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LOCALLY CONVEX SPACES NOT CONTAINING l^1

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: The criteria for non-containment of l^1 for the classes of Banach and Fréchet spaces are extended to the class of locally complete locally convex spaces the bounded sets of which are metrizable.

Keywords: locally convex spaces, non-containment of l^1, limited sets.

1. Introduction

For the classes of Banach and of Fréchet spaces X, non-containment of l^1 has been characterized both (a) by internal conditions on sequences in either X – such as all bounded sequences having weak Cauchy subsequences (cf. [20], [8, 17]) – or in the dual – such as Mackey nullsequences being strongly null ([9, 23]–, and (b) by compactness criteria for subsets of certain spaces of compact linear operators from X into a locally convex space Y ([18, Thm. 1], [22, Prop. 2.1]).
In this paper, it is shown that these criteria extend to the class of all locally complete locally convex spaces the bounded sets of which are metrizable.

Notation and Terminology. Given a locally convex space (lcs) (X, τ), X'_b, respectively X'_τ, will denote the dual of X endowed with the strong $\beta(X', X)$, respectively the Mackey $\tau(X', X)$-topology.

Given a subset C of X, $C^\circ := \{x' \in X' \mid |\langle x', x\rangle| \leqslant 1 \text{ for all } x \in C\}$ will denote its (absolute) polar in X'.

A subset H of X' will be called *limited* if any weak nullsequence in X converges to zero uniformly over H (cf. [13, Ch. V.3, Ex. suppl.]).

A subset B of an lcs X that is absolutely convex will be called a *disk*. The closed absolutely convex hull of a subset B of X will be denoted by $clacB$.

Given any bounded disk B in an lcs X, we denote by (X_B, B) the linear span of B in X, endowed with the norm of the gauge q_B of B. An lcs (X, τ) is called

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locally complete if (X_B, B) is complete for all closed bounded disks in (X, τ), and (X, τ) is said to satisfy the *strict Mackey convergence condition* (*sMcc*) if, given any bounded subset B of X, there exists a closed bounded disk C in X containing B such that the topologies induced on B by (X_C, C) and τ coincide.

Recall that local completeness is an invariant for all locally convex topologies compatible with the dual pair (X, X'), that local completeness is implied by sequential completeness of any such topology, and that the sMcc is inherited by closed linear subspaces as well as by countable products or direct sums.

Most importantly for the purpose of this paper, metrizable lcs satisfy the sMcc ([13, Ch. IV.2.2, Théorème 1]), so that Fréchet spaces both are locally complete and satisfy the strict Mackey convergence condition.

These notions and results are pretty classical, compare [13]; for a more recent account, we refer to [19, Chs. 3.2 and 5.1].

Finally, we recall that an lcs X is called *quasinormable* [13, Ch. IV.4.1] if for every polar U° of a zero-neighbourhood U in X there exists another such $V^\circ \supset U^\circ$ such that the topologies induced on U° by X'_b and (X'_{V°, V°) coincide.

2. Results

The following are the extensions of the non-containment-of$-l^1$ criteria from the classes of Banach or Fréchet spaces to locally convex spaces that are locally complete and have their bounded sets metrizable.

Theorem 2.1. *For a locally complete locally convex space X, the bounded sets of which are metrizable, the following are equivalent:*

(a) *X does not contain an isomorphic copy of l^1.*
(b) *Every bounded sequence in X has a weak Cauchy subsequence.*
(c) *Every limited subset H of X' is strongly ($\beta(X', X)$-) precompact.*
(d) *Every Mackey ($\tau(X', X)$-) precompact subset of X' is $\beta(X', X)$- precompact.*
(e) *Every $\tau(X', X)$-nullsequence in X' is $\beta(X', X)$-null.*

Remark 2.2. The implication "(e) $\Rightarrow$ (a)" holds for just any locally convex space [9, Lemma 1].

In order to extend the compact-operator characterization of non-containment of l^1 of [18, Thm. 1] and [22, Prop. 2.1] for Banach and for Fréchet spaces to the more general class considered here, we work in the context of the following operator space. Given locally convex spaces X and Y, the space $K_b^b(X, Y)$ is the space of all weak-to-weak-continuous linear operators from X into Y that transform bounded subsets of X into precompact subsets of Y, endowed with the topology of uniform convergence on the bounded subsets of X.

Proposition 2.3. *For a locally complete locally convex space X, the bounded sets of which are metrizable, the following are equivalent:*

(a) *X does not contain an isomorphic copy of l^1.*

(b) *Given any locally convex (equivalently, any Fréchet, equivalently, any Banach) space Y, a subset H of $K_b^b(X,Y)$ is precompact if and only if*

(i) *$H(x)$ is precompact in Y for all $x \in X$, and*

(ii) *$h(x_n) \to 0$ in Y uniformly over all $h \in H$ for any weak-nullsequence $(x_n)_n$ in X.*

Remarks 2.4.

1. *Relations to existing results. Theorem 2.1:* For Banach and Fréchet spaces, (b) is H.P. Rosenthal's (and L.E. Dor's) original result [10, 20], and its extension to Fréchet spaces in [8, 17]. For Banach spaces, (c) is [12, Thm. 2]. For Fréchet spaces, the equivalence of (e) with (a) has been shown in [4, Thm. 10] and in [23], while the implication "(e) $\Rightarrow$ (a)" for just any locally convex space is due to [9, Lemma 1].
Proposition 2.3 for Banach spaces is [18, Theorem 1], and for Fréchet spaces [22, Prop. 2.1].
2. *Further classes of locally convex spaces with their bounded sets metrizable:* Obviously, locally convex spaces with sMcc have their bounded sets metrizable. Notice that even this very special subclass includes the strong duals of quasi-barrelled quasinormable locally convex spaces, as well as the class of quasi-barrelled spaces with quasinormable strong duals.
Also, boundedly retractive (LF)-spaces (and thus, in particular, strict (LF)-spaces) fulfill the assumptions on the space X in Theorem 2.1. (An inductive limit (X,τ) of an inductive sequence (X_n,τ_n) of locally convex spaces is boundedly retractive if every bounded subset B of (X,τ) is contained and bounded in some X_n, and τ and τ_n agree on B, cf. [19, Def. 8.5.32].)
With regard to the class of (DF)-spaces, we recall from [1, 2, 3] that (a) the strong dual of a metrizable locally convex space X has its bounded sets metrizable if and only if X satisfies the *density condition* of [14], while (b) a general (DF)-space has its bounded sets metrizable if and only if it satisfies the *dual density condition* of [2]. As a further particular case, it is shown in [3, 1.6. Prop.] that the space $L_b(X,Y)$ of bounded linear operators from X into Y, endowed with the topology of uniform convergence on the bounded subsets of X, has its bounded sets metrizable in case X is a metrizable lcs with the density condition, and Y is an lcs with a fundamental sequence of bounded sets that are metrizable in the induced topology. (For further results in this context, as well as for the definitions of the density condition and the dual density condition, the reader is referred to [1, 2, 3] and [14].)

3. Proofs and related results

In order to have the subsequent proofs transparent, we first single out some technical facts about the class of locally convex spaces in question.

Lemma 3.1. *Let (X,τ) be a locally complete lcs, $(x_n)_n$ a bounded sequence in (X,τ), and $B = clac\{x_n \mid n \in \mathbb{N}\}$. Then we have:*

(a) *For any* $(\alpha_i)_i \in l^1$, *the series* $\sum_1^\infty \alpha_i x_i$ *converges in* (X_B, B), *and the linear map* $T : l^1 \to X_B$, $\{(\alpha_i)_i \mapsto \sum_1^\infty \alpha_i x_i\}$, *is continuous into* (X_B, B) *with* $T(B_{l^1}) \subset B$, *and* $B = \tau$*-closure of* $T(B_{l^1})$ (B_{l^1} *the closed unit ball of* l^1).

(b) *If, in addition,* $(x_n)_n$ *is a weak nullsequence in* X, *then the map* T *of* (a) *is* $\sigma(l^1, c_0) - \sigma(X, X')$*-continuous, and* $B = T(B_{l^1})$ *is* $\sigma(X, X')$*-compact metrizable.*

(c) *If, in addition, the bounded subsets of* (X, τ) *are metrizable, then every weakly compact disk in* X *is weakly sequentially compact.*

Remark 3.2. With regard to proposition (b) of Lemma 3.1, we note that, according to [19, Thm. 5.1.11], the fact that closed absolutely convex hulls of weak nullsequences in an lcs (X, τ) are weakly compact is actually equivalent to (X, τ) being locally complete.

Proof of Lemma 3.1. (a) As, given $(\alpha_i)_i \in l^1$, $q_B(\sum_m^n \alpha_i x_i) \leqslant \sum_m^n |\alpha_i|$, completeness of (X_B, B) implies q_B-convergence of the series $\sum_1^\infty \alpha_i x_i$ in X_B. With B a τ-closed disk, if $(\alpha_i)_i \in B_{l^1}$, $\sum_1^n \alpha_i x_i \in B$ for all $n \in \mathbb{N}$, and thus $\sum_1^\infty \alpha_i x_i \in B$ as well, as the series is q_B- and thus τ-convergent. Since $T(e_i) = x_i$, τ-closure$(T(B_{l^1})) = B$.

(b) