t_2 \leqslant T^{\frac{1}{2}}, \\ O\left(t_2^{-1} T^2 \sqrt{\log T}\right) & if \quad 2 \leqslant t_2 \leqslant \sqrt{\log T}. \end{cases} \tag{1.12}$$

with $\sigma_1 + \sigma_2 = \frac{1}{2}$.

In a similar manner to the proof of Theorem 2, we consider the integral $\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1$ for $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$, which may show the formula

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = O\left(t_2 T\right),$$

but this does not provides an improvement on the formula (1.2). Similarly, in the case of $\sigma_1 + \sigma_2 = \frac{3}{2}$, we obtain the formula

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = \frac{1}{|s_2-1|^2} T \log T + O\left(t_2 T\right)$$

for $2 \leqslant t_2 < (\log T)^{\frac{1}{3}}$, which does not give an improvement upon the formula (1.3), since the error term of this formula depends on t_2.

Theorem 3. *Suppose that* $2 \leqslant t_1 \leqslant T,\ \ 0 < \sigma_1 < 1,\ \ 0 < \sigma_2 < 1$ *and* $0 < \sigma_1 + \sigma_2 < \frac{1}{2}$. *Then, for any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + \begin{cases} O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) & if \quad T^{\frac{1-2\sigma_1-2\sigma_2}{3-2\sigma_1-2\sigma_2}} \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}, \\ O\left(t_2^{-1} T^{3-2\sigma_1-2\sigma_2}\right) & if \quad 2 \leqslant t_2 \leqslant T^{\frac{1-2\sigma_1-2\sigma_2}{3-2\sigma_1-2\sigma_2}}. \end{cases} \tag{1.13}$$

Remark 2. An average order of magnitude for the double zeta-function (1.1) derived from the asymptotic behaviour of the integrals of the double zeta-function $\zeta_2(s_1, s_2)$ with (1.9)–(1.13) holds when the ratio of the order of t_1 to that of t_2 is small. However, it is often difficult to determine their analytic behaviour for a general ratio of the order of t_1 to the order of t_2.

Sometimes it is more fruitful to study the integrals for the double zeta-function $\zeta_2(s_1, s_2)$ for $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$.

Furthermore, the asymptotic behaviour of the Ω result of the double zeta-function (1.1) was derived by Kiuchi, Tanigawa, and Zhai [10], who showed that for $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_1 + \sigma_2 \leqslant 1$ and

$$t_2 \ll t_1^{\frac{3-2\sigma_1-2\sigma_2}{7-2\sigma_1-2\sigma_2}-\varepsilon}, \tag{1.14}$$

then we have

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) \asymp \frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2} \tag{1.15}$$

with $\sigma_1 + \sigma_2 < 1$, and

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) = \Omega\left(\frac{t_1^{\frac{1}{2}} \log\log t_1}{t_2}\right) \tag{1.16}$$

with $\sigma_1 + \sigma_2 = 1$. For example, if we can take $t_2 \ll t_1^{\frac{1}{6}-\varepsilon}$, then (1.16) implies that

$$\zeta_2\left(\frac{1}{2} + it_1, \frac{1}{2} + it_2\right) = \Omega\left(t_1^{\frac{1}{3}+\varepsilon}\right).$$

In view of this, it is of some interest to try to determine the Ω result of the double zeta-function under the other region. From Theorems 1, 2, and 3, we are able to expand the region $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_1 + \sigma_2 \leqslant 1$ of their result into $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < \frac{3}{2}$, and also improve a relation between t_1 and t_2 with the inequality (1.14). Thus we deduce the following Ω results.

Corollary 2. *Suppose that $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, and $2 \leqslant t_1 \leqslant T$. Then we have*

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) = \Omega\left(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\right) \tag{1.17}$$

with $0 < \sigma_1 + \sigma_2 < 1$ and $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}-\varepsilon}$, and

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) = \Omega\left(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\right) \tag{1.18}$$

with $1 \leqslant \sigma_1 + \sigma_2 < \frac{3}{2}$ and $2 \leqslant t_2 \leqslant T^{1-\frac{2}{3}(\sigma_1+\sigma_2)-\varepsilon}$, where ε is any small positive constant.

Remark 3. For the region $T^{\frac{3-2\sigma_1-2\sigma_2}{7-2\sigma_1-2\sigma_2}-\varepsilon} \ll t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}-\varepsilon}$, the Ω result of (1.17) holds, hence this observation expands (1.15) into (1.17). For the case of $\sigma_1+\sigma_2=1$, the formula (1.16) holds for $t_2 \ll t_1^{\frac{1}{5}-\varepsilon}$, but the formula (1.18) holds for $t_2 \leqslant T^{\frac{1}{3}-\varepsilon}$. Furthermore, in the case where $1 < \sigma_1+\sigma_2 < \frac{3}{2}$, the formula (1.18) holds for $2 \leqslant t_2 \leqslant T^{1-\frac{2}{3}(\sigma_1+\sigma_2)-\varepsilon}$. Therefore, this corollary implies an improvement upon the result of Kiuchi, Tanigawa and Zhai.

Remark 4. Now, we assume that $0<\sigma_1<1$, $0<\sigma_2<1$, $\sigma_1+\sigma_2=1$, $2<t_1<T$ and $2<t_2<T^{\frac{1}{2}}$. If the order of magnitude of the function $E(s_1,s_2)$ below is estimated as $O\left(t_2^{\varepsilon}\right)$ with ε being any small positive constant by more elaborate analysis. Then, the formula (1.9) can be improved by

$$\int_2^T |\zeta_2(s_1,s_2)|^2\,dt_1 = \frac{\zeta(2)}{4\pi|s_2-1|^2}\,T^2 + O\left(t_2^{-1+\varepsilon}T^{\frac{3}{2}}\right) \tag{1.19}$$

for any sufficiently large positive number $T>2$. Taking $t_2 = T^{\frac{1}{2}-\varepsilon}$ into (1.19), we obtain the estimation

$$\int_2^T |\zeta_2(s_1,s_2)|^2\,dt_1 = O\left(T^{1+\varepsilon}\right). \tag{1.20}$$

Notations. When $g(x)$ is a positive function of x for $x \geqslant x_0$, $f(x)=\Omega(g(x))$ means that $f(x)=o(g(x))$ does not hold as $x\to\infty$. In what follows, ε denotes any arbitrarily small positive number, not necessarily the same ones at each occurrence.

2. Some lemmas

Let a be any complex number. The generalized divisor function is defined by $\sigma_a(n) = \sum_{d|n} d^a$. We use a weak form of the approximate formula of Kiuchi, Tanigawa, and Zhai [10] to prove our theorems. Suppose that $0<\sigma_1<1$ and $0<\sigma_2<1$. They showed that

$$\begin{aligned}\zeta_2(s_1,s_2) &= \frac{\zeta(s_1+s_2-1)}{s_2-1} - \frac{1}{2}\zeta(s_1+s_2)\\ &\quad + \chi(s_2)\sum_{n\leqslant\frac{|t_2|}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O\left(|t_2|^{\max(0,1-\sigma_1-\sigma_2)+\varepsilon}\right)\end{aligned} \tag{2.1}$$

where

$$\chi(s_2) = 2(2\pi)^{s_2-1}\sin\left(\frac{\pi}{2}s_2\right)\Gamma\left(1-s_2\right).$$

For our purpose, it is enough to quote the following weak form of (2.1):

$$\zeta_2(s_1,s_2) = \frac{\zeta(s_1+s_2-1)}{s_2-1} - \frac{1}{2}\zeta(s_1+s_2) + E(s_1,s_2) \tag{2.2}$$

with the error term $E(s_1, s_2)$. Using the partial summation formula, the estimate $|\chi(s_2)| \asymp |t_2|^{\frac{1}{2}-\sigma_2}$ and the partial sums of the generalized divisor function:

$$\sum_{n\leqslant x}\sigma_\alpha(n) = \begin{cases} \frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1} + O\left(x^{\max(1,\alpha)}\right) & \text{if } \alpha > 0, \\ x\log x + O(x) & \text{if } \alpha = 0, \\ O(x) & \text{if } \alpha < 0 \end{cases}$$

for any sufficiently large positive number $x > 2$, we have the following.

