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NON-VANISHING OF DERIVATIVES OF CERTAIN MODULAR *L*-FUNCTIONS

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Abstract: This paper is to show a non-vanishing property of the derivatives of certain class of L-functions. We study the non-vanishing and transcendence of special values of L-functions and their derivatives, attached to (cuspidal) Siegel-Hecke eigenforms of genus 2, quadratic twists of classical Hecke eigenforms, and half-integral weight modular forms.

Keywords: Siegel modular forms, twists of cusp forms, half-integral weight modular forms, L-functions, non-vanishing, special values, transcendence.

1. Introduction

The special values of L-functions have been the subject of much study, both because of their intrinsic interest and also of their prominent role they have played in various places. For example, the central critical values play an important role in the Kolyvagin's work on the Birch and Swinnerton-Dyer conjecture, and Böcherer's conjecture concerning central critical values of odd quadratic twists of spinor zeta functions attached to cuspidal Siegel-Hecke eigenforms of genus 2.

In [8], Gun, Murty, and Rath investigated the non-vanishing and transcendence of special values of a varying class of L-functions and their derivatives. In [16], Tanobe extended these results to Hilbert modular forms.

In this paper, we show that the non-vanishing and transcendence of special values of L-functions and their derivatives continues to hold for (cuspidal) Siegel-Hecke eigenforms of genus 2 (cf. Section 2), quadratic twists of classical Hecke eigenforms (cf. Section 3), and half-integral weight modular forms (cf. Section 4). Some applications in transcendental number theory are deduced from these results (cf. Section 5).

Throughout this paper, we let F, f, and h to denote (cuspidal) Siegel-Hecke eigenforms of genus 2, classical Hecke eigenforms, and half-integral weight modular forms.

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2. For (cuspidal) Siegel-Hecke eigenforms of genus 2

Let $\Gamma_2 = \mathrm{Sp}_4(\mathbb{Z}) \subset \mathrm{GL}_4(\mathbb{Z})$ be the Siegel modular group of genus 2. (This consists of all four by four symplectic matrices with integer entries). Let F be a nonzero (cuspidal) Siegel-Hecke eigenform of integral weight k on Γ_2, in particular, F is a simultaneous eigenform for all the Hecke operators T_n ($n \in \mathbb{N}$). For the definitions of Siegel modular forms and Hecke operators, see [1], [2].

2.1. Spinor zeta functions of Siegel-Hecke eigenforms

Let F be a nonzero (cuspidal) Siegel-Hecke eigenform of integral weight k on Γ_2 (or of genus 2) with Hecke eigenvalues $\lambda(n)$ ($n \in \mathbb{N}$). We shall recall some facts about the spinor zeta function attached to F (cf. [1] for more details). Consider the Dirichlet series

$$D(F,s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$$

which converges for $\mathrm{Re}(s) \gg 0$.

Theorem 2.1 (Andrianov). *The spinor zeta function of F is*

$$Z(F,s) = \zeta(2s-2k+4)D(F,s)$$

has a meromorphic continuation to $\mathbb{C}$. More precisely, the completed function is

$$Z^*(F,s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z(F,s)$$

has a meromorphic continuation to $\mathbb{C}$, with at most two simple poles at the points $s = k-2$ and $s = k$, and satisfies the functional equation

$$Z^*(F, 2k-2-s) = (-1)^k Z^*(F,s). \tag{2.1}$$

2.2. Non-vanishing result for (cuspidal) Siegel-Hecke eigenforms

Now, we show that if the spinor zeta function $Z(F,s)$ is non-zero at center of symmetry of the functional equation, then so is the derivative. Here, we prove the following theorem, which is analogous to [8, Thm. 4.1].

Theorem 2.2. *Suppose F is a (cuspidal) Siegel-Hecke eigenform of weight k of genus 2. Assume that $Z(F,k-1) \neq 0$. Then one has*

$$\frac{Z'(F,k-1)}{Z(F,k-1)} = -\psi(k-1) - \psi(1) + 2\log(2\pi),$$

where ψ is the logarithmic derivative of the gamma function Γ. Further, for such an F, $Z'(F,k-1) \neq 0$.

Proof. By (2.1), we have

$$(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z(F,s) = (2\pi)^{-2(2k-2-s)}\Gamma(2k-2-s)\Gamma(k-s)Z(F,2k-2-s).$$

Taking the logarithmic derivative with respect to s, we see that

$$\begin{aligned} &-2\log(2\pi) + \psi(s) + \psi(s-k+2) + \frac{Z'(F,s)}{Z(F,s)} \\ &\qquad = 2\log(2\pi) - \psi(2k-2-s) - \psi(k-s) - \frac{Z'(F,2k-2-s)}{Z(F,2k-2-s)}. \end{aligned}$$

Since $Z(F,k-1) \neq 0$, putting $s = k-1$ in the expression above, we get that

$$\frac{Z'(F,k-1)}{Z(F,k-1)} = -\psi(k-1) - \psi(1) + 2\log(2\pi).$$

Now, if $\frac{Z'(F,k-1)}{Z(F,k-1)} = 0$, then we will have

$$\gamma + \log(2\pi) = \frac{1}{2}\sum_{n=1}^{k-2}\frac{1}{n}, \tag{2.2}$$

by noting that $\psi(k) = -\gamma + \sum_{n=1}^{k-1}\frac{1}{n}$, where γ is Euler's constant, and $0.577215 < \gamma < 0.577216$.

The equality in (2.2) cannot happen, because $2.41421 < \gamma + \log 2\pi < 2.41511$, and $\frac{1}{2}\sum_{n=1}^{k-2}\frac{1}{n}$ an increasing function of k such that $\frac{1}{2}\sum_{n=1}^{69}\frac{1}{n} = 2.4093$ and $\frac{1}{2}\sum_{n=1}^{70}\frac{1}{n} = 2.4164$. This proves the theorem. ■

Remark 2.3. Suppose that the cuspidal Siegel-Hecke eigenform F of weight k is the Maass lift of a classical Hecke eigenform f of weight $2k-2$ and level 1. Then the L-functions of F and f are related by the formula

$$Z(F,s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s),$$

where $L(f,s)$ is the Hecke L-function of f (cf. [12, §2, §3] for more details). Hence, the non-vanishing of $Z(F,s)$ at $s = k-1$ could be closely related to the non-vanishing of $L(f,s)$, for cusp forms f, at $s = k-1$ (as in [8, Thm. 4.1]).

Now, we have some corollaries to Theorem 2.2.

Let $E(k)$ denote the set of all (cuspidal) Siegel-Hecke eigenforms F of weight k of genus 2 such that $Z(F,k-1) \neq 0$.

Corollary 2.4. *For $F_k \in E(k)$, the function $\frac{Z'(F_k,k-1)}{Z(F_k,k-1)} + \psi(k-1)$ is independent of k, and its value belongs to the interval* $[4.252, 4.253]$.

Proof. The assertions follow from Theorem 2.2, and from the estimates of γ, $\log(2\pi)$. ■

Corollary 2.5. *For $F_k \in E(k)$, the function* $\frac{Z'(F_k,k-1)}{Z(F_k,k-1)} \to -\infty$ *as* $k \to \infty$.

Proof. By Corollary 2.4, we have $\frac{Z'(F_k,k-1)}{Z(F_k,k-1)} = c - \psi(k-1)$, for some $c \in \mathbb{R}_{>0}$. Since the digamma function $\psi(k) \to \infty$ as $k \to \infty$, the corollary follows. ■

2.3. Transcendence of the special values

Recall that, $E(k)$ denotes the set of all (cuspidal) Siegel-Hecke eigenforms F of weight k of genus 2 such that $Z(F, k-1) \neq 0$.

