

## ON INTEGRALS AND DIRICHLET SERIES OBTAINED FROM THE ERROR TERM IN THE CIRCLE PROBLEM

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**Abstract:** In this paper, we shall investigate several properties of integrals defined by  $\int_1^\infty t^{-\theta} P(t) \log^j t dt$  with a complex variable  $\theta$  and a non-negative integer  $j$ , where  $P(x)$  is the error term in the circle problem of Gauss. We shall also study the analytic continuation of several types of the Dirichlet series related with the circle problem, and study a proof of the functional equation of the Dedekind zeta-function associated with the Gaussian number field  $\mathbb{Q}(\sqrt{-1})$ .

**Keywords:** analytic continuation, the circle problem, the Dedekind zeta-function, periodic Bernoulli functions.

### 1. Introduction and Statement of Results

For a natural number  $n$ , let  $r(n)$  be the number of ways to write  $n$  as the sum of two squares:

$$r(n) = \#\{(\xi, \eta) \in \mathbb{Z}^2 \mid \xi^2 + \eta^2 = n\}$$

and  $P(x)$  be the error term in the circle problem defined by

$$P(x) = \sum_{1 \leq n \leq x} r(n) - \pi x. \quad (1.1)$$

The important problem is to determine the best-possible estimate of  $P(x)$  as  $x \rightarrow \infty$ . As far this problem, Gauss proved that  $P(x) = O(x^{1/2})$  and after him it has been improved by many researchers. The best estimate to date is

$$P(x) = O(x^{131/416} (\log x)^{18627/8320})$$

due to Huxley [10]. See [8], [11] and [15] for the topics of the circle problem and the several related results.

The mean value of  $P(x)$  is also of great interest. For example it is well-known that

$$\int_1^x P(t)dt = -x + O(x^{3/4}) \tag{1.2}$$

and

$$\int_1^x P(t)^2 dt = \left( \frac{1}{3\pi^2} \sum_{n=1}^{\infty} r(n)^2 n^{-3/2} \right) x^{3/2} + O(x \log^2 x). \tag{1.3}$$

The latter was proved by Kátai [14] and Preissmann [20] independently. The more precise form of (1.2) by means of the expansion of the Bessel function of the first kind (cf. e.g. [15, Theorem 3.11]) and the sharper estimate of the error term in (1.3) are known [19].

The purpose of this paper is to study several properties of the functions related to  $P(x)$ . Especially in the former half of this paper, we treat the integral defined by

$$\int_1^x t^{-\theta} P(t) \log^j t dt, \tag{1.4}$$

where  $\theta$  is a complex variable and  $j$  is a fixed non-negative integer. We are interested in its convergence properties as  $x \rightarrow \infty$  and the explicit representations of (1.4) in the range  $0 < \Re\theta \leq 1$ .

For preliminaries, we shall derive the growth of orders of the integral (1.4) with respect to  $x$  from (1.2). By integration by parts we find easily that

$$\int_1^x t^{-\theta} P(t) dt = \begin{cases} C_1 + O(x^{1-\Re\theta}) & \text{if } \Re\theta > 1, \\ -\log x + C_2 + O(x^{-1/4}) & \text{if } \theta = 1, \\ -(1-\theta)^{-1} x^{1-\theta} + C_1 + O(x^{3/4-\Re\theta}) & \text{if } 3/4 < \Re\theta \leq 1 \text{ and } \theta \neq 1, \\ -(1-\theta)^{-1} x^{1-\theta} + O(\log x) & \text{if } \Re\theta = 3/4, \\ -(1-\theta)^{-1} x^{1-\theta} + O(x^{3/4-\Re\theta}) & \text{if } \Re\theta < 3/4 \end{cases} \tag{1.5}$$

and

$$\int_1^x t^{-\theta} P(t) \log^j t dt = \begin{cases} C_3 + O(x^{1-\Re\theta} \log^j x) & \text{if } \Re\theta > 1, \\ -(j+1)^{-1} \log^{j+1} x + O(1) & \text{if } \theta = 1, \\ -(1-\theta)^{-1} x^{1-\theta} \log^j x + O(x^{1-\Re\theta} \log^{j-1} x) & \text{otherwise} \end{cases} \tag{1.6}$$

for  $j \geq 1$ , where  $C_1, C_2$  and  $C_3$  are suitable constants depending at most on  $\theta$  and  $j$ . To get a certain integral representation of  $C_1$ , it is convenient to define  $P(x) = -\pi x$  for  $0 \leq x < 1$ , i.e. the empty sum in (1.1) is zero. Let

$$Q(x) = \int_0^x P(t)dt + x = x - \frac{\pi}{2} + \int_1^x P(t)dt.$$

Then the constant  $C_1$  is expressed as

$$C_1 = \frac{\theta}{1 - \theta} + \frac{\pi}{2} + \theta \int_1^\infty t^{-\theta-1} Q(t) dt, \tag{1.7}$$

From (1.5) and (1.6) it follows easily that the function defined by

$$\mathfrak{J}_j(\theta) = \int_1^\infty t^{-\theta} P(t) \log^j t dt \tag{1.8}$$

converges for  $\Re\theta > 1$  and diverges for  $\Re\theta \leq 1$ . Moreover by (1.3) it is convergent absolutely and uniformly on every compact subset in the region  $\Re\theta > 5/4$ , For  $j = 0$  it is known that

$$\mathfrak{J}_0(\theta) = -\frac{\pi}{\theta - 2} + \frac{4}{\theta - 1} \zeta_K(\theta - 1), \tag{1.9}$$

where  $K = \mathbb{Q}(\sqrt{-1})$  is the Gaussian number field and  $\zeta_K(s) = \zeta(s)L(s, \chi)$  is the Dedekind zeta-function associated with  $K$ . Here  $\zeta(s)$  denotes the Riemann zeta-function and  $L(s, \chi)$  the Dirichlet  $L$ -function associated with the primitive Dirichlet character  $\chi \pmod{4}$  ([7, Section 6]). The expression on the right hand side gives the analytic continuation of  $\mathfrak{J}_0(\theta)$  to the whole  $\theta$ -plane as a meromorphic function with a simple pole at  $\theta = 1$ . (We should note that the point  $\theta = 2$  is not a pole of  $\mathfrak{J}_0(\theta)$ .) Since the integral  $\mathfrak{J}_0(\theta)$  is absolutely convergent for  $\Re\theta > 5/4$ , we can differentiate  $\mathfrak{J}_0(\theta)$  under the integral sign (see e.g. [17, Theorem 3.4 in p.340] and the ‘‘added in proof’’ in [5] for the possibility of the termwise differentiation of the integral), and get

$$\begin{aligned} \mathfrak{J}_j(\theta) &= (-1)^j \frac{d^j}{d\theta^j} \int_1^\infty t^{-\theta} P(t) dt \\ &= -\frac{\pi j!}{(\theta - 2)^{j+1}} + 4 \sum_{m=0}^j \binom{j}{m} \frac{(-1)^{j+m} m!}{(\theta - 1)^{m+1}} (\zeta_K(\theta - 1))^{(j-m)}. \end{aligned} \tag{1.10}$$

Here  $\binom{j}{m}$  denotes the binomial coefficient with  $j \geq m$ , and we define  $\binom{j}{0} = 1$  for every  $j \geq 0$ . Here and in the further part, we make use of the notation

$$(f(a))^{(\nu)} = \left. \frac{d^\nu}{ds^\nu} f(s) \right|_{s=a}$$

for short, where  $\nu$  is a non-negative integer and  $f(s)$  is a (at least)  $\nu$  times continuously differentiable function.

On the special value of  $\mathfrak{J}_j(\theta)$  at  $\theta = 2$ , by (1.10) we have:

**Proposition 1.** *The special value of (1.8) at  $\theta = 2$  is given by*

$$\mathfrak{J}_j(2) = 4j! \sum_{m=0}^j (-1)^{j-m} \left\{ \frac{(L(1, \chi))^{(j-m+1)}}{(j - m + 1)!} + \sum_{\nu=0}^{j-m} \frac{\gamma_{j-m-\nu}(L(1, \chi))^{(\nu)}}{\nu!} \right\} - \pi j!.$$

Here  $\gamma_n$  are the coefficients of the Laurent expansion of  $\zeta(s)$  at  $s = 1$  defined by

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \gamma_n (s-1)^n,$$

especially  $\gamma_0$  is the Euler constant.

We give one remark on the absolute convergence of  $\mathfrak{J}_j(\theta)$ . Let  $A$  be the constant defined by

$$A = \inf\{\alpha \mid P(x) = O(x^\alpha)\}.$$

By the trivial estimation, the range on the absolute convergence of  $\mathfrak{J}_0(\theta)$  is  $\Re\theta > A + 1$ . By (1.3)  $A \geq 1/4$ , hence, this range “ $\Re(\theta) > 5/4$ ” is best-possible.

Our first theorem concerns convergence property of the integral (1.8) in  $1 < \Re\theta \leq 5/4$ .

**Theorem 1.** *The integral (1.8) is convergent uniformly on every compact subset in  $1 < \Re\theta \leq 5/4$ , and it can be expressed by the right-hand side in (1.10) in this region.*

The proof is carried out by the way similar to that in [5]. However it turns out that the case of the circle problem is more difficult and complicated than that of the divisor problem. For instance, in the case of the divisor problem we repeatedly apply an identity for the sum  $\sum_{n \leq x} n^{1-\theta}$ , while in the case of the circle problem we have to treat the sum of the form  $\sum_{n \leq x} \chi(n)n^{1-\theta}$ . Since the Euler-Maclaurin summation formula is not so effective for this kind of sum, we divide it into two parts according to the residues mod 4.

Our next interest is to study the magnitude of the integral  $\int_1^x t^{-\theta} P(t) \log^j t dt$  with respect to  $x$  in the range  $0 < \Re\theta \leq 1$ . We obtain the following theorem.

**Theorem 2.** *For  $0 < \Re\theta \leq 1$  with  $\theta \neq 1$ , we have*

$$\int_1^x t^{-\theta} P(t) dt = -\frac{1}{1-\theta} x^{1-\theta} + \frac{4\zeta_K(\theta-1)}{\theta-1} + \frac{\pi}{2-\theta} + O(x^{3/4-\Re\theta}) \tag{1.11}$$

and

$$\begin{aligned} \int_1^x t^{-\theta} P(t) \log^j t dt &= x^{1-\theta} \sum_{m=0}^j \binom{j}{m} \frac{m!}{(\theta-1)^{m+1}} \log^{j-m} x + O(1) \\ &\quad + O(x^{3/4-\Re\theta} \log^j x) \end{aligned} \tag{1.12}$$

for  $j \geq 1$ . Besides this, for  $\theta = 1$  we have

$$\int_1^x t^{-1} P(t) dt = -\log x + \{\pi - \log 2\pi - 2L'(0, \chi)\} + O(x^{-1/4}) \tag{1.13}$$

and

$$\int_1^x t^{-1} P(t) \log^j t dt = -\frac{1}{j+1} \log^{j+1} x + O(1) \tag{1.14}$$

for  $j \geq 1$ .

We remark that (1.14) is equivalent to (1.6) for the case  $\theta = 1$ .