Lemma 1. *Let the notation be as above. We have*

$$E(s_1, s_2) \ll \begin{cases} |t_2|^{\frac{3}{2}-\sigma_1-\sigma_2} & \text{if } 0 < \sigma_1 + \sigma_2 < 1, \\ |t_2|^{\frac{1}{2}}\log|t_2| & \text{if } \sigma_1 + \sigma_2 = 1, \\ |t_2|^{\frac{1}{2}} & \text{if } \sigma_1 + \sigma_2 > 1. \end{cases} \tag{2.3}$$

Note that the error term in this lemma is independent of t_1. Here we choosed ε in (2.1) as $0 < \varepsilon < 1/2$. This error term $E(s_1, s_2)$ is obviously **independent of** t_1. To prove our theorem, we shall establish the mean square formula for $\zeta_2(s_1, s_2)$ by using (2.2), (2.3) and Schwarz's inequality, thus we have the following.

Lemma 2. *For $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, and any sufficiently large number $T > 2$, we have*

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 = I_1 + I_2 + I_3 + O\left(I_1^{\frac{1}{2}}I_2^{\frac{1}{2}} + I_2^{\frac{1}{2}}I_3^{\frac{1}{2}} + I_3^{\frac{1}{2}}I_1^{\frac{1}{2}}\right), \tag{2.4}$$

where

$$I_1 = \frac{1}{|s_2-1|^2}\int_2^T |\zeta(s_1+s_2-1)|^2 dt_1, \tag{2.5}$$

$$I_2 = \frac{1}{4}\int_2^T |\zeta(s_1+s_2)|^2 dt_1, \tag{2.6}$$

and

$$I_3 = \int_2^T |E(s_1, s_2)|^2 dt_1. \tag{2.7}$$

To deal with the integrals I_j $(j = 1, 2, 3)$, we shall use the mean square formula of the Riemann zeta-function.

Lemma 3. *For any sufficiently large positive number $T > 2$, we have*

$$\int_2^T |\zeta(\sigma+it)|^2 dt = (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} + \zeta(2\sigma)T + O\left(T^{1-\sigma}\right) \tag{2.8}$$

with $0<\sigma<\frac{1}{2}$,

$$\int_2^T |\zeta\,(it)\,|^2 dt = \frac{\zeta(2)}{4\pi}T^2 + \frac{1}{2\pi}R(T)T + O(T) \tag{2.9}$$

with $\sigma=0$, *and*

$$\int_2^T |\zeta\,(\sigma+it)\,|^2 dt = (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} + O\left(T^{1-2\sigma}\right) \tag{2.10}$$

with $-1<\sigma<0$, *where* $R(T)$ *is given by* (1.7).

Proof. Since the functional equation $\zeta(\sigma+it)=\chi(\sigma+it)\zeta(1-\sigma-it)$, the equality $|\zeta(1-\sigma-it)|=|\zeta(1-\sigma+it)|$ and the formula (see (1.25) in [6])

$$|\chi(\sigma+it)|^2 = \left(\frac{t}{2\pi}\right)^{1-2\sigma} + O\left(t^{-2\sigma}\right) \qquad (t \geqslant t_0 > 0),$$

we can integrate by parts and use (1.6) to obtain

$$\begin{aligned}\int_2^T |\zeta(\sigma+it)|^2 dt &= \int_2^T |\chi(\sigma+it)|^2|\zeta(1-\sigma+it)|^2 dt \\ &= \left(\frac{1}{2\pi}\right)^{1-2\sigma}\int_2^T t^{1-2\sigma}|\zeta(1-\sigma+it)|^2 dt \\ &\quad + O\left(\int_2^T t^{-2\sigma}|\zeta(1-\sigma+it)|^2 dt\right) \\ &= (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} + \zeta(2\sigma)T + O\left(T^{1-\sigma}\right)\end{aligned}$$