Corollary 2.6. *For $F \in E(k)$, the real number*

$$\exp\left(\frac{Z'(F,k-1)}{Z(F,k-1)} + \psi(k-1) + \psi(1)\right)$$

is transcendental.

Surprisingly, the transcendental nature of these special values

$$\left\{\frac{Z'(F,k-1)}{Z(F,k-1)} : F \in E(k),\ \forall\ k \in \mathbb{N}\right\}$$

is quite different from the classical situation (cf. [8, Cor. 4.3]). Now, we show that the nature of these values is pure, in the sense that either all are algebraic or all are transcendental. More precisely, we have the following:

Proposition 2.7. *If* $\frac{Z'(F_0,k_0-1)}{Z(F_0,k_0-1)}$ *is algebraic (resp., transcendental) for some* $F_0 \in E(k_0)$, $k_0 \in \mathbb{N}$, *then* $\frac{Z'(F,k-1)}{Z(F,k-1)}$ *is algebraic (resp., transcendental) for all* $F \in E(k)$ *and for all* $k \in \mathbb{N}$.

Proof. By Theorem 2.2, for any $F \in E(k)$, we have

$$\frac{Z'(F,k-1)}{Z(F,k-1)} - \frac{Z'(F_0,k_0-1)}{Z(F_0,k_0-1)} = \psi(k_0-1) - \psi(k-1).$$

Now, the proposition follows immediately from the functional equation of $\psi(x)$, i. e., $\psi(x+1) - \psi(x) = 1/x$, because $\psi(k_0-1) - \psi(k-1)$ is an algebraic number. ■

In last part of this section, we study the non-vanishing of $D(F,s)$ and its derivative at the central critical value $\frac{1}{2}$ (critical, in the sense of Deligne). We set

$$\Delta(F,s) := (2\pi)^{-2s}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma(s+k-1)\Gamma(s+k-2)D(F,s).$$

Then, $\Delta(F,s)$ has a holomorphic continuation to $\mathbb{C}$, except for simple poles at $s=0$ and $s=1$, and satisfies the functional equation

$$\Delta(F,s) = \Delta(F,1-s)$$

(cf. [7, §20, §21]). We have the following theorem, whose proof is similar to the proof of Theorem 2.2.

Theorem 2.8. *Suppose F is a (cuspidal) Siegel-Hecke eigenform of weight k of genus 2. Assume that $D(F,1/2) \neq 0$, i.e., the standard zeta function of F does not vanish at the critical value $\frac{1}{2}$. Then one has*

$$\frac{D'(F,1/2)}{D(F,1/2)} = -\psi\left(k-\frac{1}{2}\right) - \psi\left(k-\frac{3}{2}\right) - \frac{1}{2}\psi\left(\frac{1}{4}\right) + \log\left(4\pi^{\frac{5}{2}}\right).$$

Further, for such an F, $D'(F,1/2) \neq 0$.

3. For quadratic twists of (classical) Hecke eigenforms

Let $k, N \geqslant 1$ be two integers. Let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized cuspidal Hecke eigenform of weight $2k$ for $\Gamma_0(N)$ with trivial nebentypus, in particular, f is a simultaneous eigenform for all the Hecke operators T_n ($n \in \mathbb{N}$). For definitions of cusp forms and Hecke operators, see [6].

3.1. *L*-functions of quadratic twists of modular forms

Let $L(f,s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, $s \in \mathbb{C}$, $\mathrm{Re}(s) > k + \frac{1}{2}$ denote the L-function associated with f. For a fundamental discriminant D, that is $D = 1$ or the discriminant of a quadratic field, with $(D,N) = 1$, we let

$$L(f,D,s) := \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) \frac{a(n)}{n^s}$$

for $\mathrm{Re}(s) > k + \frac{1}{2}$, be the L-series of f twisted with the quadratic character $\left(\frac{D}{\cdot}\right)$. The following theorem is well-known.

