ON THE SPECIAL VALUES OF ARTIN L-FUNCTIONS FOR DIHEDRAL EXTENSIONS
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Abstract: We study special values of Artin L-functions for dihedral extensions at negative integers. We give a relation between these values and orders of the χ-parts of certain étale cohomology groups.

Keywords: étale cohomology, K-group, class number, Iwasawa theory, Artin L-function.

1. Introduction and the main result

Let p and l be distinct odd primes. We denote by $D_{2l}$ the dihedral group of order $2l$. Let $L^+$ be a dihedral extension over a number field $F^+$ of degree $2l$. Suppose that both $L^+$ and $F^+$ are totally real. For a totally positive algebraic number $r \in F^+$, let $L = L^+(\sqrt{-r})$ and $F = F^+(\sqrt{-r})$. Let $O_L$ be the integer ring of $L$. Let $\chi$ be a character of $\text{Gal}(L/F^+)$. Denote by $L(L/F^+, \chi, s)$ the Artin L-function attached to $\chi$ and put $d_\chi = [\mathbb{Z}_p[\text{Im}(\chi)]: \mathbb{Z}_p]$. We say that $\chi$ is even if it is the inflation of a character of $\text{Gal}(L^+/F^+)$, while odd if it is the product of an even character with the inflation of the non-trivial character of $\text{Gal}(F/F^+)$. Moreover, $a \sim_p b$ signifies that $a$ and $b$ are two $p$-adic numbers with the same valuation. Let $H^i_{\text{ét}}(\text{Spec} O_L[1/p], \mathbb{Z}_p(n))$ be the étale cohomology group, which we will simply denote by $H^i(O'_L, \mathbb{Z}_p(n))$. The main result of this paper is the following theorem.

Theorem 1.1. Let $n \geq 2$ be an integer and $\chi$ an irreducible character of $\text{Gal}(L/F^+)$. Assume that $\chi$ is even if $n$ is even and $\chi$ is odd if $n$ is odd. Then

$$L(L/F^+, \chi, 1-n)^{\chi(1)d_\chi} \sim_p \# H^2(O'_L, \mathbb{Z}_p(n))^\chi, \ # H^1(O'_L, \mathbb{Z}_p(n))^\chi,$$

where $H^i(O'_L, \mathbb{Z}_p(n))^\chi$ means the $\chi$-part of $H^i(O'_L, \mathbb{Z}_p(n))$.

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The definition of $\chi$-part will be given in Section 2. Theorem 1.1 is close to the following known result for an abelian extension, which will be used by our proof.

**Theorem 1.2 ([3], p. 707).** Let $n \geq 2$ be an integer and $L/K$ a totally complex abelian extension of the totally real base field $K$ of degree prime to $p$. Let $\chi$ be a character of $\text{Gal}(L/K)$, such that $\chi(-1) = (-1)^n$, and view $\chi$ as a $p$-adic character. Then

$$\mathcal{L}(L/K, \chi^{-1}, 1-n)^{d_\chi} \sim_p \frac{\# H^2(O'_L, \mathbb{Z}_p(n))^{\chi}}{\# H^1(O'_L, \mathbb{Z}_p(n))^{\chi}}.$$ 

Now, we can interpret Theorem 1.1 in terms of $K$-groups. For $n \geq 2$, it is seen that the $p$-adic Chern maps

$$K_{2n-i}(O_L) \otimes \mathbb{Z}_p \to H^i(O'_L, \mathbb{Z}_p(n)) \quad (i = 1, 2)$$

are isomorphisms, which is known as the Quillen-Lichtenbaum conjecture (cf. [7], [8]). Consequently, Theorem 1.1 gives the relation

$$\mathcal{L}(L/F^+, \chi^{-1}, 1-n)^{\chi(1)} \sim_p \frac{\# K_{2n-2}(O_L)_{\text{tors}}^{\chi}}{\# K_{2n-1}(O_L)_{\text{tors}}^{\chi}} \quad (1.1)$$

for $\chi$ with the same parity of $n \geq 2$. Further, we add the fact that (1.1) is essentially valid for $n = 1$, by

$$K_0(O_L) \simeq \mathbb{Z} \oplus \text{Cl}_L, \quad K_1(O_L) \simeq O_L^{\times}$$

and the main theorem of [4] (p. 1063). Here, $\text{Cl}_L$ denotes the ideal class group of $L$.