with $E_{1-\sigma}(T)=O\left(T^{\sigma}\right)$ for $0<\sigma<\frac{1}{2}$. Similarly in the case of $\sigma=0$, we have, by integrating by parts and using (1.7), (1.8),

$$\begin{aligned}\int_2^T |\zeta(it)|^2 dt &= \int_2^T |\chi(it)|^2|\zeta(1+it)|^2 dt \\ &= \frac{1}{2\pi}\int_2^T t|\zeta(1+it)|^2 dt + O\left(\int_2^T |\zeta(1+it)|^2 dt\right) \\ &= \frac{\zeta(2)}{4\pi}T^2 + \frac{1}{2\pi}R(T)T + O(T).\end{aligned}$$

In a similar manner, we have,

$$\begin{aligned}\int_2^T |\zeta(\sigma+it)|^2 dt &= \int_2^T |\chi(\sigma+it)|^2 |\zeta(1-\sigma+it)|^2 dt \\ &= \left(\frac{1}{2\pi}\right)^{1-2\sigma} \int_2^T t^{1-2\sigma} |\zeta(1-\sigma+it)|^2 dt \\ &\quad + O\left(\int_2^T t^{-2\sigma} |\zeta(1-\sigma+it)|^2 dt\right) \\ &= (2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{2-2\sigma} T^{2-2\sigma} + O\left(T^{1-2\sigma}\right)\end{aligned}$$

for $-1 < \sigma < 0$, which gives the formula (2.10). ∎

3. Proofs of Theorem 1 and Corollary 1

We evaluate the integral $\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1$ under the condition $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $\sigma_1 + \sigma_2 = 1$, and $2 \leqslant t_2 \leqslant \frac{T^{\frac{1}{3}}}{\log T}$. From (1.7), we have

$$I_2 = \frac{1}{4}\left\{\int_2^{T+t_2} |\zeta(1+it)|^2 dt - \int_2^{2+t_2} |\zeta(1+it)|^2 dt\right\} = O(T+t_2). \tag{3.1}$$

Similarly, we have

$$I_1 = \frac{1}{|s_2-1|^2}\left\{\int_2^{T+t_2} |\zeta(it)|^2 dt - \int_2^{2+t_2} |\zeta(it)|^2 dt\right\}. \tag{3.2}$$

Inserting (2.9) into (3.2), we obtain

$$\begin{aligned} I_1 &= \frac{\zeta(2)}{4\pi|s_2-1|^2}\left\{(T+t_2)^2 - (t_2+2)^2\right\} \\ &\quad + \frac{1}{2\pi|s_2-1|^2}\left\{(T+t_2)R(T+t_2) - (t_2+2)R(t_2+2)\right\} + O\left(\frac{T+t_2}{|s_2-1|^2}\right) \\ &= \frac{\zeta(2)}{4\pi|s_2-1|^2}(T+t_2)^2 + O\left(\frac{T(\log T)^{2/3}(\log\log T)^{1/3}}{t_2^2}\right) + O\left(\frac{T}{t_2^2}\right). \end{aligned} \tag{3.3}$$

We have

$$I_3 = O\left((t_2 \log^2 t_2) T\right). \tag{3.4}$$

Substituting (3.1), (3.3) and (3.4) into (2.4), we observe that all error terms on the right-hand side of (2.4) are absorbed into $O\left(t_2^{-\frac{1}{2}}(\log t_2) T^{\frac{3}{2}}\right)$, completing the proof of (1.9).

Next, as an application of (1.9), we evaluate the double integral for the double zeta-function $\zeta_2(s_1, s_2)$: $\int_2^N \int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 dt_2$. From (1.9), we have

$$\int_2^N \int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 dt_2$$
$$= \frac{\zeta(2)}{4\pi}\left(\int_2^N \frac{1}{|s_2-1|^2} dt_2\right) T^2 + O\left(T^{\frac{3}{2}} \int_2^N t_2^{-\frac{1}{2}} \log t_2 dt_2\right)$$

for $2 \leqslant N \leqslant \frac{T^{\frac{1}{3}}}{\log T}$. It follows that for $0 < \sigma_2 < 1$

$$\int_2^N \frac{1}{|s_2-1|^2} dt_2 = \frac{1}{1-\sigma_2}\left(\frac{\pi}{2} - \mathrm{Tan}^{-1}\frac{2}{1-\sigma_2}\right) + O\left(\frac{1}{N}\right)$$

and

$$\int_2^N t_2^{-\frac{1}{2}} \log t_2 dt_2 = O\left(N^{\frac{1}{2}} \log N\right).$$

Then, we easily see that

$$\int_2^N \int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 dt_2 = \frac{\pi}{24(1-\sigma_2)}\left(\frac{\pi}{2} - \mathrm{Tan}^{-1}\frac{2}{1-\sigma_2}\right) T^2$$
$$+ O\left(T^{\frac{3}{2}} N^{\frac{1}{2}} \log N\right) + O\left(\frac{T^2}{N}\right).$$

Hence, for $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $\sigma_1 + \sigma_2 = 1$ and $2 \leqslant N \leqslant \frac{T^{\frac{1}{3}}}{\log T}$, we obtain, as $T \to \infty$

$$\frac{1}{T^2}\int_2^N \int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 dt_2 \to \frac{\pi}{24\sigma_1}\left(\frac{\pi}{2} - \mathrm{Tan}^{-1}\frac{2}{\sigma_1}\right) + O\left(\frac{1}{N}\right).$$

Therefore, we obtain the assertion of Corollary 1.

4. Proof of Theorem 2

Throughout this section, we assume that $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $\frac{1}{2} \leqslant \sigma_1 + \sigma_2 < \frac{3}{2}$, $\sigma_1 + \sigma_2 \neq 1$ and $2 \leqslant t_2 \leqslant T^{\frac{2}{3}}$. As in the proof of Theorem 1, we shall first evaluate the integral $\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1$ for $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$. Using (1.4) and (2.6), we obtain

$$I_2 = \frac{1}{4}\int_2^T |\zeta(s_1+s_2)|^2 dt_1 = \frac{1}{4}\zeta(2(\sigma_1+\sigma_2))T + O(t_2). \tag{4.1}$$

From (2.3) and (2.7), we have

$$I_3 = O\left(t_2 T\right). \tag{4.2}$$

By (2.5) and (2.8), we have

$$
\begin{aligned}
I_1 &= \frac{1}{|s_2-1|^2}\int_2^T |\zeta(\sigma_1+\sigma_2-1+i(t_1+t_2))|^2\,dt_1 \\
&= \frac{(2\pi)^{2\sigma_1+2\sigma_2-3}}{|s_2-1|^2}\frac{\zeta(4-2\sigma_1-2\sigma_2)}{4-2\sigma_1-2\sigma_2}T^{4-2\sigma_1-2\sigma_2} \\
&\quad + O\left(t_2^{-1}T^{3-2\sigma_1-2\sigma_2}\right) + O\left(t_2^{-2}T\right).
\end{aligned} \tag{4.3}
$$

Now we assume that $2 \leqslant t_2 \leqslant T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$. Substituting (4.1), (4.2) and (4.3) into (2.4), we observe that all error terms on the right-hand side of (2.4) are absorbed into $O\left(t_2^{-\frac{1}{2}}T^{\frac{5}{2}-\sigma_1-\sigma_2}\right)$. Hence, we derive the formula (1.10).