Theorem 3.1. *The function $L(f,D,s)$ has a holomorphic continuation to $\mathbb{C}$. More precisely, the completed L-function*

$$L^*(f,D,s) = (2\pi)^{-s}(ND^2)^{s/2}\Gamma(s)L(f,D,s)$$

has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$L^*(f,D,s) = (-1)^k \left(\frac{D}{-N}\right) L^*(W_N f, D, 2k-s), \tag{3.1}$$

where $W_N f := f|_k \left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right]$, an idempotent operator.

3.2. Non-vanishing result for quadratic twists of (classical) Hecke eigenforms

Now, we show that if the central critical value $L(f,D,k)$ is non-zero, then so is the derivative at $s = k$. Here, we prove the following.

Theorem 3.2. *Let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized cuspidal Hecke eigenform of weight $2k$ for $\Gamma_0(N)$ with trivial nebentypus. Assume that $L(f, D, k) \neq 0$. Then one has*

$$\frac{L'(f,D,k)}{L(f,D,k)} = -\psi(k) + \log(2\pi) - \frac{\log(ND^2)}{2}.$$

Further, for such an f, $L'(f, D, k) \neq 0$.

Proof. By (3.1), we have

$$(2\pi)^{-s}(ND^2)^{s/2}\Gamma(s)L(f,D,s) = (2\pi)^{-(2k-s)}(ND^2)^{k-\frac{s}{2}}\Gamma(2k-s)L(f,D,2k-s).$$

Taking the logarithmic derivative with respect to s, we see that

$$\begin{aligned} &-\log(2\pi) + \frac{1}{2}\log(ND^2) + \psi(s) + \frac{L'(f,D,s)}{L(f,D,s)} \\ &\qquad = \log(2\pi) - \frac{1}{2}\log(ND^2) - \psi(2k-s) - \frac{L'(f,D,2k-s)}{L(f,D,2k-s)}. \end{aligned} \tag{3.2}$$

Since $L(D, f, k) \neq 0$, putting $s = k$ in the expression above, we get that

$$\frac{L'(f,D,k)}{L(f,D,k)} = -\psi(k) + \log(2\pi) - \frac{\log(ND^2)}{2}.$$

Now, if $\frac{L'(f,D,k)}{L(f,D,k)} = 0$, then we will have

$$\gamma + \log(2\pi) = \sum_{n=1}^{k-1}\frac{1}{n} + \frac{\log(ND^2)}{2}, \tag{3.3}$$

since $\psi(k) = -\gamma + \sum_{n=1}^{k-1}\frac{1}{n}$, where $0.577215 < \gamma < 0.577216$. The equality in (3.3) cannot hold, because from the inequalities

$$2.41421 < \gamma + \log 2\pi < 2.41511, \qquad 2.418 < \frac{1}{2}\log 126, \qquad \text{and} \qquad \sum_{n=1}^{6}\frac{1}{n} = 2.45$$

we get that

$$\gamma + \log(2\pi) < \sum_{n=1}^{k-1}\frac{1}{n} + \frac{\log(ND^2)}{2}$$

is true if either $k \geqslant 7$ or $ND^2 \geqslant 126$. For the remaining cases, one can similarly check that (3.3) can never happen. ∎

Deep results of Bump, Friedberg, and Hoffstein [5], of Murty and Murty [14], of Ono and Waldspurger [13], and of others (in various cases) establish non-vanishing results for central critical values of quadratic twists of modular L-functions.

Remark 3.3. In [8, §5], the authors proved a non-vanishing result for the derivatives of symmetric square L-functions for $k > 48$. A computation similar to the above (or by calculating with the computer), one can check that Theorem 4.7 in loc. cit. continues to hold for all $k \geqslant 1$.