Recall the constant  $C_1$  appearing in (1.5) and (1.7). If we substitute the well-known Hardy identity to the integral on the right hand side of (1.7),  $C_1$  may be expressed as an infinite series involving  $J$ -Bessel functions. Comparing (1.5) and Theorem 2, we get alternative representations of  $C_1$  as well as  $C_2$  and  $C_3$  in terms of elementary functions and the values of  $\zeta_K(s)$  or its derivatives. This remark gives two applications. One is the new proof of the functional equation of  $\zeta_K(s)$ , which is derived by combining (1.11) and the mean of the Hardy identity. The other is the asymptotic representation of the log-Riesz mean of  $r(n)$  first obtained by Müller [18]. We shall discuss these topics in Sections 5 and 6, respectively.

In [5] we discussed the possibility of analytic continuation of the double series defined by

$$D(s_1, s_2) = \sum_{1 \leq m < n} \frac{d(m)d(n)}{m^{s_1}n^{s_2}},$$

where  $s_1$  and  $s_2$  are complex variables and  $d(n)$  is the divisor function. In fact we showed that it can be continued analytically as a meromorphic function to the region

$$\Re s_1 + \Re s_2 > 1/2 \tag{1.15}$$

in  $\mathbb{C}^2$ -space [5, Theorem 4]. In the last section, we shall study analogous properties of the double series  $M(s_1, s_2)$  defined by

$$M(s_1, s_2) = \sum_{1 \leq m < n} \frac{r(m)r(n)}{m^{s_1}n^{s_2}}.$$

The series on the right hand side is absolutely convergent for  $\Re s_2 > 1$  and  $\Re s_1 + \Re s_2 > 2$ , hence  $M(s_1, s_2)$  is holomorphic with respect to  $s_1$  and  $s_2$  there. As in the divisor function case, we can show that the double series  $M(s_1, s_2)$  can be continued analytically as a meromorphic function to the region (1.15) in  $\mathbb{C}^2$ -space. Moreover, by applying the first formula in (1.5), we can improve the above range.

**Theorem 3.** *The multiple series  $M(s_1, s_2)$  can be continued analytically as the meromorphic function to the region*

$$\Re s_1 + \Re s_2 > 1/4 \tag{1.16}$$

in  $\mathbb{C}^2$ -space.

It seems difficult to improve the range (1.16). See the remark in Section 8.

Finally if we use the argument similar to that of Theorem 3 we can improve the range (1.15) to (1.16) for analytic continuation of  $D(s_1, s_2)$ , namely we have

**Proposition 2.** *The multiple series  $D(s_1, s_2)$  can be continued analytically as the meromorphic function to the region (1.16) in  $\mathbb{C}^2$ -space.*

Throughout this paper, the summation  $\sum_{n \leq x}$  means that  $\sum_{1 \leq n \leq x}$ . The implied constant in the symbols  $O(\ )$  and  $\ll$  will depend at most on  $j$  and  $\theta$ . For a real number  $x$ , let  $\psi(x)$  be the first periodic Bernoulli function defined by  $\psi(x) = x - [x] - 1/2$ , where  $[x]$  is the greatest integer not exceeding  $x$ . Besides this, let  $\psi_1(x) = \int_1^x \psi(t)dt$  for  $x \geq 1$ . In the latter discussions, we usually use the estimate  $\psi_1(x) = O(1)$  uniformly in  $x$  without any remarks.

## 2. The integral $\mathfrak{J}_j(\theta)$ for $\Re\theta > 1$

As a preparation for the proofs of Theorems 1 and 2, we shall derive some identity for the integral of  $P(t)$  over the finite interval. Though the integral exists for any  $\theta$  we will consider it only for  $0 < \Re\theta \leq 5/4$  with  $\theta \neq 1$  here.

**Lemma 1.** *Suppose that  $0 < \Re\theta \leq 5/4$  and  $\theta \neq 1$ . Then we have*

$$\begin{aligned} \int_1^x t^{-\theta} P(t)dt &= \frac{4}{2-\theta} x^{2-\theta} \sum_{n \leq x} \chi(n)n^{-1} + \frac{2\theta}{(1-\theta)(2-\theta)} \sum_{n \leq x} \chi(n)n^{1-\theta} \\ &\quad - \frac{2}{1-\theta} x^{1-\theta} \sum_{n \leq x} \chi(n) - \frac{\pi}{2-\theta} x^{2-\theta} + \frac{\pi}{2-\theta} \\ &\quad - 4 \sum_{n \leq x} \chi(n)n^{1-\theta} \int_1^{x/n} t^{-\theta} \psi(t)dt. \end{aligned} \tag{2.1}$$

**Proof.** By the definition of  $P(x)$ , we have

$$\int_1^x t^{-\theta} P(t)dt = \frac{1}{\theta-1} \sum_{n \leq x} r(n)n^{1-\theta} + \frac{x^{1-\theta}}{1-\theta} P(x) + \frac{\pi}{(1-\theta)(2-\theta)} x^{2-\theta} + \frac{\pi}{2-\theta}. \tag{2.2}$$

We consider the sum on the right hand side of (2.2). By  $r(n) = 4 \sum_{d|n} \chi(d)$  and the well-known identity

$$\sum_{n \leq y} n^{1-\theta} = \frac{1}{2-\theta} y^{2-\theta} - \psi(y)y^{1-\theta} + \frac{1}{\theta-2} + \frac{1}{2} + (1-\theta) \int_1^y t^{-\theta} \psi(t)dt$$

for  $\theta \neq 2$  (see e.g. [5, Lemma 1]), we have

$$\begin{aligned} \frac{1}{4} \sum_{n \leq x} r(n)n^{1-\theta} &= \sum_{n \leq x} \chi(n)n^{1-\theta} \sum_{m \leq x/n} m^{1-\theta} \\ &= \frac{1}{2-\theta} x^{2-\theta} \sum_{n \leq x} \chi(n)n^{-1} - x^{1-\theta} \sum_{n \leq x} \chi(n)\psi\left(\frac{x}{n}\right) \\ &\quad - \frac{\theta}{2(2-\theta)} \sum_{n \leq x} \chi(n)n^{1-\theta} \\ &\quad + (1-\theta) \sum_{n \leq x} \chi(n)n^{1-\theta} \int_1^{x/n} t^{-\theta} \psi(t)dt. \end{aligned} \tag{2.3}$$

Note that (2.3) is also valid for  $\theta = 1$ , hence

$$\frac{1}{4}(\pi x + P(x)) = x \sum_{n \leq x} \chi(n)n^{-1} - \frac{1}{2} \sum_{n \leq x} \chi(n) - \sum_{n \leq x} \chi(n)\psi\left(\frac{x}{n}\right). \tag{2.4}$$

Combing (2.2), (2.3) and (2.4) we get (2.1). ■

**Proof of Theorem 1.** Suppose that  $1 < \Re\theta \leq 5/4$ . Noting that

$$\sum_{n \leq x} \chi(n)n^{-s} = L(s, \chi) + O(x^{-\Re s}), \quad (\Re s > 0) \tag{2.5}$$

$L(1, \chi) = \pi/4$  and  $\sum_{n \leq x} \chi(n) = O(1)$ , we obtain from (2.1) that

$$\begin{aligned} \int_1^x t^{-\theta} P(t) dt &= \frac{\pi}{2-\theta} + \frac{2\theta}{(1-\theta)(2-\theta)} L(\theta-1, \chi) \\ &\quad - 4 \sum_{n \leq x} \chi(n)n^{1-\theta} \int_1^{x/n} t^{-\theta} \psi(t) dt + O(x^{1-\Re\theta}). \end{aligned}$$

By the interchange of integration and summation, the third term on the right-hand side is transformed as

$$\begin{aligned} \sum_{n \leq x} \chi(n)n^{1-\theta} \int_1^{x/n} t^{-\theta} \psi(t) dt &= \frac{L(\theta-1, \chi)}{\theta-1} \left\{ \frac{\theta}{2(\theta-2)} - \zeta(\theta-1) \right\} \\ &\quad + O(x^{1-\Re\theta} \log x). \end{aligned}$$

by using the well-known formula

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s \int_1^\infty t^{-s-1} \psi(t) dt, \quad (\Re s > -1).$$

Hence we obtain

$$\int_1^x t^{-\theta} P(t) dt = -\frac{\pi}{\theta-2} + \frac{4\zeta_K(\theta-1)}{\theta-1} + O(x^{1-\Re\theta} \log x). \tag{2.6}$$

Letting  $x \rightarrow \infty$  in (2.6), we obtain the assertion of Theorem 1 for  $j = 0$ .

Let us treat the case  $j \geq 1$ . By integration by parts and (2.6), we can show that for any positive  $\varepsilon$ , there exists a sufficiently large  $M_0$  such that for all  $L > M > M_0$ ,

$$\left| \int_M^L t^{-\theta} P(t) \log^j t dt \right| < \varepsilon.$$

Hence the integral (1.8) converges uniformly in the wider sense in the region  $1 < \Re\theta \leq 5/4$ , thus we can differentiate (1.8) with respect to  $\theta$  under the integral sign and get (1.10). This completes the proof of Theorem 1. ■

**Proof of Proposition 1.** By the Leibniz rule, the Laurent expansion of  $\zeta_K(s)$  at  $s = 1$  in terms of the zeta- and the  $L$ -functions is given by

$$\begin{aligned}
 (\zeta_K(s))^{(\nu)} &= \sum_{n=\nu}^{\infty} \left\{ \frac{(L(1, \chi))^{(n+1)}}{(n+1)!} + \sum_{m=0}^n \frac{(L(1, \chi))^{(m)} \gamma_{n-m}}{m!} \right\} \binom{n}{\nu} \nu! (s-1)^{n-\nu} \\
 &\quad + \frac{(-1)^\nu \nu! \pi}{4(s-1)^{\nu+1}}.
 \end{aligned}$$

Hence, we have

$$\begin{aligned}
 \mathfrak{J}_j(\theta) &= 4j! \sum_{m=0}^j (-1)^{j-m} \left\{ \frac{(L(1, \chi))^{(j-m+1)}}{(j-m+1)!} + \sum_{n=0}^{j-m} \frac{\gamma_{j-m-n} (L(1, \chi))^{(n)}}{n!} \right\} \\
 &\quad - \frac{j! \pi}{(\theta-2)^{j+1}} + j! \pi \sum_{m=0}^j \left( \frac{1}{\theta-1} \right)^{m+1} \left( \frac{1}{\theta-2} \right)^{j-m+1} + O(|\theta-2|)
 \end{aligned}$$

near  $\theta = 2$ . Since (1.8) is absolutely convergent for  $\Re\theta > 5/4$ , we obtain the assertion of the proposition immediately by applying the following relation

$$\sum_{m=0}^a \left( \frac{1}{\theta-1} \right)^{m+1} \left( \frac{1}{\theta-2} \right)^{a-m+1} = \left( \frac{1}{\theta-2} \right)^{a+1} - \left( \frac{1}{\theta-1} \right)^{a+1}$$

for any non-negative integer  $a$  to the above formula for  $\mathfrak{J}_j(\theta)$  and by taking  $\theta \rightarrow 2$ . ■

### 3. Integral formulas involving the $\psi$ -function

For  $a = 1$  and 3, let

$$I(a) = \int_1^{x/(4-a)} t^{-1} \psi(t) \psi \left( \frac{x}{4t} + \frac{a}{4} \right) dt \tag{3.1}$$

and

$$Z(a) = \int_1^{x/(4-a)} t^{-\theta} \psi(t) \int_1^{(x/t+a)/4} \left( u - \frac{a}{4} \right)^{-\theta} \psi(u) du dt. \tag{3.2}$$

In this section, we derive certain estimates of these integrals, which will be used to prove Theorem 2 in Section 4.

Let  $x > 9$ . First we consider  $I(a)$ . Dividing the range of this integral we have

$$\begin{aligned}
 I(a) &= \left\{ \int_1^{\sqrt{x}} + \int_{\sqrt{x}}^{x/(4-a)} \right\} t^{-1} \psi(t) \psi \left( \frac{x}{4t} + \frac{a}{4} \right) dt \\
 &= \int_1^{\sqrt{x}} t^{-1} \psi(t) \psi \left( \frac{x}{4t} + \frac{a}{4} \right) dt + \int_{4-a}^{\sqrt{x}} t^{-1} \psi \left( \frac{x}{t} \right) \psi \left( \frac{t}{4} + \frac{a}{4} \right) dt \\
 &= I_1(a) + I_2(a),
 \end{aligned}$$

say. Here, for  $I_2(a)$ , we have changed the variable  $x/t$  to  $t$ . We shall show the asymptotic formulas for these two functions.



**Lemma 2.** For  $a = 1$  and  $3$ , we have

$$I_1(a) = -\frac{2}{x} \sum_{k \leq \sqrt{x}} k \bar{B}_2\left(\frac{x}{4k} + \frac{a}{4}\right) + O(x^{-1/2}), \tag{3.3}$$

where  $\bar{B}_2(x)$  is the second periodic Bernoulli function.

**Proof.** By the definition of the  $\psi$ -function, we have

$$\begin{aligned} I_1(a) &= \int_1^{\sqrt{x}} t^{-1} \left(t - [t] - \frac{1}{2}\right) \psi\left(\frac{x}{4t} + \frac{a}{4}\right) dt \\ &= \int_1^{\sqrt{x}} t^{-1} \left(t - \frac{1}{2}\right) \psi\left(\frac{x}{4t} + \frac{a}{4}\right) dt - \int_1^{\sqrt{x}} t^{-1} [t] \psi\left(\frac{x}{4t} + \frac{a}{4}\right) dt \\ &= I_{11} - I_{12}, \end{aligned} \tag{3.4}$$

say. In  $I_{11}$ , changing the variables as  $x/t = u$  and applying integration by parts, we have

$$\begin{aligned} I_{11} &= x \int_{\sqrt{x}}^x \left(u^{-2} - \frac{1}{2x} u^{-1}\right) \psi\left(\frac{u}{4} + \frac{a}{4}\right) du \\ &= -4\psi_1\left(\frac{\sqrt{x}}{4} + \frac{a}{4}\right) + 4x \int_{\sqrt{x}}^x \left(2u^{-3} - \frac{1}{2x} u^{-2}\right) \psi_1\left(\frac{u}{4} + \frac{a}{4}\right) du + O(x^{-1/2}). \end{aligned}$$

Let  $\psi_2(x) = \int_1^x \psi_1(t) dt$ . We can see easily that

$$\psi_2(x) = -\frac{1}{12}x + O(1). \tag{3.5}$$

Hence,

$$\begin{aligned} &\int_{\sqrt{x}}^x \left(2u^{-3} - \frac{1}{2x} u^{-2}\right) \psi_1\left(\frac{u}{4} + \frac{a}{4}\right) du \\ &= -\frac{1}{12} \left[ u \left(2u^{-3} - \frac{1}{2x} u^{-2}\right) \right]_{\sqrt{x}}^x - \frac{1}{12} \int_{\sqrt{x}}^x u \left(6u^{-4} - \frac{1}{x} u^{-3}\right) du + O(x^{-3/2}) \\ &= -\frac{1}{12x} + O(x^{-3/2}), \end{aligned}$$

and therefore we obtain

$$I_{11} = -4 \left\{ \frac{1}{12} + \psi_1\left(\frac{\sqrt{x}}{4} + \frac{a}{4}\right) \right\} + O(x^{-1/2}). \tag{3.6}$$

Next we consider  $I_{12}$ . We easily see that

$$\begin{aligned} I_{12} &= \sum_{k=1}^{[\sqrt{x}]-1} k \int_k^{k+1} t^{-1} \psi\left(\frac{x}{4t} + \frac{a}{4}\right) dt + [\sqrt{x}] \int_{[\sqrt{x}]}^{\sqrt{x}} t^{-1} \psi\left(\frac{x}{4t} + \frac{a}{4}\right) dt \\ &= I_{12}^{(1)} + I_{12}^{(2)}, \end{aligned}$$

say. The integral in  $I_{12}^{(1)}$  can be transformed as

$$\begin{aligned} \int_k^{k+1} t^{-1} \psi\left(\frac{x}{4t} + \frac{a}{4}\right) dt &= \int_{x/(k+1)}^{x/k} u^{-1} \psi\left(\frac{u}{4} + \frac{a}{4}\right) du \\ &= \frac{4}{x} \left\{ k \psi_1\left(\frac{x}{4k} + \frac{a}{4}\right) - (k+1) \psi_1\left(\frac{x}{4(k+1)} + \frac{a}{4}\right) \right\} \\ &\quad + 4 \int_{x/(k+1)}^{x/k} u^{-2} \psi_1\left(\frac{u}{4} + \frac{a}{4}\right) du \end{aligned}$$

by applying the change of variables and integration by parts. Hence by using this formula and the relation

$$\sum_{n=1}^{N-1} n(a_n - a_{n+1}) = \sum_{n=1}^N a_n - N a_N \tag{3.7}$$

with any sequence  $\{a_n\}_{n \in \mathbb{N}}$  and  $N \geq 2$ , we have

$$\begin{aligned} I_{12}^{(1)} &= \frac{4}{x} \left\{ \sum_{k=1}^{[\sqrt{x}]} k \psi_1\left(\frac{x}{4k} + \frac{a}{4}\right) - [\sqrt{x}]^2 \psi_1\left(\frac{x}{4[\sqrt{x}]} + \frac{a}{4}\right) \right\} \\ &\quad + 4 \sum_{k=1}^{[\sqrt{x}]-1} k \int_{x/(k+1)}^{x/k} u^{-2} \psi_1\left(\frac{u}{4} + \frac{a}{4}\right) du. \end{aligned} \tag{3.8}$$

We further transform the sum of the last part on the right-hand side in (3.8). Let us define

$$F(k) = \int_1^{x/k} u^{-2} \psi_1\left(\frac{u}{4} + \frac{a}{4}\right) du$$

briefly. Then by (3.7) we have

$$\sum_{k=1}^{[\sqrt{x}]-1} k \int_{x/(k+1)}^{x/k} u^{-2} \psi_1\left(\frac{u}{4} + \frac{a}{4}\right) du = \sum_{k=1}^{[\sqrt{x}]} F(k) - [\sqrt{x}] F([\sqrt{x}]).$$

Since  $F(k) = O(1)$  uniformly in  $k$  and  $x$ , we have, by integration by parts and (3.5),

$$\begin{aligned} F(k) &= \int_1^\infty u^{-2} \psi_1\left(\frac{u}{4} + \frac{a}{4}\right) du \\ &\quad - 4 \left\{ \left[ u^{-2} \psi_2\left(\frac{u}{4} + \frac{a}{4}\right) \right]_{x/k}^\infty + 2 \int_{x/k}^\infty u^{-3} \psi_2\left(\frac{u}{4} + \frac{a}{4}\right) du \right\} \\ &= \int_1^\infty u^{-2} \psi_1\left(\frac{u}{4} + \frac{a}{4}\right) du + \frac{k}{12x} + O\left(\frac{k^2}{x^2}\right), \end{aligned}$$

and hence

$$\sum_{k=1}^{[\sqrt{x}] - 1} k \int_{x/(k+1)}^{x/k} u^{-2} \psi_1 \left( \frac{u}{4} + \frac{a}{4} \right) du = -\frac{1}{24} + O(x^{-1/2}).$$

Therefore we obtain

$$I_{12}^{(1)} = \frac{4}{x} \left\{ \sum_{k=1}^{[\sqrt{x}]} k \psi_1 \left( \frac{x}{4k} + \frac{a}{4} \right) - [\sqrt{x}]^2 \psi_1 \left( \frac{x}{4[\sqrt{x}]} + \frac{a}{4} \right) \right\} - \frac{1}{6} + O(x^{-1/2}).$$

Similarly we have

$$\begin{aligned} I_{12}^{(2)} &= -[\sqrt{x}] \int_{x/[\sqrt{x}]}^{\sqrt{x}} u^{-1} \psi \left( \frac{u}{4} + \frac{a}{4} \right) du \\ &= -4[\sqrt{x}] \left\{ \frac{1}{\sqrt{x}} \psi_1 \left( \frac{\sqrt{x}}{4} + \frac{a}{4} \right) - \frac{\sqrt{x}}{x} \psi_1 \left( \frac{x}{4[\sqrt{x}]} + \frac{a}{4} \right) \right\} + O(x^{-1/2}). \end{aligned}$$

Adding  $I_{12}^{(1)}$  and  $I_{12}^{(2)}$  we have

$$I_{12} = \frac{4}{x} \sum_{k=1}^{[\sqrt{x}]} k \psi_1 \left( \frac{x}{4k} + \frac{a}{4} \right) - \frac{1}{6} - 4\psi_1 \left( \frac{\sqrt{x}}{4} + \frac{a}{4} \right) + O(x^{-1/2}). \quad (3.9)$$

Combining (3.4), (3.6) and (3.9), we obtain

$$I_1(a) = -\frac{4}{x} \sum_{k=1}^{[\sqrt{x}]} k \psi_1 \left( \frac{x}{4k} + \frac{a}{4} \right) - \frac{1}{6} + O(x^{-1/2}).$$

If we recall the relation  $\bar{B}_2(x) = 2\psi_1(x) + 1/6$ , we obtain (3.3). ■

**Lemma 3.** For  $a = 1$  and 3, we have

$$I_2(a) = -\frac{1}{2x} \sum_{k \leq (\sqrt{x} + a)/4} (4k - a) \bar{B}_2 \left( \frac{x}{4k - a} \right) + O(x^{-1/2}). \quad (3.10)$$

**Proof.** We begin our proof with the transformation

$$\begin{aligned} I_2(a) &= \int_{4-a}^{\sqrt{x}} t^{-1} \left( \frac{t}{4} + \frac{a}{4} - \left[ \frac{t}{4} + \frac{a}{4} \right] - \frac{1}{2} \right) \psi \left( \frac{x}{t} \right) dt \\ &= \frac{1}{4} \int_{4-a}^{\sqrt{x}} \left( 1 + \frac{a-2}{t} \right) \psi \left( \frac{x}{t} \right) dt - \int_{4-a}^{\sqrt{x}} t^{-1} \left[ \frac{t}{4} + \frac{a}{4} \right] \psi \left( \frac{x}{t} \right) dt \\ &= I_{21} - I_{22}, \end{aligned}$$

say. Similarly to  $I_{11}$  in Lemma 2, we find that

$$I_{21} = -\frac{1}{4}\psi_1(\sqrt{x}) - \frac{1}{48} + O(x^{-1/2}). \tag{3.11}$$