Similarly, in the case of $\frac{1}{2}<\sigma_1+\sigma_2<1$, we obtain

$$
I_2 = \frac{1}{4}\zeta(2(\sigma_1+\sigma_2))T + A(\sigma_1+\sigma_2)T^{2-2\sigma_1-2\sigma_2} + O\left(t_2\right) \tag{4.4}
$$

by using (1.6) and (2.6). From (2.3) and (2.7), we have

$$
I_3 = O\left(t_2^{3-2\sigma_1-2\sigma_2}T\right). \tag{4.5}
$$

From (2.5) and (2.10), we have

$$
I_1 = \frac{(2\pi)^{2\sigma_1+2\sigma_2-3}}{|s_2-1|^2}\frac{\zeta(4-2\sigma_1-2\sigma_2)}{4-2\sigma_1-2\sigma_2}T^{4-2\sigma_1-2\sigma_2} + O\left(t_2^{-1}T^{3-2\sigma_1-2\sigma_2}\right). \tag{4.6}
$$

Now we assume that $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. Substituting (4.4), (4.5) and (4.6) into (2.4), we observe that all error terms on the right-hand side of (2.4) are absorbed into $O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2}T^{\frac{5}{2}-\sigma_1-\sigma_2}\right)$. Hence, we derive the formula (1.11).

Similarly, in the case of $\sigma_1+\sigma_2=\frac{1}{2}$, we obtain, by (1.5) and (2.6)

$$
I_2 = T\log\frac{T}{2\pi} + (2\gamma-1)T + O\left(t_2\log T + T^{\frac{1}{2}}\right). \tag{4.7}
$$

From (2.3) and (2.7), we have

$$
I_3 = O\left(t_2^2 T\right). \tag{4.8}
$$

From (2.5) and (2.10), we have

$$
I_1 = \frac{\zeta(3)}{12\pi^2|s_2-1|^2}T^3 + O\left(t_2^{-1}T^2\right). \tag{4.9}
$$

Now we assume that $2 \leqslant t_2 \leqslant T^{\frac{1}{2}}$. Substituting (4.7), (4.8) and (4.9) into (2.4), we observe that all error terms on the right-hand side of (2.4) are absorbed into $O\left(t_2^{-1}T^2\sqrt{\log T}\right)$ if $2 \leqslant t_2 \leqslant \sqrt{\log T}$, or into $O\left(T^2\right)$ if $\sqrt{\log T} \leqslant t_2 \leqslant T^{\frac{1}{2}}$. Hence, we obtain the formula (1.12).

5. Proof of Theorem 3

Let $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $2 \leqslant t_2 \leqslant T^{\frac{2}{3}}$. As in the proof of Theorem 1, we shall evaluate the integral $\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1$ for $0 < \sigma_1 + \sigma_2 < \frac{1}{2}$. From (2.6) and (2.8), we have

$$I_2 = O(T^{2-2\sigma_1-2\sigma_2}). \tag{5.1}$$

By (2.3) and (2.7), we have

$$I_3 = O\left(t_2^{3-2\sigma_1-2\sigma_2} T\right). \tag{5.2}$$

By (2.5) and (2.10), we obtain

$$I_1 = \frac{(2\pi)^{2\sigma_1+2\sigma_2-3}}{|s_2-1|^2} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{4-2\sigma_1-2\sigma_2} T^{4-2\sigma_1-2\sigma_2} + O\left(t_2^{-1} T^{3-2\sigma_1-2\sigma_2}\right). \tag{5.3}$$

Now we assume that $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. Substituting (5.1), (5.2) and (5.3) into (2.4), we observe that all error terms on the right-hand side of (2.4) are absorbed into

$$O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) \qquad \text{if } T^{\frac{1-2\sigma_1-2\sigma_2}{3-2\sigma_1-2\sigma_2}} \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}},$$

or into

$$O\left(t_2^{-1} T^{3-2\sigma_1-2\sigma_2}\right) \qquad \text{if } 2 \leqslant t_2 \leqslant T^{\frac{1-2\sigma_1-2\sigma_2}{3-2\sigma_1-2\sigma_2}}.$$

Therefore, we obtain the formula (1.13).

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