3.3. Transcendence of the special values

Fix a fundamental discriminant D. For $N \in \mathbb{N}$ with $(N, D) = 1$, we let $E_D(N, k)$ to denote the set of all normalized cuspidal Hecke eigenforms f of weight $2k$ for $\Gamma_0(N)$ with trivial nebentypus such that the twisted L-value $L(f, D, k) \neq 0$.

Corollary 3.4. *For $f \in E_D(N, k)$, the real number*

$$\exp\left(\frac{L'(f, D, k)}{L(f, D, k)} + \psi(k)\right)$$

is transcendental.

Proof. The corollary follows from Theorem 3.2. ∎

We finish this section with a generalization of [8, Cor. 4.3] to the quadratic twists of cuspidal Hecke eigenforms.

Proposition 3.5. *For $k \geqslant 1$, the set*

$$\left\{\frac{L'(f, D, k)}{L(f, D, k)} : f \in E_D(N, k),\ \forall\ N \geqslant 1 \text{ such that } (N, D) = 1\right\}$$

has at most one algebraic element.

Proof. For $i = 1, 2$, let $f_i \in E_D(N_i, k)$ be two Hecke eigenforms of weights $2k$ for $\Gamma_0(N_i)$. If $N_1 = N_2$, then $\frac{L'(f_1,D,k)}{L(f_1,D,k)} = \frac{L'(f_2,D,k)}{L(f_2,D,k)}$. In the other case, i.e., $N_1 \neq N_2$, if both $\frac{L'(f_1,D,k)}{L(f_1,D,k)}$ and $\frac{L'(f_2,D,k)}{L(f_2,D,k)}$ are algebraic, then their difference $\log N_2/N_1$ is also algebraic, which is a contradiction. This proves the proposition. ∎

4. For half-integral weight (cuspidal) modular forms

For $k \in \mathbb{N}$, N odd square-free positive integer, we let $S_{k+1/2}(\Gamma_0(4N))$ to denote the space of cusp forms of weight $k + \frac{1}{2}$ on $\Gamma_0(4N)$ with trivial nebentypus (cf. [15] for definitions). The results in this section are quite similar to the ones obtained before, so we will sketch the results (and also only for the subgroup $\Gamma_0(4)$).

4.1. *L*-functions of half-integral weight (cuspidal) modular forms

Let $h = \sum_{n=1}^{\infty} c(n) q^n$ be a cuspidal Hecke eigenform of weight $k+1/2$, in particular, h is a simultaneous eigenform for all the Hecke operators. For the definitions of cusp forms and Hecke operators, see [15].

Consider the Dirichlet series

$$L(h, s) := \sum_{n \geqslant 1} \frac{c(n)}{n^s}$$

which converges for $\mathrm{Re}(s) > k/2+5/4$. Then by means of the usual Mellin formula and using some standard arguments one proves that the complete L-function

$$L^*(h,s) = \pi^{-s}\Gamma(s)L(h,s)$$

has a holomorphic continuation to the complex plane $\mathbb{C}$ and satisfies the functional equation

$$L^*(h, k+1/2-s) = L^*(W_4 h, s),$$

where $W_4 h(z) := (-2iz)^{-k-1/2} h(-1/4z) \in S_{k+1/2}(\Gamma_0(4))$, an idempotent operator (cf. [15, §5] or [11, pg. 429]).

4.2. Non-vanishing result for half-integral weight cusp forms

Now, we show that if the L-function $L(h,s)$ is non-zero at center of symmetry of the functional equation, then so is the derivative. Here, we prove the following theorem, which is analogous to Theorem 2.2 and Theorem 3.2.

Theorem 4.1. *Let $h = \sum_{n=1}^{\infty} c(n)q^n$ be a cuspidal Hecke eigenform of weight $k+1/2$ for $\Gamma_0(4)$ with trivial nebentypus. Assume that $L(h, k/2+1/4) \neq 0$. Then one has*

$$\frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)} = -\psi(k/2+1/4) + \log(\pi)$$

Further, for such an h, $L'(h, k/2+1/4) \neq 0$.