For  $I_{22}$ , we note that

$$I_{22} = 4 \int_1^{(\sqrt{x}+a)/4} (4u - a)^{-1} [u] \psi \left( \frac{x}{4u - a} \right) du.$$

Then we easily see with an abbreviation  $N_a = (\sqrt{x} + a)/4$  that

$$\begin{aligned} I_{22} &= 4 \sum_{k=1}^{[N_a]-1} k \int_k^{k+1} (4u - a)^{-1} \psi \left( \frac{x}{4u - a} \right) du \\ &\quad + 4[N_a] \int_{[N_a]}^{N_a} (4u - a)^{-1} \psi \left( \frac{x}{4u - a} \right) du \\ &= I_{22}^{(1)} + I_{22}^{(2)}, \end{aligned}$$

say. The function  $I_{22}^{(1)}$  can be rewritten as

$$\begin{aligned} I_{22}^{(1)} &= \frac{1}{x} \left\{ \sum_{k=1}^{[N_a]} (4k - a) \psi_1 \left( \frac{x}{4k - a} \right) - [N_a] (4[N_a] - a) \psi_1 \left( \frac{x}{4[N_a] - a} \right) \right\} \\ &\quad + \sum_{k=1}^{[N_a]-1} k \int_{x/(4(k+1)-a)}^{x/(4k-a)} t^{-2} \psi_1(t) dt \end{aligned}$$

by integration by parts and (3.7). We put

$$\tilde{F}(k) = \int_1^{x/(4k-a)} t^{-2} \psi_1(t) dt.$$

for simplicity. Since

$$\tilde{F}(k) = \int_1^\infty t^{-2} \psi_1(t) dt + \frac{4k - a}{12x} + O\left(\frac{k^2}{x^2}\right),$$

we have (by (3.7) again)

$$\sum_{k=1}^{[N_a]-1} k \int_{x/(4(k+1)-a)}^{x/(4k-a)} t^{-2} \psi_1(t) dt = -\frac{1}{96} + O(x^{-1/2}).$$

Therefore we have

$$\begin{aligned} I_{22}^{(1)} &= \frac{1}{x} \left\{ \sum_{k=1}^{[N_a]} (4k - a) \psi_1 \left( \frac{x}{4k - a} \right) - [N_a] (4[N_a] - a) \psi_1 \left( \frac{x}{4[N_a] - a} \right) \right\} \\ &\quad - \frac{1}{96} + O(x^{-1/2}). \end{aligned}$$

It is easy to see that the function  $I_{22}^{(2)}$  is transformed as

$$\begin{aligned} I_{22}^{(2)} &= [N_a] \int_{x/(4N_a-a)}^{x/(4[N_a]-a)} t^{-1} \psi(t) du \\ &= [N_a] \left\{ \frac{4[N_a]-a}{x} \psi_1\left(\frac{x}{4[N_a]-a}\right) - \frac{4N_a-a}{x} \psi_1\left(\frac{x}{4N_a-a}\right) \right\} + O(x^{-1/2}). \end{aligned}$$

Therefore we get the asymptotic formula

$$I_{22} = \frac{1}{x} \sum_{k=1}^{[N_a]} (4k-a) \psi_1\left(\frac{x}{4k-a}\right) - \frac{1}{96} - \frac{1}{4} \psi_1(\sqrt{x}) + O(x^{-1/2}). \tag{3.12}$$

By combining (3.11) and (3.12) and using the relation  $\bar{B}_2(x) = 2\psi_1(x) + 1/6$  again, we complete the proof of Lemma 3. ■

The estimate of  $I(a)$  is given in the following lemma.

**Lemma 4.** *For  $a = 1$  and  $3$ , we have*

$$I(a) \ll x^{-1/4}.$$

**Proof.** Substituting the well-known Fourier expansion

$$\bar{B}_2(x) = \frac{1}{2\pi^2} \sum_{m \neq 0} \frac{e(mx)}{m^2}$$

to the right-hand sides of (3.3) and (3.10), and interchanging summations, we have

$$\begin{aligned} I(a) &= -\frac{1}{\pi^2 x} \sum_{m \neq 0} \frac{e(ma/4)}{m^2} \sum_{k \leq \sqrt{x}} ke\left(\frac{mx}{4k}\right) \\ &\quad - \frac{1}{\pi^2 x} \sum_{m \neq 0} \frac{1}{m^2} \sum_{k \leq (\sqrt{x}+a)/4} ke\left(\frac{mx}{4k-a}\right) + O(x^{-1/2}). \end{aligned}$$

To treat the innermost sum, we use the estimate of the exponential sum as [13] (in fact, our case is easier than theirs). Let  $M > 0$  and  $M < M' \leq 2M$ . Then we have

$$\sum_{M < k \leq M'} e\left(\frac{z}{k}\right) \ll M^{-1/2} z^{1/2} + M^{3/2} z^{-1/2} \tag{3.13}$$

and

$$\sum_{M < k \leq M'} e\left(\frac{z}{4k-a}\right) \ll M^{-1/2} z^{1/2} + M^{3/2} z^{-1/2}, \tag{3.14}$$

where (3.13) is given in [13, (7)] and (3.14) is derived easily by [24, Theorem 5.9]. Note that these exponential sums have the same upper bounds.

By partial summation, (3.13) and (3.14), we see that the exponential sums

$$M^{-1} \sum_{M < k \leq M'} ke\left(\frac{z}{k}\right) \quad \text{and} \quad M^{-1} \sum_{M < k \leq M'} ke\left(\frac{z}{4k-a}\right)$$

can be also estimated by the right-hand side in (3.13) and (3.14). Hence, by the splitting argument on the range of  $k$ , we obtain

$$\sum_{k \leq \sqrt{x}} ke\left(\frac{mx}{k}\right) \quad \text{and} \quad \sum_{k \leq (\sqrt{x+a})/4} ke\left(\frac{mx}{4k-a}\right) \ll \sqrt{x} \left(x^{-1/4}(mx)^{1/2} + x^{3/4}(mx)^{-1/2}\right) \ll x^{3/4}m^{1/2}.$$

Since the sum with respect to  $m$  is absolutely convergent, we have the assertion of the lemma. ■

To derive the asymptotic formula for  $Z(a)$  with an error of negative powers, we shall make use of the integral version of the Dirichlet hyperbola method, which we recall now. Let  $f(t)$  and  $g(t)$  denote integrable functions in  $[1, x]$  and  $y$  denote a parameter with  $1 \leq y \leq x$ . Then we have, for a positive number  $c$  with  $c \geq 1$ ,

$$\begin{aligned} \int_1^{x/c} f(t) \int_c^{x/t} g(u) du dt &= \int_1^y f(t) \int_c^{x/t} g(u) du dt + \int_y^{x/c} f(t) \int_c^{x/t} g(u) du dt \\ &= \int_1^y f(t) \int_c^{x/t} g(u) du dt + \int_c^{x/y} g(u) \int_y^{x/u} f(t) dt du \\ &= \int_1^y f(t) \int_c^{x/t} g(u) du dt + \int_c^{x/y} g(u) \int_1^{x/u} f(t) dt du \\ &\quad - \left(\int_1^y f(t) dt\right) \left(\int_c^{x/y} g(u) du\right). \end{aligned} \tag{3.15}$$

**Lemma 5.** For  $a = 1$  and  $3$ , let  $Z(a)$  be the integral defined by (3.2). For  $\Re\theta > 0$ , we have

$$Z(a) = c_a(\theta)c(\theta) + O(x^{-\Re\theta/2}) + O(x^{1/2-\Re\theta}),$$

where we put

$$c(\theta) = \int_1^\infty t^{-\theta} \psi(t) dt \tag{3.16}$$

and

$$c_a(\theta) = \int_1^\infty \left(t - \frac{a}{4}\right)^{-\theta} \psi(t) dt. \tag{3.17}$$

**Proof.** Changing the variable as  $v = 4u - a$  in the innermost integral in  $Z(a)$ , we find that

$$Z(a) = 4^{\theta-1} \int_1^{x/(4-a)} t^{-\theta} \psi(t) \int_{4-a}^{x/t} v^{-\theta} \psi\left(\frac{v+a}{4}\right) dv dt.$$

We apply the Dirichlet hyperbola method to the integral on the right-hand side above by putting  $f(t) = t^{-\theta} \psi(t)$ ,  $g(u) = u^{-\theta} \psi((u+a)/4)$ ,  $y = \sqrt{x}$  and  $c = 4 - a$  in (3.15). Furthermore if note that

$$\int_1^x t^{-\theta} \psi(t) dt = c(\theta) + O(x^{-\Re\theta})$$

and

$$\int_1^x \left(t - \frac{a}{4}\right)^{-\theta} \psi(t) dt = c_a(\theta) + O(x^{-\Re\theta})$$

for  $\Re\theta > 0$ , we get the assertion of this lemma immediately. ■

The function  $c(\theta)$  can be written as

$$c(\theta) = \frac{1}{\theta - 2} - \frac{1}{2(\theta - 1)} - \frac{\zeta(\theta - 1)}{\theta - 1}, \tag{3.18}$$

particularly,  $c(\theta)$  is regular at  $\theta = 2$  (see e.g. [5]).

For later use, we derive an explicit evaluation of  $c_1(\theta) - c_3(\theta)$  by means of the Dirichlet  $L$ -function.

**Lemma 6.** *For  $\Re\theta > 0$ , we have*

$$4^{1-\theta}(c_1(\theta) - c_3(\theta)) = \frac{L(\theta - 1, \chi)}{\theta - 1} + \frac{3 - 2\theta - 3^{1-\theta}(1 - 2\theta)}{4(1 - \theta)(2 - \theta)}.$$

Remark that the function on the right-hand side is holomorphic at  $\theta = 1$  and 2.

**Proof.** From the definition of  $\chi(n)$  we have

$$\sum_{n \leq x} \chi(n) = \frac{1}{2} - \psi\left(\frac{x+3}{4}\right) + \psi\left(\frac{x+1}{4}\right). \tag{3.19}$$

Then, by partial summation and (3.19), we have

$$\sum_{n \leq x} \chi(n)n^{-s} = x^{-s} \sum_{n \leq x} \chi(n) + s \int_1^x t^{-s-1} \left\{ \frac{1}{2} - \psi\left(\frac{t+3}{4}\right) + \psi\left(\frac{t+1}{4}\right) \right\} dt.$$

Hence, by noting

$$c_a(\theta) = 4^{\theta-1} \int_{4-a}^{\infty} t^{-\theta} \psi\left(\frac{t+a}{4}\right) dt,$$

and taking  $x \rightarrow \infty$  with  $s = \theta - 1$ , we have

$$L(\theta - 1, \chi) = \frac{1}{2} + (\theta - 1) \left\{ 4^{1-\theta}(c_1(\theta) - c_3(\theta)) + \int_1^3 t^{-\theta} \psi \left( \frac{t+1}{4} \right) dt \right\} \quad (3.20)$$

for  $\Re\theta > 1$ , and for  $\Re\theta > 0$  by analytic continuation of  $c_a(\theta)$ .

