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SET OF UNIQUENESS OF SHIFTED GAUSSIAN PRIMES

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Abstract: In this paper, we show that any additive complex valued function over non-zero Gaussian integers which vanishes on the shifted Gaussian primes is necessarily identically zero.

Keywords: additive functions, shifted Gaussian primes, set of uniqueness.

1. Introduction

A function $f : \mathbb{N} \to \mathbb{R}$ is said to be additive if

$$f(mn) = f(m) + f(n) \tag{1}$$

for all $m, n \in \mathbb{N}$ with $(m, n) = 1$ and is said to be completely additive if (1) holds for all $m, n \in \mathbb{N}$. Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the set of all such additive and completely additive functions respectively.

A set $A \subset \mathbb{N}$ is said to be a *set of uniqueness* for additive (or completely additive) functions if for all $f \in \mathcal{A}$ (or $\mathcal{A}^*$), we have

$$f(A) = \{0\} \Rightarrow f(\mathbb{N}) = \{0\} \quad \text{i.e.} \quad f \equiv 0.$$

The notion of the set of uniqueness was introduced by I. Kátai (see [10]).

A set $A \subset \mathbb{N}$ is said to be a *set of quasi-uniqueness* if there exists a suitable finite set $B \subset \mathbb{N}$ such that $A \cup B$ is a set of uniqueness. Let $\mathcal{P}$ denote the set of rational primes. I. Kátai [10] proved that the set $\mathcal{P} + 1 := \{p + 1 \mid p \in \mathcal{P}\}$ is a set of quasi-uniqueness assuming validity of the Riemann-Piltz conjecture. In [11], he again proved the same result without using any unproven hypothesis and conjectured that $\mathcal{P} + 1$ is in fact a set of uniqueness. Using sieve methods, P.D.T.A. Elliott proved a stronger result ([1], Theorem 2) that settled the conjecture of Kátai completely.

D. Wolke [16] proved that, in the case of completely additive functions, every $n \in \mathbb{N}$ can be expressed as a finite product of rational powers of elements of a set

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of uniqueness A. More precisely, a set $A \subset \mathbb{N}$ is a set of uniqueness with respect to completely additive functions if and only if each positive integer n can be written as

$$n = a_1^{r_1} \cdots a_k^{r_k}; \qquad (a_i \in A,\ r_i \in \mathbb{Q},\ 1 \leqslant i \leqslant k,\ k \in \mathbb{N}_0).$$

Many other interesting results related to behaviour of arithmetical functions at shifted primes can be found in the literature, for example Hildebrand [7], Elliott [2, 3], Wirsing [15], etc.

J. Mehta and G. K. Viswanadham [12] extended the notion of the set of uniqueness for completely additive complex valued functions over non-zero Gaussian integers. The authors proved the set $P[i]+1 := \{p+1 \mid p \in \mathcal{P}[i]\}$, where $\mathcal{P}[i]$ denote the set of all Gaussian primes, is a set of quasi-uniqueness for completely additive complex valued functions over the set of Gaussian integers. However their proof can be made to work for any shift k, $k \in \mathbb{Z}[i]$, by choosing the finite set of Gaussian primes appropriately.

In this paper, we prove a stronger result which would imply that the set of shifted Gaussian primes is a set of uniqueness for additive functions. More precisely, we have the following theorem:

Theorem 1. *Let $N \in \mathbb{N}$ and let $f : \mathbb{Z}[i]^* \to \mathbb{C}$ be an additive function such that $f(p+1) = 0$ for all Gaussian primes p with $\mathrm{N}(p) \geqslant N$, then $f \equiv 0$ (f is identically zero).*

The basic idea of the proof of the above theorem comes from Elliott [1]. Note that the above theorem is very strong in the sense that here we consider just additive functions instead of completely additive functions. Further, we assume that f vanishes only on Gaussian primes with sufficiently large norm rather than assuming on all shifted primes.

Corollary 1. *The set $\mathcal{P}[i] + 1$ is a set of uniqueness for additive functions over $\mathbb{Z}[i]^*$.*

As a consequence of Theorem 1, along the lines of D. Wolke's result mentioned earlier, we have the following corollary:

Corollary 2. *Every $\alpha \in \mathbb{Z}[i]^*$ can be written in the following form:*

$$\alpha = \prod_{j=1}^{k} (p_j + 1)^{l_j},$$

where $p_j \in \mathcal{P}[i]$ and $l_j \in \mathbb{Q}$.