Proof. The proof of this theorem is quite similar to the proofs of Theorem 2.2 and Theorem 3.2. We will skip the proof, but we remark that in the final step of the proof, one need to use the fact that $\log(\pi) \sim 1.144729885$ and $\psi(k/2+1/4)$ is an increasing function for $k \geqslant 3$ such that $\psi(13/4) = 1.016991$ and $\psi(15/4) = 1.182537$. ■

We have a few corollaries to the theorem above, which are similar to Corollary 2.4 and Corollary 2.5.

Let $E(k+1/2)$ denote the set of all cuspidal Hecke eigenforms h of weight $k+1/2$ for $\Gamma_0(4)$ with trivial nebentypus, such that the L-value $L(h, k/2+1/4) \neq 0$.

Corollary 4.2. *For $h_k \in E(k+1/2)$, the function $\frac{L'(h_k, k/2+1/4)}{L(h_k, k/2+1/4)} + \psi(k/2+1/4)$ is independent of k, and its value belongs to the interval* $[1.144, 1.145]$.

Corollary 4.3. *For $h_k \in E(k+1/2)$, the function $\frac{L'(h_k, k/2+1/4)}{L(h_k, k/2+1/4)} \to -\infty$ as $k \to \infty$.*

Finally, we will make one remark. By Waldspurger's theorem, the central critical values $L(f, D, k)$ are essentially proportional to the squares of Fourier coefficients of the modular form h of weight $k+1/2$ corresponding to f under Shimura correspondence.

Corollary 4.4. *Let $h = \sum_{n=1}^{\infty} c(n)q^n$ be a half-integral weight cuspidal newform of weight $k+1/2$ in the Kohnen's $+$-subspace, and let f denote the corresponding classical cusp form of weight $2k$ under the Shimura correspondence. Let D be a fundamental discriminant such that $(-1)^k D > 0$. If $c(|D|) \neq 0$, then $L'(f, D, k) \neq 0$.*

4.3. Transcendence of the special values

Recall that, $E(k+1/2)$ denotes the set of all cuspidal Hecke eigenforms h of weight $k+1/2$ for $\Gamma_0(4)$ with trivial nebentypus, such that the L-value $L(h, k/2+1/4) \neq 0$.

Corollary 4.5. *For $h \in E(k+1/2)$, the real number*

$$\exp\left(\frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)} + \psi(k/2+1/4)\right)$$

is transcendental.

Surprisingly, the transcendental nature of these special values

$$\left\{\frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)} : h \in E(k+1/2)\right\}$$

is quite different from the Propositions 2.7 and 3.5. Now, we show that the nature of these values is pure with respect to the parity of k. More precisely, we have the following:

Proposition 4.6. *If $\frac{L'(h_0, k_0/2+1/4)}{L(h_0, k_0/2+1/4)}$ is algebraic (resp., transcendental) for some $h_0 \in E(k_0+1/2)$, $k_0 \in \mathbb{N}$, then $\frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)}$ is algebraic (resp., transcendental) for all $h \in E(k+1/2)$ and for all $k \in \mathbb{N}$ with $k \equiv k_0 \pmod 2$.*

Proof. By Theorem 4.1, for any $h_1 \in E(k+1/2)$ and $h_2 \in E(\ell+1/2)$, we will have

$$\frac{L'(h_1, k/2+1/4)}{L(h_1, k/2+1/4)} - \frac{L'(h_2, \ell/2+1/4)}{L(h_2, \ell/2+1/4)} = \psi(\ell/2+1/4) - \psi(k/2+1/4).$$

The proposition is immediate from the functional equation of $\psi(x)$, i.e., $\psi(x+1) - \psi(x) = 1/x$, because $\psi(\ell/2+1/4) - \psi(k/2+1/4)$ is an algebraic number if and only if $\ell \equiv k \pmod 2$. ∎

5. Applications in transcendental number theory

In this penultimate section, we briefly recall one of the consequences of Schanuel's conjecture from the transcendental number theory (cf. [8, §2] for more details). We will apply this to understand the transcendental nature of the special values of (cuspidal) Siegel-Hecke eigenforms of genus 2, quadratic twists of classical Hecke eigenforms.