**2. Proof of the main theorem**

Let $D_{2l} = \langle a, b \rangle$ with $a^l = b^2 = 1$ and $bab^{-1} = a^{-1}$. It is known that $D_{2l}$ has the two one-dimensional representations and the $(l-1)/2$ irreducible two-dimensional representations. The character table is as follows:

<table>
<thead>
<tr>
<th>$\chi_k (1 \leq k \leq \frac{l-1}{2})$</th>
<th>$1_{D_{2l}}$</th>
<th>$a^i (1 \leq i \leq \frac{l-1}{2})$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_k$</td>
<td>2</td>
<td>$\zeta^i_k + \zeta^{-i}_{l-k}$</td>
<td>0</td>
</tr>
</tbody>
</table>

where $\zeta_l = \exp(2\pi \sqrt{-1}/l)$.

Take $\sigma \in \text{Hom}(\langle a \rangle, \overline{\mathbb{Q}}^{\times})$ satisfying $\sigma(a) = \zeta_l$, and write $\sigma_i = \sigma^i (0 \leq i \leq l-1)$. Then, the characters $\chi_k$ are induced from $\sigma_k$ and $\sigma_{l-k}$, namely,

$$\chi_k = \text{Ind} \sigma_k = \text{Ind} \sigma_{l-k} \quad (2.1)$$

for all $k \in \{1, \cdots, \frac{l-1}{2}\}$. 
Fix an embedding $\overline{\mathbb{Q}}^\times \hookrightarrow \overline{\mathbb{Q}}_p^\times$ and regard any character as $p$-adic one. Let $\text{Irr}(D_{2l})$ be the set of all irreducible characters of $D_{2l}$. For $\chi \in \text{Irr}(D_{2l})$, put $\mathcal{O}_\chi = \mathbb{Z}_p[\text{Im} \chi]$ and define
\[
e_{\chi} = \frac{\chi(1)}{2l} \sum_{g \in D_{2l}} \chi(g^{-1})g \in \mathcal{O}_\chi[D_{2l}].
\]

Let $M$ be a module over $\mathbb{Z}_p[D_{2l}]$. We call $e_{\chi}(M \otimes \mathcal{O}_\chi)$ the $\chi$-part of $M$ and simply denote this by $M^{\chi}$. Put $\mathcal{O} = \mathbb{Z}_p[\xi_l]$. Since \(\{ e_{\xi} \}_{\chi \in \text{Irr}(D_{2l})}\) is orthogonal idempotents of $\mathcal{O}[D_{2l}]$ and \(1_{\mathcal{O}[D_{2l}]} = \sum_{\chi \in \text{Irr}(D_{2l})} e_\chi\), we may write
\[
M \otimes \mathcal{O} = \bigoplus_{\chi \in \text{Irr}(D_{2l})} M^{\chi},
\]
where $\tilde{M}^{\chi} = e_\chi(M \otimes \mathcal{O})$. On the other hand, it is well-known that
\[
M \otimes \mathcal{O} = \bigoplus_{i=0}^{l-1} M^{\sigma_i},
\]
as an $\mathcal{O}[(a)]$-module where $M^{\sigma_i} = \{ x \in M \otimes \mathcal{O} \mid ax = \sigma_i(a)x \}$. In particular, when $M$ is finite, we have
\[
\frac{\sum_{k=1}^{l-1} M^{\chi_k}}{\sum_{k=1}^{l-1} M^{\sigma_k}} = \frac{\sum_{k=1}^{l-1} M^{\chi_k}}{\sum_{k=1}^{l-1} M^{\sigma_k}} = \frac{\sum_{k=1}^{l-1} M^{\chi_k}}{\sum_{k=1}^{l-1} M^{\sigma_k}}, \quad (2.2)
\]
since $\tilde{M}^{\xi} \oplus \tilde{M}^{\eta} = \{ x \in M \otimes \mathcal{O} \mid ax = x \} = M^{\sigma_0}$.

**Lemma 2.1.** Let $d_k = [\mathcal{O} : \mathcal{O}_{\chi_k}]$. If $M$ is a finite $\mathbb{Z}_p[D_{2l}]$-module, then
\[
(\sum_{k=1}^{l-1} M^{\chi_k})^{d_k} = \left( \sum_{k=1}^{l-1} M^{\sigma_k} \right)^2
\]
for all $k \in \{1, \ldots, \frac{l-1}{2}\}$.

**Proof.** Since $e_{\chi_k} = e_{\sigma_k} + e_{\sigma_1-k}$ in $\mathcal{O}[D_{2l}]$, we have the natural homomorphism
\[
f : \tilde{M}^{\chi_k} \longrightarrow M^{\sigma_k} \oplus M^{\sigma_1-k}, \quad e_{\chi_k}x \mapsto (e_{\sigma_k}x, e_{\sigma_1-k}x)
\]
as abelian groups. Take $x \in M \otimes \mathcal{O}$ with $(e_{\sigma_k}x, e_{\sigma_1-k}x) = (0, 0)$. This yields $e_{\chi_k}x = e_{\sigma_k}x + e_{\sigma_1-k}x = 0$, which implies that $f$ is injective. Thus the equation (2.2) leads to
\[
\#\tilde{M}^{\chi_k} = \#(M^{\sigma_k} \oplus M^{\sigma_1-k})
\]
for each $k$, therefore $f$ is also surjective. Note that $be_{\sigma_k} = e_{\sigma_1-k}b$ and $be_{\sigma_1-k} = e_{\sigma_k}b$. The homomorphism
\[
M^{\sigma_k} \longrightarrow M^{\sigma_1-k}, \quad x \mapsto bx
\]
is an isomorphism because

\[ M^\sigma_{t-k} \longrightarrow M^{\sigma_k}, \quad x \mapsto bx \]

is its inverse map. It follows that \( \#M^{\sigma_k} = \#M^\sigma_{t-k} \), so \( \#M^{\sigma_k} = (\#M^{\sigma_k})^2 \). On the other hand, we know \( \#M^{\sigma_k} = (\#M^{\sigma_k})^{d_k} \) by

\[ M \otimes \mathcal{O} \simeq M \otimes (\mathcal{O}^{d_k}) \simeq (M \otimes \mathcal{O}^{d_k})^{d_k} \]

as \( \mathcal{O}_{\chi_k}[D_{2l}] \)-modules. This completes the proof. \( \blacksquare \)

Now we give a proof of Theorem 1.1. In the following arguments, we identify \( \text{Gal}(L^+/F^+) \) with \( D_{2l} = \langle a, b \rangle \). Let \( K^+ \) be the fixed field of \( \langle a \rangle \) in \( L^+ \) and \( K = K^+(\sqrt{-r}) \). For an irreducible character \( \psi \) of \( \text{Gal}(L^+/F^+) \), we define the characters \( \psi^+ \) and \( \psi^- \) of \( \text{Gal}(L/F^+) \) by

\[ \psi^+(g) = \psi(g|_{L^+}), \quad \psi^-(g) = \gamma(g|_F)\psi(g|_{L^+}), \]

respectively, where \( \gamma \) is the non-trivial character of \( \text{Gal}(F/F^+) \). In fact, we know that \( \psi^+ \) is even while \( \psi^- \) is odd. For a character \( \sigma \) of \( \text{Gal}(L^+/K^+) \), define the characters \( \sigma^\pm \) of \( \text{Gal}(L/K^+) \) in the same manner. Using these notations and Theorem 4.21 of [2], we obtain

\[ \text{Irr}(\text{Gal}(L/F^+)) = \{ \varepsilon^\pm, \eta^\pm, \chi^\pm_1, \cdots, \chi^\pm_{l-1} \} \]

and

\[ \text{Hom} \left( \text{Gal}(L/K^+), \mathbb{Q}_p^\times \right) = \{ \sigma^\pm_0, \cdots, \sigma^\pm_{l-1} \} . \]

First, we treat the characters of two-dimensional representations. For a finite \( \mathbb{Z}_p[\text{Gal}(L/F^+)] \)-module \( M \), we have

\[ \left( \#M^{\chi^\pm_k} \right)^{d_{\chi^\pm_k}/d_{\chi^\pm_k}} = \left( \#M^{\sigma^\pm_k} \right)^2, \]

by Lemma 2.1, and therefore

\[ \left( \#H^2(O_L', \mathbb{Z}_p(n))^{\chi^\pm_k} \right)^{d_{\chi^\pm_k}/d_{\chi^\pm_k}} = \left( \#H^2(O_L', \mathbb{Z}_p(n))^{\sigma^\pm_k} \right)^2. \quad (2.3) \]

We remark that characters of dihedral groups take real values. Since \( \chi^\pm_k = \chi^\pm_k \) by (2.1), it follows from Chapter VII, Proposition 10.4 (iv) of [5] that

\[ \mathcal{L} \left( L/F^+, \chi^\pm_k, 1-n \right) = \mathcal{L} \left( L/K^+, \sigma^\pm_k, 1-n \right). \quad (2.4) \]
By the way, we can apply Theorem 1.2 to $L/K^+$ because $\text{Gal}(L/K^+)$ is the direct product of two cyclic groups of order $l$ and 2. Hence,

$$\mathcal{L} \left( \frac{L}{K^+}, (\sigma_k^{(n)})^{-1}, 1 - n \right) d_{\sigma_k^{(n)}} \sim_p \frac{\#H^2(O'_L, \mathbb{Z}_p(n))\sigma_k^{(n)}}{\#H^1(O'_L, \mathbb{Z}_p(n))\sigma_k^{(n)}} \tag{2.5}$$

where $\sigma_k^{(n)} = \sigma_k^+$ if $n$ is even and $\sigma_k^{(n)} = \sigma_k^-$ if $n$ is odd. Since $\chi_k^+(1) = 2$, the relationship (2.5) is equivalent to

$$\mathcal{L} \left( \frac{L}{K^+}, (\sigma_k^{(n)})^{-1}, 1 - n \right) d_{\sigma_k^{(n)}} \chi_k^{(1)-d_{\sigma_k^{(n)}}} \sim_p \frac{(\#H^2(O'_L, \mathbb{Z}_p(n))\sigma_k^{(n)})^2}{(\#H^1(O'_L, \mathbb{Z}_p(n))\sigma_k^{(n)})^2}.$$ 

Combining this with (2.3) and (2.4), we deduce that

$$\mathcal{L} \left( \frac{L}{K^+}, \chi_k^{(n)}, 1 - n \right) d_{\chi_k^{(n)}} \chi_k^{(1)-d_{\sigma_k^{(n)}}} \sim_p \frac{\#H^2(O'_L, \mathbb{Z}_p(n))\chi_k^{(n)}}{\#H^1(O'_L, \mathbb{Z}_p(n))\chi_k^{(n)}}.$$ 

i.e.

$$\mathcal{L} \left( \frac{L}{K^+}, \chi_k^{(n)}, 1 - n \right) d_{\sigma_k^{(n)}} \sim_p \frac{\#H^2(O'_L, \mathbb{Z}_p(n))\chi_k^{(n)}}{\#H^1(O'_L, \mathbb{Z}_p(n))\chi_k^{(n)}}.$$ 

This completes the proof for the case $\chi = \chi_k^\pm$.

We next explain the cases $\chi = \varepsilon^\pm$ that are linear characters. For this purpose we prepare the following lemma, which seems folklore for experts.

**Lemma 2.2.** Let $L/K$ be a finite Galois extension of number fields and suppose $p$ is prime to $[L : K]$. Then the canonical homomorphism

$$H^i(O'_K, \mathbb{Z}_p(n)) \to H^i(O'_L, \mathbb{Z}_p(n))^\text{Gal}(L/K)$$

is bijective for any $i$ and any $n$.

**Proof.** We write $A = O_K[1/p]$, $B = O_L[1/p]$ and $\Gamma = \text{Gal}(L/K)$. Let $\mu_{p^r}$ denote the group scheme of $p^r$-th root of unity over $A$. Then $\mu_{p^r}$ is étale and finite over $A$ since $p$ is invertible in $A$, and the Tate twist $\mu_{p^r}^{\otimes n}$ is also representable by an étale finite group scheme over $A$. Put $G = \mu_{p^r}^{\otimes n}$ and let $\text{Res}_{B/A}G$ denote the Weil restriction with respect to the finite extension $B/A$. We have the natural inclusion $\iota : G \to \text{Res}_{B/A}G$ and the natural norm homomorphism $\text{Nr} : \text{Res}_{B/A}G \to G$. Furthermore, it is readily seen that

1. $\text{Nr} \circ \iota$ is equal to the multiplication-by-$[L : K]$ map over $G$;
2. $\iota \circ \text{Nr}$ is equal to $\sum_{\gamma \in \Gamma} \gamma$ over $\text{Res}_{B/A}G$. 

Note that the Weil restriction is nothing but the direct image of the étale sheaf on \( \text{Spec} B \) by the morphism \( \pi : \text{Spec} B \to \text{Spec} A \). Therefore, the canonical homomorphism

\[
H^i(A, \text{Res}_{B/A} G) \to H^i(B, G)
\]
is bijective since \( \pi \) is finite (cf. [1], Expo VIII, Cor 5.6). Moreover, the homomorphism \( \iota : G \to \text{Res}_{B/A} G \) gives rise to a homomorphism

\[
\iota : H^i(A, G) \to H^i(A, \text{Res}_{B/A} G) \simeq H^i(B, G),
\]
which is nothing but the homomorphism induced by \( \pi : \text{Spec} B \to \text{Spec} A \). On the other hand, \( \text{Nr} : \text{Res}_{B/A} G \to G \) gives rise to a homomorphism

\[
\text{Nr} : H^i(B, G) \simeq H^i(A, \text{Res}_{B/A} G) \to H^i(A, G).
\]

It follows from (1) and (2) that

- (1)’ \( \text{Nr} \circ \iota \) is equal to the multiplication-by-[\( L : K \)] map over \( H^i(A, G) \);
- (2)’ \( \iota \circ \text{Nr} \) is equal to \( \sum_{\gamma \in \Gamma} \gamma \) over \( H^i(B, G) \).

Passing to the limit, we obtain homomorphisms

\[
\iota : H^i(A, \mathbb{Z}_p(n)) \to H^i(B, \mathbb{Z}_p(n))
\]
and

\[
\text{Nr} : H^i(B, \mathbb{Z}_p(n)) \to H^i(A, \mathbb{Z}_p(n)).
\]

It follows again from (1)’ and (2)’ that

- (1)” \( \text{Nr} \circ \iota \) is equal to the multiplication-by-[\( L : K \)] map over \( H^i(A, \mathbb{Z}(n)) \);
- (2)” \( \iota \circ \text{Nr} \) is equal to \( \sum_{\gamma \in \Gamma} \gamma \) over \( H^i(B, \mathbb{Z}(n)) \),

and therefore \( \iota \circ \text{Nr} \) is equal to the multiplication-by-[\( L : K \)] map over \( H^i(B, \mathbb{Z}_p(n)) \)\( \Gamma \). Note that the two multiplication-by-[\( L : K \)] maps \( \text{Nr} \circ \iota : H^i(A, \mathbb{Z}_p(n)) \to H^i(A, \mathbb{Z}_p(n)) \) and \( \iota \circ \text{Nr} : H^i(B, \mathbb{Z}_p(n)) \to H^i(B, \mathbb{Z}_p(n)) \)\( \Gamma \) are bijective because \( p \) does not divide \( [L : K] \). This implies that \( \iota : H^i(A, \mathbb{Z}_p(n)) \to H^i(B, \mathbb{Z}_p(n)) \)\( \Gamma \) is bijective.

Let \( \gamma^+ : \text{Gal}(F/F^+) \to \overline{\mathbb{Q}}_p^\times \) and \( \gamma^- : \text{Gal}(F/F^+) \to \overline{\mathbb{Q}}_p^\times \) be the trivial and non-trivial character, respectively. Note that \( d_{\gamma^\pm} = 1, (\gamma^\pm)^{-1} = \gamma^\pm, \) and \( \bar{\varepsilon} = \varepsilon^\pm \).

We can apply Theorem 1.2 to the quadratic extension \( F/F^+ \), so

\[
\mathcal{L} \left( F/F^+, (\gamma^n)^{-1}, 1-n \right) \sim_p \frac{\#H^2(O_F', \mathbb{Z}_p(n))_{\gamma^n}}{\#H^1(O_F', \mathbb{Z}_p(n))_{\gamma^n}}, \quad (2.6)
\]

For the left side of (2.6), it follows from Chapter VII, Proposition 10.4 (iii) of [5] that

\[
\mathcal{L} \left( L/F^+, \varepsilon^\pm, 1-n \right) = \mathcal{L} \left( F/F^+, (\gamma^\pm)^{-1}, 1-n \right). \quad (2.7)
\]
Since \( ge_{\varepsilon} = e_{\varepsilon} \) for all \( g \in \text{Gal}(L/F) \), we find

\[
H^i(O'_F, \mathbb{Z}_p(n))^{\gamma^\pm} \oplus H^i(O'_F, \mathbb{Z}_p(n))^{\gamma^-} \simeq H^i(O'_L, \mathbb{Z}_p(n))^{\text{Gal}(L/F)} \\
\simeq H^i(O'_L, \mathbb{Z}_p(n))^{\epsilon^+} \oplus H^i(O'_L, \mathbb{Z}_p(n))^{\epsilon^-}
\]

and

\[
H^i(O'_F, \mathbb{Z}_p(n))^{\gamma^+} \simeq H^i(O'_F, \mathbb{Z}_p(n))^\text{Gal}(F/F^+)} \\
\simeq H^i(O'_{F^+}, \mathbb{Z}_p(n)) \\
\simeq H^i(O'_L, \mathbb{Z}_p(n))^\text{Gal}(L/F^+) \\
\simeq H^i(O'_L, \mathbb{Z}_p(n))^{\epsilon^+}
\]

by Lemma 2.2. Thus, the following equations

\[
\#H^1(O'_F, \mathbb{Z}_p(n))^{\gamma^\pm} = \#H^1(O'_L, \mathbb{Z}_p(n))^{\epsilon^\pm}
\]

(2.8)

hold for \( i = 1, 2 \). These (2.6), (2.7) and (2.8) lead to

\[
\mathcal{L} \left( \frac{L}{F^+, \overline{\varepsilon(n)}, 1-n} \right) \sim_p \frac{\#H^2(O'_L, \mathbb{Z}_p(n))^{\epsilon(n)}}{\#H^1(O'_L, \mathbb{Z}_p(n))^{\epsilon(n)}}.
\]

This completes the proof for the case \( \chi = \varepsilon^\pm \).

Similarly, by [5, Proposition 10.4 (iii) in Ch. VII], we can apply Theorem 1.2 to \( K/F^+ \) to obtain the desired result for the case \( \chi = \eta^\pm \).

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References


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