On the other hand, because of  $\psi((t+1)/4) = (t-1)/4$  for  $1 \leq t < 3$ , we have

$$\int_1^3 t^{-\theta} \psi \left( \frac{t+1}{4} \right) dt = \frac{1}{4} \int_1^3 t^{-\theta} (t-1) dt = \frac{1}{4} \left\{ \frac{3^{2-\theta} - 1}{2-\theta} - \frac{3^{1-\theta} - 1}{1-\theta} \right\}.$$

Substituting the above into (3.20) we obtain the assertion of this lemma. ■

### 4. Proof of Theorem 2

We used the inequality (2.5) in the proof of Theorem 1, but this is not enough for the proof of Theorem 2. To get more precise estimate, we return to (2.1).

Let  $0 < \Re\theta \leq 1$  and  $\theta \neq 1$ . On the right hand side of (2.1), the sum  $\sum_{n \leq x} \chi(n)$  was already treated in (3.19). As for the sum  $\sum_{n \leq x} \chi(n)n^{-1}$ , we have

$$\sum_{n \leq x} \chi(n)n^{-1} = \frac{\pi}{4} - \frac{1}{x} \left\{ \psi \left( \frac{x+3}{4} \right) - \psi \left( \frac{x+1}{4} \right) \right\} + O(x^{-2}) \quad (4.1)$$

by partial summation and (3.19). Note that (4.1) is a refinement of the formula (2.5) for  $s = 1$ .

As for the sum  $\sum_{n \leq y} \chi(n)n^{1-\theta}$ , by the definition of  $\chi(n)$ , we divide it as

$$\sum_{n \leq y} \chi(n)n^{1-\theta} = 4^{1-\theta} \left\{ \sum_{1 \leq m \leq (y+3)/4} \left( m - \frac{3}{4} \right)^{1-\theta} - \sum_{1 \leq m \leq (y+1)/4} \left( m - \frac{1}{4} \right)^{1-\theta} \right\}.$$

Applying the Euler-Maclaurin summation formula to each sum above, we get

$$\begin{aligned} \sum_{1 \leq m \leq (y+a)/4} \left( m - \frac{a}{4} \right)^{1-\theta} &= \frac{1}{2-\theta} \left\{ \left( \frac{y}{4} \right)^{2-\theta} - \left( \frac{4-a}{4} \right)^{2-\theta} \right\} \\ &\quad - \psi \left( \frac{y+a}{4} \right) \left( \frac{y}{4} \right)^{1-\theta} + \frac{1}{2} \left( \frac{4-a}{4} \right)^{1-\theta} \\ &\quad + (1-\theta) \int_1^{(y+a)/4} \left( t - \frac{a}{4} \right)^{-\theta} \psi(t) dt \end{aligned} \quad (4.2)$$

for  $a = 1$  and  $3$ , hence

$$\begin{aligned} \sum_{n \leq y} \chi(n)n^{1-\theta} &= \frac{3-2\theta-(1-2\theta)3^{1-\theta}}{4(2-\theta)} - \left\{ \psi \left( \frac{y+3}{4} \right) - \psi \left( \frac{y+1}{4} \right) \right\} y^{1-\theta} \\ &\quad + 4^{1-\theta} (1-\theta) \{ c_3(\theta) - c_1(\theta) + O(y^{-\Re\theta}) \}. \end{aligned} \quad (4.3)$$



The implied constant in the  $O$ -symbol in (4.3) is bounded uniformly in  $\theta$  for  $0 \leq \Re\theta \leq 5/4$ .

It remains to evaluate the last sum on the right-hand side of (2.1). We have similarly that

$$\begin{aligned} & \sum_{n \leq x} \chi(n)n^{1-\theta} \int_1^{x/n} t^{-\theta} \psi(t) dt \\ &= \int_1^x t^{-\theta} \psi(t) \left\{ \sum_{1 \leq m \leq (x/t+3)/4} (4m-3)^{1-\theta} - \sum_{1 \leq m \leq (x/t+1)/4} (4m-1)^{1-\theta} \right\} dt. \end{aligned}$$

If we note that the second sum in the integrand on the right-hand side above is empty for  $x/3 < t \leq x$ , we find that

$$\sum_{n \leq x} \chi(n)n^{1-\theta} \int_1^{x/n} t^{-\theta} \psi(t) dt = R_3(x) - R_1(x), \tag{4.4}$$

where

$$R_a(x) = \int_1^{x/(4-a)} t^{-\theta} \psi(t) \sum_{1 \leq m \leq (x/t+a)/4} (4m-a)^{1-\theta} dt.$$

Substituting (4.2) with  $R_a(x)$  and applying Lemmas 4 and 6, we obtain

$$\begin{aligned} R_a(x) &= \frac{x^{2-\theta}}{4(2-\theta)} \int_1^{x/(4-a)} t^{-2} \psi(t) dt + \frac{(4-a)^{1-\theta}(a-2\theta)}{4(2-\theta)} \int_1^{x/(4-a)} t^{-\theta} \psi(t) dt \\ &\quad - x^{1-\theta} \int_1^{x/(4-a)} t^{-1} \psi(t) \psi\left(\frac{x}{4t} + \frac{a}{4}\right) dt \\ &\quad + 4^{1-\theta}(1-\theta) \int_1^{x/(4-a)} t^{-\theta} \psi(t) \int_1^{(x/t+a)/4} \left(u - \frac{a}{4}\right)^{-\theta} \psi(u) du dt \\ &= \frac{x^{2-\theta}}{4(2-\theta)} (c(2) + O(x^{-2})) + \frac{(4-a)^{1-\theta}(a-2\theta)}{4(2-\theta)} (c(\theta) + O(x^{-\Re\theta})) \\ &\quad - x^{1-\theta} I(a) + 4^{1-\theta}(1-\theta) Z(a) \\ &= \frac{x^{2-\theta}}{4(2-\theta)} c(2) + \frac{(4-a)^{1-\theta}(a-2\theta)}{4(2-\theta)} c(\theta) \\ &\quad + 4^{1-\theta}(1-\theta) c_a(\theta) c(\theta) + O(x^{3/4-\Re\theta}), \end{aligned}$$

where the implied constant is bounded uniformly in  $\theta$  for  $0 \leq \Re\theta \leq 5/4$ . Subtracting  $R_1(x)$  from  $R_3(x)$  we get

$$\begin{aligned} \sum_{n \leq x} \chi(n)n^{1-\theta} \int_1^{x/n} t^{-\theta} \psi(t) dt &= \frac{(3-2\theta) - (1-2\theta)3^{1-\theta}}{4(2-\theta)} c(\theta) \\ &\quad + 4^{1-\theta}(1-\theta) \{c_3(\theta) - c_1(\theta)\} c(\theta) + O(x^{3/4-\Re\theta}). \tag{4.5} \end{aligned}$$

Now substitute (3.19), (4.1), (4.3) and (4.5) with (2.1). Thus we get

$$\begin{aligned} \int_1^x t^{-\theta} P(t) dt &= \frac{4}{2-\theta} x^{2-\theta} \left\{ \frac{\pi}{4} - \frac{1}{x} \left( \psi \left( \frac{x+3}{4} \right) - \psi \left( \frac{x+1}{4} \right) \right) \right\} \\ &+ \frac{2\theta}{(1-\theta)(2-\theta)} \cdot \frac{(3-2\theta) - (1-2\theta)3^{1-\theta}}{4(2-\theta)} \\ &- \frac{2\theta}{2-\theta} 4^{1-\theta} (c_1(\theta) - c_3(\theta)) \\ &- \frac{2\theta}{(1-\theta)(2-\theta)} x^{1-\theta} \left\{ \psi \left( \frac{x+3}{4} \right) - \psi \left( \frac{x+1}{4} \right) \right\} \\ &- \frac{2}{1-\theta} x^{1-\theta} \left\{ \frac{1}{2} - \psi \left( \frac{x+3}{4} \right) + \psi \left( \frac{x+1}{4} \right) \right\} - \frac{\pi}{2-\theta} x^{2-\theta} \\ &+ \frac{\pi}{2-\theta} - \frac{(3-2\theta) - (1-2\theta)3^{1-\theta}}{2-\theta} c(\theta) \\ &- 4^{2-\theta} (1-\theta) \{c_3(\theta) - c_1(\theta)\} c(\theta) + O(x^{3/4-\Re\theta}). \end{aligned}$$

But the sum of the coefficients of  $\psi((x+3)/4) - \psi((x+1)/4)$  in the above formula vanishes, hence we have

$$\begin{aligned} \int_1^x t^{-\theta} P(t) dt &= -\frac{x^{1-\theta}}{1-\theta} + \frac{\pi}{2-\theta} + \left( \frac{2\theta}{(1-\theta)(2-\theta)} - 4c(\theta) \right) \\ &\times \left\{ \frac{(3-2\theta) - (1-2\theta)3^{1-\theta}}{4(2-\theta)} - 4^{1-\theta} (1-\theta) (c_1(\theta) - c_3(\theta)) \right\} \\ &+ O(x^{3/4-\Re\theta}). \end{aligned}$$

Using (3.18) and Lemma 6, we finally obtain

$$\int_1^x t^{-\theta} P(t) dt = -\frac{x^{1-\theta}}{1-\theta} + \frac{\pi}{2-\theta} + \frac{4\zeta(\theta-1)L(\theta-1, \chi)}{\theta-1} + O(x^{3/4-\Re\theta}). \tag{4.6}$$

This proves Theorem 2 for the case  $\theta \neq 1$  and  $j = 0$ .

The formula (1.12) is obtained by partial summation, (1.11) and

$$\int_1^x t^{-\theta} \log^j t dt = \frac{j!}{(\theta-1)^{j+1}} - x^{1-\theta} \sum_{m=0}^j \binom{j}{m} \frac{m!}{(\theta-1)^{m+1}} \log^{j-m} x,$$

which is valid for a fixed non-negative integer  $j$  and a complex number  $\theta$  with  $\theta \neq 1$  [5, the formula (5.10)]. This completes the proof of Theorem 2 in the case  $0 < \Re\theta \leq 1$  and  $\theta \neq 1$ .

For the proof of (1.13), we let  $\theta \rightarrow 1$  in (4.6) and get

$$\begin{aligned} \int_1^x t^{-1} P(t) dt &= -\log x + \pi + 4 \{ \zeta'(0)L(0, \chi) + \zeta(0)L'(0, \chi) \} + O(x^{-1/4}) \\ &= -\log x + \{ \pi - \log 2\pi - 2L'(0, \chi) \} + O(x^{-1/4}). \end{aligned}$$

The formula (1.14) can be derived by using integration by parts and (1.13). This completes the proof of Theorem 2.

**5. On the constant  $C_1$**

As we saw in Section 1, we have two kinds of representation of the constant  $C_1$ ; one is (1.7) and the other is (1.11). Using these representations we shall give a new proof of the functional equation of  $\zeta_K(s)$  for  $K = \mathbb{Q}(\sqrt{-1})$ .

In order to get the explicit form of  $\int_0^x P(t)dt$  we need the following lemma.

**Lemma 7.** *For  $x \geq 0$ , let  $R(x)$  denote the number of lattice points which lie on or inside the circle  $\xi^2 + \eta^2 = x$  in  $(\xi, \eta)$ -plane. Then we have*

$$\int_0^x R(t)dt = \frac{\pi}{2}x^2 + \frac{x}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} J_2(2\pi\sqrt{nx}), \tag{5.1}$$

where  $J_\nu(x)$  is the Bessel function of the first kind of order  $\nu$ . The infinite series on the right hand side of (5.1) is absolutely convergent.

The proof of this formula can be found in, e.g., [9] and [15, Theorem 3.11]. In particular, in [15, Theorem 3.11], (5.1) is proved as follows: first represent  $\int_0^x R(t)dt$  by the integral of Bernoulli functions, called the ‘‘convolution’’ of the Bernoulli functions. And then substitute the Fourier expansion of  $\psi(x)$  into that integral and apply the integral formula of  $J_\nu(x)$ . We should stress that the proof given there is independent of the Dedekind zeta function  $\zeta_K(s)$ .

Now we consider the error function  $P(x)$  defined by (1.1) for  $x < 1$ . In order to relate it to the function  $R(x)$ , we define  $P(x) = -\pi x$  for  $0 \leq x < 1$ . Then

$$R(x) = 1 + \pi x + P(x)$$

holds true for  $x \geq 0$ . Under this interpretation, the above lemma leads

$$\int_0^x P(t)dt = -x + \frac{x}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} J_2(2\pi\sqrt{nx}). \tag{5.2}$$

Suppose that  $3/4 < \Re\theta < 1$ , and let

$$f(x) = \int_0^x t^{-\theta} P(t)dt, \tag{5.3}$$

which we shall calculate in two ways.

Firstly, by (1.11) and the above definition of  $P(x)$ , we have

$$f(x) = -\frac{1}{1-\theta}x^{1-\theta} + \frac{4\zeta_K(\theta-1)}{\theta-1} + O(x^{3/4-\Re\theta}).$$

Secondly, by applying integration by parts and substituting (5.2) into (5.3), we have

$$\begin{aligned} f(x) &= -x^{1-\theta} + \frac{x^{1-\theta}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} J_2(2\pi\sqrt{nx}) \\ &\quad + \theta \int_0^x t^{-\theta-1} \left\{ -t + \frac{t}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} J_2(2\pi\sqrt{nt}) \right\} dt \\ &= -\frac{x^{1-\theta}}{1-\theta} + \frac{\theta}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n} \int_0^x t^{-\theta} J_2(2\pi\sqrt{nt}) dt + O(x^{3/4-\Re\theta}), \end{aligned}$$

where the interchange of summation and integration can be justified by  $J_\nu(x) = O(1/\sqrt{x})$  as  $x \rightarrow \infty$  and the formula (5.5) below. Recall the well-known integral formula

$$\int_0^\infty t^{\rho-1} J_\nu(at) dt = 2^{\rho-1} a^{-\rho} \Gamma\left(\frac{\nu+\rho}{2}\right) / \Gamma\left(1 + \frac{\nu-\rho}{2}\right) \tag{5.4}$$

for  $-\Re\nu < \Re\rho < 3/2$  and  $a > 0$  (cf. e.g. [3, eq. (19) in p.49]). Then by changing the variable of integration as  $\sqrt{t} = u$  and applying (5.4) with  $a = 2\pi\sqrt{n}$ ,  $\nu = 2$  and  $\rho = 2 - 2\theta$ , we get

$$\int_0^\infty t^{-\theta} J_2(2\pi\sqrt{nt}) dt = \frac{\pi^{2\theta-2} \Gamma(2-\theta)}{n^{1-\theta} \Gamma(1+\theta)}, \tag{5.5}$$

and therefore

$$\begin{aligned} f(x) &= -\frac{x^{1-\theta}}{1-\theta} + \frac{\theta}{\pi} \pi^{2\theta-2} \frac{\Gamma(2-\theta)}{\Gamma(1+\theta)} \sum_{n=1}^{\infty} \frac{r(n)}{n^{2-\theta}} + O(x^{3/4-\Re\theta}) \\ &= -\frac{x^{1-\theta}}{1-\theta} + 4\pi^{2\theta-3} \frac{\Gamma(2-\theta)}{\Gamma(\theta)} \zeta_K(2-\theta) + O(x^{3/4-\Re\theta}). \end{aligned}$$

Comparing these representations of  $f(x)$  and letting  $x \rightarrow \infty$ , we have

$$\frac{\zeta_K(\theta-1)}{\theta-1} = \pi^{2\theta-3} \frac{\Gamma(2-\theta)}{\Gamma(\theta)} \zeta_K(2-\theta).$$

The change of the variable  $\theta = s + 1$  derives that

$$\pi^{-s} \Gamma(s) \zeta_K(s) = \pi^{s-1} \Gamma(1-s) \zeta_K(1-s),$$

firstly for  $-1/4 < \Re s < 0$ , and for all  $s$  by analytic continuation. This gives

**Corollary 1.** *The formula (1.11) in Theorem 2 and (5.1) in Lemma 7 give the proof of the functional equation of the Dedekind zeta-function  $\zeta_K(s)$  associated with  $K = \mathbb{Q}(\sqrt{-1})$ .*

**Remark 1.** It is appropriate to give some remarks on the relation between the analytic continuation and the functional equation of  $\zeta_K(s)$ . In the standard text-book such as [16, §3 in Chapter XIII], they are proved simultaneously by using the theta transformation formula. But in the case that we are concerned with,  $\zeta_K(s)$  is represented as the product of  $\zeta(s)$  and  $L(s, \chi)$ , whose analytic continuations can be proved independently from their functional equations. In this sense, we can say that if we assume the analytic continuation of  $\zeta_K(s)$ , the functional equation is proved by (1.11) via the Hardy identity.

### 6. On the log-Riesz mean of $r(n)$

Using the theory of elliptic and theta functions, Müller [18] proved that

$$\sum_{n \leq x} r(n) \log \frac{x}{n} = \pi x - \log x - \log \frac{\Gamma^4(1/4)}{4\pi} + O(x^{-1/4}) \tag{6.1}$$

and a little later Carlitz [2] gave a simpler proof of (6.1) by using the Abel transformation formula. (But Carlitz did not give the explicit form of the constant term.)

We shall give an alternative proof of (6.1) as an application of (1.13). In fact, by partial summation, we have

$$\sum_{n \leq x} r(n) \log \frac{x}{n} = \int_1^x t^{-1} \sum_{n \leq t} r(n) dt = \pi(x-1) + \int_1^x t^{-1} P(t) dt. \tag{6.2}$$

Hence the formula (1.13) implies that

$$\sum_{n \leq x} r(n) \log \frac{x}{n} = \pi x - \log x - \log 2\pi - 2L'(0, \chi) + O(x^{-1/4}). \tag{6.3}$$

However, it is well-known that

$$L'(0, \chi) = \log \Gamma^2(1/4) - \log \pi - \frac{3}{2} \log 2 \tag{6.4}$$

(see e.g. [1, p.344]). Therefore the substitution of (6.4) into (6.3) reproduces Müller's result (6.1).

We remark that Ayoub and Chowla [1, eqs.(3) and (4)] also derived the same formula as (6.3), in fact they considered the general imaginary quadratic field cases by the use of (the logarithmic version of) Perron's formula [1, eq.(2)]. Redmond [22] and [23] considered the generalization of the results of Ayoub and Chowla [1]. Our proof for (6.2) is different from theirs.

**7. Dirichlet series related with  $P(x)$**

In this section we study some properties of Dirichlet series and integrals related to  $P(x)$ . First define  $\mathcal{D}_j(s)$  and  $\mathcal{I}_j(s)$  by

$$\mathcal{D}_j(s) = \sum_{n=1}^{\infty} P(n)^j n^{-s} \tag{7.1}$$

and

$$\mathcal{I}_j(s) = \int_1^{\infty} t^{-s} P(t)^j dt \tag{7.2}$$

for  $j = 1$  and  $2$ . It is easy to see that the right hand sides of (7.1) and (7.2) converge absolutely for  $\Re s > 1 + j/4$ . Further, we note that  $\mathcal{I}_1(s) = \mathfrak{J}_0(s)$ . In our previous paper [7] we studied the possibility of analytic continuation of  $\mathcal{D}_j(s)$  and  $\mathcal{I}_j(s)$ , which we recall by adding the exact evaluation of the residues.

**Lemma 8 ([7, Theorem 3]).** *We have*

- (1) *The Dirichlet series  $\mathcal{D}_1(s)$  can be continued analytically to the whole complex plane. It has a simple pole at  $s = 1$  with the residue*

$$\operatorname{Res}_{s=1} \mathcal{D}_1(s) = \frac{\pi}{2} - 1.$$

- (2) *The Dirichlet series  $\mathcal{D}_2(s)$  can be continued analytically to the right-half plane  $\Re s > 2/3$ . It has simple poles at  $s = 1$  and  $3/2$  with the residues*

$$\operatorname{Res}_{s=1} \mathcal{D}_2(s) = \frac{\pi(\pi - 6)}{6} \quad \text{and} \quad \operatorname{Res}_{s=3/2} \mathcal{D}_2(s) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}},$$

*respectively.*

**Lemma 9 ([7, Section 6]).** *The function  $\mathcal{I}_2(s)$  can be continued analytically to the right-half plane  $\Re s > 2/3$ . It has a simple pole at  $s = 3/2$  with the residue*

$$\operatorname{Res}_{s=3/2} \mathcal{I}_2(s) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}}.$$

*In particular, the function  $\mathcal{I}_2(s)$  is regular at  $s = 1$ .*

In [7], we did not give the exact evaluation of residue at each pole, and furthermore, the assertion of the location of the pole of  $\mathcal{I}_2(s)$  at  $s = 1$  is misprinted. So we would like to give a sketch of the proofs of Lemmas 8 and 9 with the corrections on the poles of  $\mathcal{I}_2(s)$ .

For that purpose, it is convenient to use the definition of the error term different from (1.1), namely, let  $\tilde{P}(x)$  denote the function defined by

$$\tilde{P}(x) = \sum'_{n \leq x} r(n) - \pi x + 1 \tag{7.3}$$

for  $x \geq 1$ , where the symbol  $\sum'_{n \leq x}$  means that the last term is to be halved if  $x$  is an integer. This definition is also used frequently.

Put

$$\mathcal{L}(p) = \int_1^\infty \tilde{P}(t)^2 e^{-pt} dt \quad (p > 0)$$

and

$$\tilde{\mathcal{I}}_2(s) = \int_1^\infty t^{-s} \tilde{P}(t)^2 dt.$$

In the region  $\Re s > 3/2$ , the defining integral for  $\tilde{\mathcal{I}}_2(s)$  is absolutely convergent, and in this region we have

$$\int_0^\infty \mathcal{L}_F(p) p^{s-1} dp = \Gamma(s) \tilde{\mathcal{I}}_2(s). \tag{7.4}$$

Ivić [12] showed that

$$\mathcal{L}(p) = \left( \frac{1}{4\pi^{3/2}} \sum_{n=1}^\infty \frac{r(n)^2}{n^{3/2}} \right) p^{-3/2} - p^{-1} + O(p^{-2/3-\varepsilon}) \tag{7.5}$$

as  $p \rightarrow +0$ . So substituting (7.5) with the left hand side of (7.4), we have

$$\Gamma(s) \tilde{\mathcal{I}}_2(s) = \left( \frac{1}{4\pi^{3/2}} \sum_{n=1}^\infty \frac{r(n)^2}{n^{3/2}} \right) \frac{1}{s-3/2} - \frac{1}{s-1} + \mathcal{H}(s)$$

for  $\Re s > 3/2$ . Here  $\mathcal{H}(s)$  represents a holomorphic function in the region  $\Re s > 2/3$ . Hence, this expression gives an analytic continuation of  $\tilde{\mathcal{I}}_2(s)$  to the region  $\Re s > 2/3$ . Now it is easy to see that  $\tilde{\mathcal{I}}_2(s)$  has poles at  $s = 3/2$  and  $s = 1$  with the residues

$$\operatorname{Res}_{s=3/2} \tilde{\mathcal{I}}_2(s) = \frac{1}{2\pi^2} \sum_{n=1}^\infty \frac{r(n)^2}{n^{3/2}} \quad \text{and} \quad \operatorname{Res}_{s=1} \tilde{\mathcal{I}}_2(s) = -1,$$

respectively.

For the proof of Lemma 9, we first note that

$$\int_1^\infty t^{-s} \tilde{P}(t)^2 dt = \int_1^\infty t^{-s} (P(t) + 1)^2 dt$$

for  $\Re s > 3/2$ , and from which we deduce that

$$\mathcal{I}_2(s) = \tilde{\mathcal{I}}_2(s) - 2\mathcal{I}_1(s) - \frac{1}{s-1}. \tag{7.6}$$

This gives the analytic continuation of  $\mathcal{I}_2(s)$  to  $\Re s > 2/3$ .

The possible poles of  $\mathcal{I}_2(s)$  are  $s = 3/2$  and  $s = 1$ , whose orders are at most one. The point  $s = 3/2$  is in fact a pole and the residue at  $s = 3/2$  is given immediately from the formula (7.6). But  $s = 1$  is not a pole of  $\mathcal{I}_2(s)$ . To see this, note that  $\text{Res}_{s=1} \mathcal{I}_1(s) = -1$  by (1.9) and the result  $\zeta_K(0) = -1/4$ , therefore we have

$$\text{Res}_{s=1} \mathcal{I}_2(s) = \text{Res}_{s=1} \widetilde{\mathcal{I}}_2(s) + 1 = 0.$$

Hence the function  $\mathcal{I}_2(s)$  is regular at  $s = 1$ . This completes the proof of Lemma 9.

Next we consider the residues of  $\mathcal{D}_2(s)$ . By [7, Lemma 7], we have

$$\mathcal{D}_1(s) = \mathcal{I}_1(s) + s \int_1^\infty \left(\frac{1}{2} - \psi(t)\right) P(t)t^{-s-1} dt + \pi \int_1^\infty \left(\frac{1}{2} - \psi(t)\right) t^{-s} dt$$

and thus

$$\begin{aligned} & \int_1^\infty \left(\frac{1}{2} - \psi(t)\right) P(t)t^{-s} dt \\ &= \frac{1}{s-1} \left\{ \mathcal{D}_1(s-1) - \mathcal{I}_1(s-1) - \pi \left( \frac{1}{s-2} - \frac{1}{s-3} + \frac{\zeta(s-2)}{s-2} \right) \right\} \end{aligned} \tag{7.7}$$

for  $\Re s > 5/4$ . Furthermore, we again apply [7, Lemma 7] and use (7.7) to obtain

$$\begin{aligned} \mathcal{D}_2(s) &= \mathcal{I}_2(s) + s \int_1^\infty \left(\frac{1}{2} - \psi(t)\right) P(t)^2 t^{-s-1} dt \\ &+ \frac{2\pi}{s-1} \left\{ \mathcal{D}_1(s-1) - \mathcal{I}_1(s-1) - \pi \left( \frac{1}{s-2} - \frac{1}{s-3} + \frac{\zeta(s-2)}{s-2} \right) \right\} \end{aligned}$$

for  $\Re s > 5/4$ , and thus for  $\Re s > 2/3$  by analytic continuation. From this relation, we trivially obtain  $\text{Res}_{s=3/2} \mathcal{D}_2(s) = \text{Res}_{s=3/2} \mathcal{I}_2(s)$ , since all terms except for  $\mathcal{I}_2(s)$  are regular at  $s = 3/2$ . The residue of  $\mathcal{D}_2(s)$  at  $s = 1$  can also be derived easily from this formula and the properties  $\mathcal{I}_1(0) = \pi/2$ ,  $\zeta(-1) = -1/12$  and  $L(-1, \chi) = 0$ . This completes the proof of Lemma 8.

Under these preparations, we shall introduce another type of Dirichlet series related to  $P(x)$ . Let

$$\mathcal{Y}(s) = \sum_{n=1}^\infty \frac{r(n)P(n)}{n^s}.$$

This series is convergent absolutely for  $\Re s > 5/4$  by the mean square formula for  $P(x)$ . The analytic continuation of  $\mathcal{Y}(s)$  is given by the following theorem.

**Theorem 4.** *The Dirichlet series  $\mathcal{Y}(s)$  can be continued analytically to the right-half plane  $\Re s > -1/3$ , whose explicit form is given by*

$$\mathcal{Y}(s) = \frac{8\zeta_K(s)^2}{(1+2^{-s})\zeta(2s)} - \frac{1}{2}\pi^2 + \pi\mathcal{I}_1(s) + \frac{1}{2}s\mathcal{I}_2(s+1). \tag{7.8}$$



In the region  $\Re s \geq 1/2$ , it has a simple pole at  $s = 1/2$  with the residue

$$\operatorname{Res}_{s=1/2} \mathcal{Y}(s) = \frac{1}{8\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}},$$

and it also has a double pole at  $s = 1$ , whose Laurent expansion is of the form

$$\mathcal{Y}(s) = \frac{2}{(s-1)^2} + \frac{1}{s-1} \left\{ 4\gamma + \frac{16}{\pi} L'(1, \chi) - \frac{24}{\pi^2} \zeta'(2) + \frac{2}{3} \log 2 - \pi \right\} + O(1).$$

**Proof.** Let

$$\mathcal{Y}_N(s) = \sum_{n \leq N} r(n)P(n)n^{-s},$$

where  $N$  is a sufficiently large integer. By using (1.1) and partial summation, we have

$$\begin{aligned} 2\mathcal{Y}_N(s) - \sum_{n \leq N} r(n)^2 n^{-s} &= \left( \sum_{n \leq N} r(n)n^{-s} \right) \left( \sum_{m \leq N} r(m) \right) - \pi \frac{2-s}{1-s} \sum_{n \leq N} r(n)n^{-s+1} \\ &\quad + \frac{\pi s}{1-s} \sum_{n \leq N} r(n) - s \sum_{n \leq N} r(n) \int_1^n t^{-s-1} P(t) dt. \end{aligned}$$

Furthermore, by noting

$$\begin{aligned} \sum_{n \leq N} r(n) \int_1^n t^{-s} P(t) dt &= \left( \sum_{n \leq N} r(n) \right) \int_1^N t^{-s} P(t) dt - \pi \int_1^N t^{-s+1} P(t) dt \\ &\quad - \int_1^N t^{-s} P(t)^2 dt, \end{aligned}$$

and by partial summation again, we have

$$\begin{aligned} &2\mathcal{Y}_N(s) - \sum_{n \leq N} r(n)^2 n^{-s} \\ &= \left\{ \sum_{n \leq N} \frac{r(n)}{n^s} - s \int_1^N t^{-s-1} P(t) dt - \pi \frac{2-s}{1-s} N^{1-s} + \frac{\pi s}{1-s} \right\} \left( \sum_{n \leq N} r(n) \right) \\ &\quad + \pi(2-s) \int_1^N t^{-s} \sum_{n \leq t} r(n) dt + \pi s \int_1^N t^{-s} P(t) dt + s \int_1^N t^{-s-1} P(t)^2 dt \\ &= -\pi^2 + 2\pi \int_1^N t^{-s} P(t) dt + s \int_1^N t^{-s-1} P(t)^2 dt + P(N)^2 N^{-s}. \end{aligned}$$

Assuming that  $\Re s > 5/4$ , we let  $N \rightarrow \infty$ , then we get (7.8), since

$$\sum_{n \leq x} r(n)^2 n^{-s} = \frac{16\zeta_K(s)^2}{(1+2^{-s})\zeta(2s)} + O(x^{1-\Re s} \log x)$$

(for  $\Re s > 1$ ) (cf. e.g. [21, p.187]) and  $P(x) = O(x^{1/3})$ . The right hand side of (7.8) gives the analytic continuation of  $\mathcal{Y}(s)$  to  $\Re s > -1/3$ . The other assertions of this lemma also follow easily from this formula and the known results of  $\mathcal{I}_j(s)$  and  $\zeta(s)$ . This completes the proof of the theorem. ■

Theorem 4 can be applied to the study of the Dirichlet series  $\tilde{\mathcal{D}}_j(s)$  defined by

$$\tilde{\mathcal{D}}_j(s) = \sum_{n=1}^{\infty} \tilde{P}(n)^j n^{-s}. \tag{7.9}$$

Since  $\tilde{P}(n) = P(n) - \frac{1}{2}r(n) + 1$  for a positive integer  $n$ , we find that

$$\tilde{\mathcal{D}}_1(s) = \mathcal{D}_1(s) - 2\zeta_K(s) + \zeta(s)$$

and

$$\tilde{\mathcal{D}}_2(s) = \mathcal{D}_2(s) + 2\mathcal{D}_1(s) - 4\zeta_K(s) + \zeta(s) + \frac{4\zeta_K(s)^2}{(1 + 2^{-s})\zeta(2s)} - \mathcal{Y}(s)$$

for  $\Re s > 1 + j/4$ . Hence from Lemma 8 and Theorem 4, we obtain the following corollary.

**Corollary 2.** *Let  $\tilde{\mathcal{D}}_j(s)$  be the Dirichlet series defined by (7.9) for  $j = 1$  and  $2$ . Then we have*

- (1) *The Dirichlet series  $\tilde{\mathcal{D}}_1(s)$  can be continued analytically as an entire function in whole complex plane, especially the point  $s = 1$  is not a pole of  $\tilde{\mathcal{D}}_1(s)$ .*
- (2) *The Dirichlet series  $\tilde{\mathcal{D}}_2(s)$  can be continued analytically as a meromorphic function to the right-half plane  $\Re s > 2/3$ . In the region  $\Re s > 2/3$ , it has a simple pole at  $s = 3/2$ , whose residue is the same as that of  $\mathcal{D}_2(s)$ , and it also has a double pole at  $s = 1$ .*

Other properties of  $\tilde{\mathcal{D}}_j(s)$ , for instance, the Laurent expansion at the poles and the special values at negative integers, can be obtained similarly, but we omit the details for these topics here.

**Remark 2.** The order of growth of  $\mathcal{D}_2(s)$  and  $\mathcal{Y}(s)$  can be derived by the same way as in [4, Theorems 2 and 3]. We only state the results here.

**Proposition 3.** *For  $t = \Im s$ , we have*

$$\mathcal{D}_2(s) \ll \begin{cases} 1 & \text{for } \Re s > 3/2, \\ \log |t| & \text{for } \Re s = 3/2, \\ |t|^{3-2\Re s} \log^2 |t| & \text{for } 1 < \Re s < 3/2 \end{cases}$$

and

$$\mathcal{Y}(s) \ll \begin{cases} 1 & \text{for } \Re s > 5/4, \\ |t|^{(5-4\Re s)/3} \log^{3/2} |t| & \text{for } 1/2 \leq \Re s \leq 5/4. \end{cases}$$

### 8. Proof of Theorem 3

In this section we shall prove Theorem 3. For this purpose, we first consider the partial sum

$$\sum_{m \leq n \leq N} \frac{r(m)r(n)}{m^{s_1}n^{s_2}}.$$

It suffices to consider the behavior of this sum as  $N \rightarrow \infty$ , since

$$\sum_{m < n \leq N} \frac{r(m)r(n)}{m^{s_1}n^{s_2}} = \sum_{m \leq n \leq N} \frac{r(m)r(n)}{m^{s_1}n^{s_2}} - \sum_{n \leq N} \frac{r(n)^2}{n^{s_1+s_2}}.$$

Firstly, we suppose that  $\Re s_1 > 1$  and  $\Re s_2 > 1$ . We can easily see that

$$\begin{aligned} \sum_{m \leq n \leq N} \frac{r(m)r(n)}{m^{s_1}n^{s_2}} &= \sum_{n \leq N} \frac{r(n)}{n^{s_2}} \sum_{m \leq n} \frac{r(m)}{m^{s_1}} \\ &= \pi \sum_{n \leq N} \frac{r(n)}{n^{s_1+s_2-1}} + \sum_{n \leq N} \frac{r(n)P(n)}{n^{s_1+s_2}} + \frac{\pi s_1}{1-s_1} \sum_{n \leq N} \frac{r(n)}{n^{s_2}} (n^{1-s_1} - 1) \\ &\quad + s_1 \left( \sum_{n \leq N} \frac{r(n)}{n^{s_2}} \right) \int_1^N t^{-s_1-1} P(t) dt - s_1 \sum_{n \leq N} \frac{r(n)}{n^{s_2}} \int_n^N t^{-s_1-1} P(t) dt. \end{aligned} \tag{8.1}$$

The first four terms on the right-hand side of (8.1) can be reduced to

$$\lim_{N \rightarrow \infty} \pi \sum_{n \leq N} \frac{r(n)}{n^{s_1+s_2-1}} = 4\pi \zeta_K(s_1 + s_2 - 1),$$

$$\lim_{N \rightarrow \infty} \sum_{n \leq N} \frac{r(n)P(n)}{n^{s_1+s_2}} = \mathcal{Y}(s_1 + s_2),$$

$$\lim_{N \rightarrow \infty} \frac{\pi s_1}{1-s_1} \sum_{n \leq N} \frac{r(n)}{n^{s_2}} (n^{1-s_1} - 1) = \frac{4\pi s_1}{1-s_1} \{ \zeta_K(s_1 + s_2 - 1) - \zeta_K(s_2) \}$$

and

$$\lim_{N \rightarrow \infty} s_1 \left( \sum_{n \leq N} \frac{r(n)}{n^{s_2}} \right) \int_1^N t^{-s_1-1} P(t) dt = 4s_1 \zeta_K(s_2) \mathcal{J}_0(s_1 + 1),$$

which are convergent absolutely for  $\Re s_1 + \Re s_2 > 2$ ,  $\Re s_1 + \Re s_2 > 5/4$ ,  $\Re s_1 + \Re s_2 > 2$  and  $\Re s_2 > 1$ , and  $\Re s_1 > 1/4$  and  $\Re s_2 > 1$ , respectively. However the right hand sides of the above formulas are known to be continued analytically to the right-half plane  $\Re s_1 + \Re s_2 > -1/3$ .

Consider the last term of (8.1). By partial summation, we have

$$\begin{aligned} & \sum_{n \leq N} \frac{r(n)}{n^{s_2}} \int_n^N t^{-s_1-1} P(t) dt \\ &= \int_1^N \left\{ t^{-s_2} \sum_{n \leq t} r(n) + s_2 \int_1^t u^{-s_2-1} \sum_{n \leq u} r(n) du \right\} t^{-s_1-1} P(t) dt \\ &= \pi \int_1^N t^{-s_1-s_2} P(t) dt + \int_1^N t^{-s_1-s_2-1} P(t)^2 dt \\ & \quad + \frac{\pi s_2}{1-s_2} \int_1^N t^{-s_1-1} P(t) (t^{1-s_2} - 1) dt + s_2 \int_1^N t^{-s_1-1} P(t) \int_1^t u^{-s_2-1} P(u) du dt \\ &=: H_{1,N} + H_{2,N} + H_{3,N} + H_{4,N}, \end{aligned}$$

say. We find that

$$\lim_{N \rightarrow \infty} H_{1,N} = \pi \mathcal{I}_1(s_1 + s_2) \quad (\Re s_1 + \Re s_2 > 5/4),$$

$$\lim_{N \rightarrow \infty} H_{2,N} = \mathcal{I}_2(s_1 + s_2 + 1) \quad (\Re s_1 + \Re s_2 > 1/2)$$

and

$$\lim_{N \rightarrow \infty} H_{3,N} = \frac{\pi s_2}{1-s_2} (\mathcal{I}_1(s_1 + s_2) - \mathcal{I}_1(s_1 + 1)) \quad (\Re s_1 + \Re s_2 > 5/4, \Re s_1 > 1/4),$$

where the functions on the right-hand sides are convergent absolutely in the given regions. But by Lemma 9 and Theorem 4, we can see that they can be continued analytically to the region  $\Re s_1 + \Re s_2 > -1/3$ .

It remains to treat  $H_{4,N}$ . Let

$$J_N(s_1, s_2) = \int_1^N t^{-s_1-1} P(t) \int_1^t u^{-s_2-1} P(u) du dt$$

and  $J(s_1, s_2) = \lim_{N \rightarrow \infty} J_N(s_1, s_2)$  briefly. It is easy to see that

$$J_N(s_1, s_2) = \left( \int_1^N t^{-s_1-1} P(t) dt \right) \left( \int_1^N u^{-s_2-1} P(u) du \right) - K_N(s_1, s_2),$$

where

$$K_N(s_1, s_2) = \int_1^N t^{-s_1-1} P(t) \int_t^N u^{-s_2-1} P(u) du dt.$$

Suppose that  $\Re s_2 > 0$  and  $\Re s_1 + \Re s_2 > 1/4$ . Then from the first formula of (1.5) with  $\theta = s_2 + 1$ , we see that

$$K(s_1, s_2) := \lim_{N \rightarrow \infty} K_N(s_1, s_2),$$

exists and defines a holomorphic function there. Since

$$J(s_1, s_2) = \mathcal{I}_1(s_1 + 1)\mathcal{I}_1(s_2 + 1) - K(s_1, s_2),$$

$J(s_1, s_2)$  has a meromorphic continuation to  $\Re s_2 > 0$  and  $\Re s_1 + \Re s_2 > 1/4$ . We should note here that  $J_N(s_1, s_2)$  has the symmetric property

$$J_N(s_1, s_2) + J_N(s_2, s_1) = \int_1^N t^{-s_1-1}P(t)dt \int_1^N t^{-s_2-1}P(t)dt,$$

which is similar to the case of the divisor problem. By the same argument above, we have

$$J(s_2, s_1) = \mathcal{I}_1(s_1 + 1)\mathcal{I}_1(s_2 + 1) - K(s_2, s_1).$$

for  $\Re s_1 > 0$  and  $\Re s_1 + \Re s_2 > 1/4$  and  $K(s_2, s_1)$  is holomorphic in this region. It shows that the function  $J(s_1, s_2)$  has a meromorphic continuation, and so does  $\lim_{N \rightarrow \infty} H_{4,N}$  to the region  $\Re s_1 + \Re s_2 > 1/4$ . The proof of Theorem 3 is complete.

To sum up, the analytic continuation of  $M(s_1, s_2)$  is explicitly given by

$$\begin{aligned} M(s_1, s_2) &= 4\pi\zeta_K(s_1 + s_2 - 1) - \frac{16\zeta_K(s_1 + s_2)^2}{(1 + 2^{-s_1-s_2})\zeta(2(s_1 + s_2))} + \mathcal{Y}(s_1 + s_2) \\ &+ \frac{4\pi s_1}{1 - s_1} \{ \zeta_K(s_1 + s_2 - 1) - \zeta_K(s_2) \} + 4s_1\zeta_K(s_2)\mathcal{J}_0(s_1 + 1) \\ &+ \pi\mathcal{I}_1(s_1 + s_2) + \mathcal{I}_2(s_1 + s_2 + 1) + s_2\mathcal{I}_1(s_1 + 1)\mathcal{I}_1(s_2 + 1) \\ &+ \frac{\pi s_2}{1 - s_2} \{ \mathcal{I}_1(s_1 + s_2) - \mathcal{I}_1(s_1 + 1) \} - s_2K(s_1, s_2) \end{aligned}$$

for  $\Re s_1 + \Re s_2 > 1/4$ . We note that we can determine the possible singularities of  $M(s_1, s_2)$  in the domain  $\Re s_1 + \Re s_2 > 1/4$  by using this representation. But we omit the details for this topic here.

**Remark 3.** As we have seen above, all terms except  $K(s_1, s_2)$  can be continued into the wider range  $\Re s_1 + \Re s_2 > -1/3$ . However it seems to be difficult to extend the region of convergence of  $K(s_1, s_2)$  into this region by our present method. For instance, if we substitute the second estimate of (1.5), we see that

$$K_N(s_1, s_2) = -C_1 \log N + O(1), \tag{8.2}$$

provided that  $s_2 = 0$  and  $\Re s_1 > 0$ . (Note that  $C_1$  in (8.2) is a function of  $s_1$  and is not identically zero.) Similarly  $K_N(s_1, s_2)$  diverges when  $N \rightarrow \infty$  if we use the third estimate of (1.5) for  $-1/4 < \Re s_2 \leq 0$  with  $s_2 \neq 0$  and  $\Re s_1 + \Re s_2 > 0$ . Hence we cannot obtain the estimate  $K_N(s_1, s_2) = O(1)$  as  $N \rightarrow \infty$  under these conditions.

The authors believe that  $M(s_1, s_2)$  can be continued analytically beyond the line  $\Re s_1 + \Re s_2 = 1/4$ . But in order to get such an analytic continuation, it should be necessary to develop a theory different from that used in this paper. In view of the formula (8.2), we may be able to make the following conjecture on the behaviour of  $M(s_1, s_2)$  for  $\Re s_1 + \Re s_2 \leq 1/4$ :

**Conjecture 1.** *The analytic continuation of  $M(s_1, s_2)$  has singularities on the hyper-plane  $\Re s_1 + \Re s_2 = 1/4$ , especially these singularities come from the function  $K(s_1, s_2)$ .*

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