Schanuel's conjecture states that, for any $\{\alpha_i\}_{i=1}^n$ in $\mathbb{C}$ that are linearly independent over $\mathbb{Q}$, the transcendence degree of the field $\mathbb{Q}(\{\alpha_i\}, \{e^{\alpha_i}\})_{i=1}^n$ over $\mathbb{Q}$ is at least n. As a consequence of this conjecture, one sees that, for a non-zero algebraic number α, the two numbers e^α and π are algebraically independent, and so are their logarithms (cf. §2 of loc. cit.).

Corollary 5.1. *Let F be as in Theorem 2.2, and assume that Schanuel's conjecture is true. Then*

$$\frac{L'(F,k-1)}{L(F,k-1)} - 2\gamma \qquad \text{and} \qquad \exp\left(\frac{L'(F,k-1)}{L(F,k-1)} - 2\gamma\right)$$

are transcendental.

Proof. By Theorem 2.2, we have

$$\frac{L'(F,k-1)}{L(F,k-1)} - 2\gamma = 2\log(2\pi) - \alpha$$

and

$$\exp\left(\frac{L'(F,k-1)}{L(F,k-1)} - 2\gamma\right) = 4\pi^2 e^{-\alpha}, \qquad \text{where } \alpha = \sum_{n=1}^{k-2} \frac{1}{n} \in \mathbb{Q}.$$

Both of these numbers are transcendental, by Schanuel's conjecture. ■

We have a similar result in the quadratic twists of classical Hecke eigenforms.

Corollary 5.2. *Let f be as in Theorem 3.2, and assume that Schanuel's conjecture is true. Then*

$$\frac{L'(f,D,k)}{L(f,D,k)} - \gamma \qquad \text{and} \qquad \exp\left(\frac{L'(f,D,k)}{L(f,D,k)} - \gamma\right)$$

are transcendental.

Proof. By Theorem 3.2, we have

$$\frac{L'(f,D,k)}{L(f,D,k)} - \gamma = \log(2\pi) - \frac{1}{2}\log(ND^2) - \alpha$$

and

$$\exp\left(\frac{L'(f,D,k)}{L(f,D,k)} - \gamma\right) = e^{-\alpha}\frac{2\pi}{N^{1/2}D}, \qquad \text{where } \alpha = \sum_{n=1}^{k-1} \frac{1}{n} \in \mathbb{Q}.$$

Both of these numbers are transcendental, again by Schanuel's conjecture. ■

We end this article with an application, which is a special case of Theorem 3.2, to elliptic curves. Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N, and let D be a fundamental discriminant such that $(D,N) = 1$. Since every elliptic curve over $\mathbb{Q}$ is modular (cf. [4], [17]), we have the following corollary.

Corollary 5.3. *Suppose that $L(E,D,1) \neq 0$. Then $L'(E,D,1) \neq 0$. Further,*

$$\exp\left(\frac{L'(E,D,1)}{L(E,D,1)} - \gamma\right)$$

is transcendental.

Finally, we ask the following:

Question. The non-vanishing results of special values of L-functions and their derivatives are known to be true for cuspidal Hecke eigenforms and for their symmetric square L-functions [8], for Hilbert Modular forms [16]. Now, the current article shows that these results continue to hold for (cuspidal) Siegel-Hecke eigenforms of genus 2, quadratic twists of cuspidal Hecke eigenforms, and half-integral weight modular forms. Is it just a coincidence? or do these non-vanishing results hold in much more generality, i. e., for a “special" class of motivic L-functions?

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