2. Preliminaries

In this section, we will state some lemmas which will be used in the proof of the main theorem.

Let Φ denote the Euler's phi function for the ring of Gaussian integers, defined by $\Phi(\alpha) := \#\left(\mathbb{Z}[i]/(\alpha)\right)^*$, for $\alpha \in \mathbb{Z}[i]^*$. One can see that

$$\Phi(\alpha) = \mathrm{N}(\alpha) \prod_{\substack{p|\alpha \\ p\in\mathcal{P}[i]}} \left(1 - \frac{1}{\mathrm{N}(p)}\right).$$

Throughout this paper, we assume that for any additive function f over $\mathbb{Z}[i]^*$ and for any unit ϵ in $\mathbb{Z}[i]$, $f(\epsilon) = 0$. Let $\overline{\mu}$ denote the Möbius function on Gaussian integers defined in the same way as the standard Möbius function μ. Let $\pi_{\mathbb{Q}[i]}(x)$ denote the number of Gaussian primes p with $\mathrm{N}(p) \leqslant x$ and we have

$$\pi_{\mathbb{Q}[i]}(x) = 1 + 2\pi(x, 1, 4) + \pi_(\sqrt{x}, -1, 4) \sim \frac{x}{\log x}\ , \tag{2}$$

where $\pi(x, a, q)$ denotes the number of rational primes $p \leqslant x$ such that $p \equiv a \pmod{q}$. For $d, l \in \mathbb{Z}[i]$ such that $(d, l) = 1$ and $x \in \mathbb{R}$, let

$$\pi_{\mathbb{Q}[i]}(x, d, l) := \sum_{\substack{p\in\mathcal{P}[i], \mathrm{N}(p)\leqslant x \\ p\equiv d \pmod{l}}} 1.$$

Then for each non-zero Gaussian integer d, clearly

$$\pi_{\mathbb{Q}[i]}(x, d, l) \ll 1 + \frac{x}{d}. \tag{3}$$

Let $E^*(x, d) := \sup_{(l,d)=1} \sup_{y\leqslant x} \left|\pi_{\mathbb{Q}[i]}(y, d, l) - \frac{Li(y)}{\Phi(d)}\right|$. Then clearly,

$$E^*(x, d) \ll 1 + \frac{x}{\Phi(d)}. \tag{4}$$

Lemma 1. $\sum_{\mathrm{N}(d)\leqslant x^{\frac{1}{5}} (\log x)^3} E^*(x, d) \ll x(\log x)^{-5}$.

Proof. The proof follows from the corollary of Theorem 3 in [9]. ■

Lemma 2. *Let $M(x, k)$ denote the number of pairs of primes (p, q) satisfying the conditions $p + 1 = kq$, $\mathrm{N}(p) \leqslant x$. Then*

$$M(x, k) \ll \frac{x}{\Phi(k) \log^2 x}.$$

Proof. See Lemma 2.1. of [12]. ■

Lemma 3. *Let $0 < \delta < 1$. Then for any $d \in \mathbb{Z}[i]$ with $\mathrm{N}(d) \leqslant x^{1-\delta}$,*

$$\pi_{\mathbb{Q}[i]}(x, d, -1) \ll_\delta \frac{x}{\Phi(d) \log x}.$$

Proof. The proof is a particular case of Theorem 4 in [8]. ■

Let x be a sufficiently large positive real number and let d be a non-zero Gaussian integer. Let

$$N(d,x) = \#\left\{(p,q) \in \mathcal{P}[i] \times \mathcal{P}[i] \,\middle|\, \substack{p+1=d(q+1),\\ (d,q+1)=1\\ \mathrm{N}(p)\leqslant x,\ x^{\frac{1}{6}}<\mathrm{N}(q)<x^{\frac{1}{5}}}\right\}. \tag{5}$$

Lemma 4. *For all Gaussian integers d with $\mathrm{N}(d) \leqslant x^{\frac{5}{6}}$, we have*

$$N(d,x) \ll \frac{x}{\mathrm{N}(d)(\log x)^2} \prod_{p \mid d(d-1)} \left(1 - \frac{1}{\mathrm{N}(p)}\right)^{-1}.$$

Proof. By using the Selberg sieve for algebraic number fields in [13], one can easily deduce the lemma along the lines of Theorem 2.3 in [6]. ■

3. Proof of Theorem 1

Let $N(d,x)$ be as in (5). Before we begin the proof of the theorem we obtain the following estimate.

Lemma 5.

$$\sum_{\substack{x^{\frac{3}{5}}<\mathrm{N}(d)<x^{\frac{5}{6}}\\ N(d,x)>0}} \frac{1}{\mathrm{N}(d)} \gg \log x.$$

Proof. We have

$$\begin{aligned}
\sum_{x^{\frac{3}{5}}<\mathrm{N}(d)<x^{\frac{5}{6}}} N(d,x) &\geqslant \sum_{x^{\frac{1}{6}}<\mathrm{N}(q)<x^{\frac{1}{5}}} \sum_{\substack{x^{\frac{4}{5}}<\mathrm{N}(p)<x\\ p\equiv -1 \pmod{q+1}\\ \left(\frac{p+1}{q+1},q+1\right)=1}} 1 \\
&\geqslant \sum_{x^{\frac{1}{6}}<\mathrm{N}(q)<x^{\frac{1}{5}}} \sum_{\substack{\mathrm{N}(p)<x\\ p\equiv -1 \pmod{q+1}}} \sum_{r\mid\left(\frac{p+1}{q+1},q+1\right)} \overline{\mu}(r) \\
&\quad - \pi_{\mathbb{Q}[i]}(x^{\frac{1}{5}})\pi_{\mathbb{Q}[i]}(x^{\frac{4}{5}}) \\
&\geqslant \sum_{x^{\frac{1}{6}}<\mathrm{N}(q)<x^{\frac{1}{5}}} \sum_{r\mid(q+1)} \frac{\overline{\mu}(r)\,\mathrm{Li}(x)}{\Phi(r(q+1))} \\
&\quad - \sum_{x^{\frac{1}{6}}<\mathrm{N}(q)<x^{\frac{1}{5}}} \sum_{r\mid(q+1)} E^*(x, r(q+1)) + O(x(\log x)^{-2}) \\
&= \sum{}' - \sum{}'' + O(x(\log x)^{-2}).
\end{aligned}$$

Now, we determine lower bound and upper bound for $\sum'$ and $\sum''$ respectively.

$$\begin{aligned}
\sum{}' &= \mathrm{Li}(x) \sum_{x^{\frac{1}{6}}<\mathrm{N}(q)<x^{\frac{1}{5}}} \sum_{r|(q+1)} \frac{1}{\mathrm{N}(r)\,\mathrm{N}(q+1)\prod_{p|r(q+1)}\left(1-\frac{1}{\mathrm{N}(p)}\right)} \\
&= \mathrm{Li}(x) \sum_{x^{\frac{1}{6}}<\mathrm{N}(q)<x^{\frac{1}{5}}} \frac{1}{\mathrm{N}(q+1)} \sum_{r|(q+1)} \frac{\overline{\mu}(r)}{\mathrm{N}(r)\prod_{p|(q+1)}\left(1-\frac{1}{\mathrm{N}(p)}\right)} \\
&\geqslant \mathrm{Li}(x) \sum_{x^{\frac{1}{6}}<\mathrm{N}(q)<x^{\frac{1}{5}}} \frac{1}{\mathrm{N}(q+1)} \\
&= \mathrm{Li}(x)\log\frac{6}{5} + O\left(\frac{x}{(\log x)^2}\right). \qquad (6)
\end{aligned}$$

By using Lemma 1 and (4), we have

$$\begin{aligned}
\sum{}'' &= \sum_{\mathrm{N}(r)<(\log x)^3} \sum_{x^{\frac{1}{6}}<\mathrm{N}(q)<x^{\frac{1}{5}}} E^*(x, r(q+1)) \\
&\quad + \sum_{\substack{\mathrm{N}(rs)<x^{\frac{1}{5}} \\ \mathrm{N}(r)>(\log x)^3}} \left(1+\frac{x}{\Phi(r^2 s)}\right) + \frac{x}{(\log x)^2} \\
&\ll \frac{x}{(\log x)^2} + x(\log\log x)^2 \sum_{\mathrm{N}(s)<x^{\frac{1}{5}}} \frac{1}{\mathrm{N}(s)} \sum_{y>\mathrm{N}(s)>(\log x)^3} \frac{1}{\mathrm{N}(r^2)} \\
&\ll x(\log x)^{-\frac{3}{2}}, \qquad (7)
\end{aligned}$$

since $\Phi(m) \gg \frac{m}{(\log\log m)^2}$ for m with sufficiently large norm. Hence,

$$\sum_{x^{\frac{3}{5}}<\mathrm{N}(d)<x^{\frac{5}{6}}} N(d,x) \gg \frac{x}{\log x}. \qquad (8)$$

Applying Cauchy-Schwarz inequality we have,

$$\sum_{x^{\frac{3}{5}}<\mathrm{N}(d)<x^{\frac{5}{6}}} N(d,x) \leqslant \left(\sum_{\substack{x^{\frac{3}{5}}<\mathrm{N}(d)<x^{\frac{5}{6}} \\ N(d,x)>0}} \frac{1}{\mathrm{N}(d)}\right)^{\frac{1}{2}} \left(\sum_{2\leqslant\mathrm{N}(d)\leqslant x^{\frac{5}{6}}} \mathrm{N}(d)N^2(d,x)\right)^{\frac{1}{2}}.$$

If we assume that

$$\sum_{2\leqslant\mathrm{N}(d)\leqslant x^{\frac{5}{6}}} \mathrm{N}(d)N^2(d,x) \ll \frac{x^2}{(\log x)^3}, \qquad (9)$$

then we have

$$\frac{x}{(\log x)} \ll \sum_{x^{\frac{3}{5}}<\mathrm{N}(d)<x^{\frac{5}{6}}} N(d,x) \ll \left(\sum_{\substack{x^{\frac{3}{5}}<\mathrm{N}(d)<x^{\frac{5}{6}} \\ \mathrm{N}(d,x)>0}} \frac{1}{\mathrm{N}(d)} \right)^{\frac{1}{2}} \frac{x}{(\log x)^{\frac{3}{2}}}.$$

Hence, the lemma. So it suffices to prove (9). Define

$$\eta(d) = \mathrm{N}(d)^{\frac{1}{2}} \prod_{p|d} \left(1 - \frac{1}{\mathrm{N}(p)}\right)^{-1}.$$

Then by Lemma 4, we have

$$\sum_{2<\mathrm{N}(d)\leqslant x^{\frac{5}{6}}} \mathrm{N}(d) N^2(d,x) \ll \frac{x^2}{(\log x)^4} \sum_{2\leqslant \mathrm{N}(d)\leqslant x^{\frac{5}{6}}} \eta(d)\eta(d-1).$$

Applying Cauchy-Schwarz inequality again and since $\eta(d)$ is multiplicative, we have

$$\begin{aligned}
\sum_{2<\mathrm{N}(d)\leqslant x} \eta(d)\eta(d-1) &\leqslant \left(\sum_{\mathrm{N}(d)\leqslant x} \eta^2(d) \right)^{\frac{1}{2}} \left(\sum_{2\leqslant \mathrm{N}(d)\leqslant x} \eta^2(d-1) \right)^{\frac{1}{2}} \\
&\leqslant \prod_{\mathrm{N}(p)\leqslant x} (1 + \eta^2(p) + \eta^2(p^2) + \cdots) \\
&= \prod_{\mathrm{N}(p)\leqslant x} \left(1 + \frac{1}{\mathrm{N}(p)} \left(1 - \frac{1}{\mathrm{N}(p)}\right)^{-5} \right) \\
&\ll \left(\prod_{\substack{p\in\mathcal{P},\ p\leqslant x \\ p\equiv 1 \pmod 4}} \left(1 + \frac{1}{p}\right)^2 \right)^{\frac{1}{2}} \prod_{\substack{p\in\mathcal{P},\ p\leqslant x \\ p\equiv 3 \pmod 4}} \left(1 + \frac{1}{p^2}\right) \\
&\ll \left(\log^2 x\right)^{\frac{1}{2}} = \log x. \qquad \blacksquare
\end{aligned}$$

Now, we give the proof of Theorem 1. Define

$$D(u) = \sum_{\substack{x^{\frac{3}{5}}<\mathrm{N}(d)\leqslant y \\ \mathrm{N}(d,x)>0}} 1, \qquad \beta = \sup_{x^{\frac{3}{5}}<y\leqslant x^{\frac{5}{6}}} \frac{D(y)}{y}. \tag{10}$$

Now, using Lemma 5 and integration by parts, we have

$$\begin{aligned}\log x &\ll \sum_{\substack{x^{\frac{3}{5}}<\mathrm{N}(d)<x^{\frac{5}{6}}\\ N(d,x)>0}} \frac{1}{\mathrm{N}(d)} = \sum_{\substack{x^{\frac{3}{5}}<n<x^{\frac{5}{6}}\\ N(d,x)>0\\ \mathrm{N}(d)=n}} \frac{a_n}{n} = \int_{x^{\frac{3}{5}}}^{x^{\frac{5}{6}}} y^{-1}dD(y)\\ &= x^{-\frac{5}{6}}D(x^{-\frac{5}{6}}) + \int_{x^{\frac{3}{5}}}^{x^{\frac{5}{6}}} D(y)y^{-2}dy\\ &\leqslant \beta + \beta\int_{x^{\frac{3}{5}}}^{x^{\frac{5}{6}}} y^{-1}dy = \beta\left(1+\left(\frac{5}{6}-\frac{3}{5}\right)\log x\right),\end{aligned}$$

where a_n denotes the number of d with $x^{\frac{3}{5}} < \mathrm{N}(d) < x^{\frac{5}{6}}$, $\mathrm{N}(d) = n$. From the definition of β it follows that there exists a constant $c_1 > 0$ such that in each interval $(x^{\frac{3}{5}}, x^{\frac{5}{6}}]$, there is a y_0 such that $D(y_0) > c_1y_0$. Choosing x aptly, we can find a sequence $\{y\}_n$ such that $D(y_n) > c_1y_n$. For each real number z and integer n, define

$$F_n(z) = \frac{1}{\pi\,\mathrm{N}(n)} \sum_{\substack{\mathrm{N}(m)\leqslant n\\ |f(m)|\leqslant z}} 1.$$

Lemma 6. *If $\sum_{f(p)\neq 0} \frac{1}{\mathrm{N}(p)}$ diverges, then for each z we have*

$$\limsup_{\delta\to 0^+}\limsup_{n\to\infty}(F_n(z+\delta) - F_n(z-\delta)) = 0.$$

Note: One can see that the converse is also true (see [4] for the classical case). Before we prove the lemma, we state the following theorem.

Proposition 1. *Let $f : \mathbb{Z}[i]^* \to \mathbb{C}$ be an additive function such that $\sum_{\substack{f(p)\neq 0\\ p\in\mathcal{P}[i]}} \frac{1}{\mathrm{N}(p)}$ diverges. Then to every $\epsilon > 0$, there exists a $0 < \delta < 1$ such that if $\{a_i\}_{i=1}^{x}$ is a sequence of Gaussian integers with $\mathrm{N}(a_1) < \mathrm{N}(a_2) < \cdots < \mathrm{N}(a_x) \leqslant n$ and $|f(a_i) - f(a_j)| < \delta$, then $x < \epsilon\pi n$ for sufficiently large n.*

The proof of the above proposition for the functions $f : \mathbb{N} \to \mathbb{C}$ was given by Erdös in [5]. The proof for the functions on non-zero Gaussian integers can be found in [14].

Proof of Lemma 6. Assume that

$$\limsup_{\delta\to 0^+}\limsup_{n\to\infty}(F_n(z+\delta) - F_n(z-\delta)) > 0. \tag{11}$$

Then we have to show that $\sum_{f(p)\neq 0} \frac{1}{\mathrm{N}(p)}$ is convergent. Suppose on the contrary $\sum_{f(p)\neq 0} \frac{1}{\mathrm{N}(p)}$ diverges. Since (11) holds, there exists a decreasing sequence $\delta_1 \geqslant \delta_2 \geqslant \cdots > 0$ and a sequence $z_1(\delta_1), z_2(\delta_2), \ldots$ such that

$$\limsup_{n\to\infty}(F_n(z_k+\delta_k) - F_n(z_k-\delta_k)) \geqslant \gamma > 0.$$

Thus, we obtain a further sequence of integers $n_1 < n_2 < \cdots$ so that n_l is sufficiently large and $F_{n_l}(z_k) - F_{n_l}(z_k) \geqslant \frac{\gamma}{2}$. Thus, the discs $1 \leqslant \mathrm{N}(m) \leqslant n_l$ contains at least $\pi\frac{\gamma}{2}n_l$ Gaussian integers a_i for which

$$|f(a_i) - f(a_j)| \leqslant \delta_k,$$

which is a contradiction to Proposition 1 if we take $\epsilon = \frac{\gamma}{2}$. ∎

Now, we continue the proof of our theorem. In our case we take $z = 0$, so for every $\delta > 0$,

$$\limsup_{n\to\infty}(F_n(\delta) - F_m(-\delta)) \geqslant \limsup_{y_0\to\infty} y_0^{-1}D(y_0) \geqslant c_1 > 0.$$

Thus from Lemma 6, it follows that $f(p) = 0$ for almost all Gaussian primes p. Let $\mathrm{N}(q_1) \leqslant \mathrm{N}(q_2) \leqslant \cdots$ denote the Gaussian primes q with $\mathrm{N}(q)$ odd and for which $f(q) \neq 0$. Let d be a fixed Gaussian integer. Let P be a sufficiently large positive integer and let $T(x)$ be the number of Gaussian primes p with $\mathrm{N}(p) \leqslant x$ such that

$$\begin{aligned} p+1 &= (1+i)dk, && (k, d(1+i)) = 1 \\ q_i &\nmid k, && \forall i \\ q &\nmid k, && \forall q,\ \mathrm{N}(q) \leqslant P \\ q^2 &\nmid (p+1), && \forall q,\ \mathrm{N}(q) > P. \end{aligned} \tag{12}$$

Now, we obtain a lower bound for $T(x)$. Let r be a positive integer. Define

$$Q = (1+i)d\prod_{i=1}^{r} q_i \prod_{\mathrm{N}(p)\leqslant P} p.$$

Let α be a real number such that $\frac{3}{4} < \alpha < 1$. Then

$$\begin{aligned} T(x) \geqslant & \sum_{\substack{\mathrm{N}(p)\leqslant x \\ p\equiv -1 \pmod{(1+i)d} \\ \left(\frac{p+1}{(i+i)d},Q\right)=1}} 1 - \sum_{i>r} \sum_{\substack{\mathrm{N}(p)\leqslant x \\ p\equiv -1 \pmod{(1+i)dq_i}}} 1 - \sum_{\mathrm{N}(q)>P} \sum_{\substack{\mathrm{N}(p)\leqslant x \\ p\equiv -1 \pmod{q^2}}} 1 \\ = & \sum_{\substack{\mathrm{N}(p)\leqslant x \\ p\equiv -1 \pmod{(1+i)d}}} \sum_{s\left|\left(\frac{p+1}{(i+i)d},Q\right)\right.} \mu(s) - \sum_{\substack{i>r \\ \mathrm{N}((1+i)dq_i)\leqslant x^\alpha}} \sum_{\substack{\mathrm{N}(p)\leqslant x \\ p\equiv -1 \pmod{(1+i)dq_i}}} 1 \\ & - \sum_{\substack{i>r \\ \mathrm{N}((1+i)dq_i)>x^\alpha}} \sum_{\substack{\mathrm{N}(p)\leqslant x \\ p\equiv -1 \pmod{(1+i)dq_i}}} 1 - \sum_{\mathrm{N}(q)>P} \sum_{\substack{\mathrm{N}(p)\leqslant x \\ p\equiv -1 \pmod{q^2}}} 1 \\ = & \sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3 - \sum\nolimits_4. \end{aligned}$$

Now, we estimate the above four sums.

$$\begin{aligned}
\sum\nolimits_1 &= \sum_{\substack{\mathrm{N}(p)\leqslant x\\ p\equiv -1 \pmod{(1+i)d}}} \sum_{s|\left(\frac{p+1}{(i+i)d},Q\right)} \mu(s)\\
&= \sum_{s|Q}\mu(s) \sum_{\substack{\mathrm{N}(p)\leqslant x\\ p\equiv -1 \pmod{(1+i)ds}}} 1\\
&= \frac{x}{\log x}\sum_{s|Q}\frac{\mu(s)}{\Phi((1+i)ds)} + o\left(\frac{x}{\log x}\right)\\
&\geqslant (1+o(1))\frac{x}{\log x}\frac{1}{2\,\mathrm{N}(d)}\prod_{\substack{q\nmid d(1+i)\\ q|Q}}\left(1-\frac{1}{\mathrm{N}(q)}\right).
\end{aligned}$$

By Lemma 3, $\sum_2 \leqslant c_3(\alpha)\frac{x}{\log x}\sum_{i>r}\frac{1}{\Phi(q_i)}$.
An application of Lemma 2 gives

$$\begin{aligned}
\sum\nolimits_3 &\leqslant \sum_{\substack{\mathrm{N}(p)\leqslant x\\ p+1=(1+i)dq_i m\\ \mathrm{N}((1+i)dq_i)>x^{\alpha}}} 1 \leqslant \sum_{\mathrm{N}(m)\leqslant x^{1-\alpha}} \sum_{\substack{p+1=(1+i)dmq\\ \mathrm{N}(p),\mathrm{N}(q)\leqslant x}} 1\\
&\leqslant \sum_{\mathrm{N}(D)\leqslant x^{2(1-\alpha)}} \sum_{\substack{p+1=Dq\\ \mathrm{N}(p),\mathrm{N}(q)\leqslant x}} 1 \ll \frac{x}{(\log x)^2}\sum_{\mathrm{N}(D)\leqslant x^{2(1-\alpha)}}\frac{1}{\Phi(D)}\\
&\ll (1-\alpha)\frac{x}{\log x}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\sum\nolimits_4 &= \sum_{\mathrm{N}(p)>P} \sum_{\substack{\mathrm{N}(p)\leqslant x\\ p\equiv -1 \pmod{q^2}}} 1\\
&\leqslant \sum_{P<\mathrm{N}(q)\leqslant x^{\frac14}} \sum_{\substack{\mathrm{N}(p)\leqslant x\\ p\equiv -1 \pmod{q^2}}} 1 + \sum_{\mathrm{N}(q)>x^{\frac14}} \sum_{\substack{\mathrm{N}(m)\leqslant x+1\\ m\equiv 0 \pmod{q^2}}} 1\\
&\ll \frac{x}{\log x}\sum_{\mathrm{N}(q)>P}\frac{1}{\mathrm{N}(q^2)} + \sum_{\mathrm{N}(q)>x^{\frac14}}\frac{x}{\mathrm{N}(q^2)}\\
&\ll \frac{x}{P\log x} + O(x^{\frac34}).
\end{aligned}$$

Let $\lambda := \liminf_{x\to\infty}\frac{T(x)\log x}{x}$. Using the above estimates and choosing P sufficiently large, we get

$$\lambda \geqslant \prod_{\substack{q\nmid(1+i)d\\ q|Q}}\left(1-\frac{1}{N(q)}\right) + O\left(\sum_{i>r}\frac{1}{N(q_i)}\right) + O(1-\alpha) + O(P^{-1}).$$

Letting $r \to \infty$ and $\alpha \to 1^-$, it follows that

$$\lambda \geqslant \frac{1}{2\,\mathrm{N}(d)} \prod_{3 \leqslant \mathrm{N}(q) \leqslant P} \left(1 - \frac{1}{\mathrm{N}(q)}\right) \prod_{i=1}^{\infty} \left(1 - \frac{1}{\mathrm{N}(q_i)}\right) + O(P^{-1})$$
$$\geqslant c(\log P)^{-1} + O(P^{-1}) > 0.$$

Since we can find infinitely many Gaussian primes which satisfies all the conditions of (12), we can choose a sufficiently large $p \in \mathcal{P}[i]$ such that

$$f((1+i)d) + f(k) = f(p+1) = 0.$$

Since k is square free and has no prime factor of the form q_i for which $f(q_i) \neq 0$, we have $f(k) = 0$ and hence $f((1+i)d) = 0$ for every non-zero Gaussian integer d. Taking $d = (1+i)^n$, $n = 0, 1, 2, \ldots$, we have $f((1+i)^n) = 0$ for all n. Next, taking $d = q^v$ for any Gaussian prime q with odd norm and any positive integer v, we get $f(d) = 0$ by additivity of f.

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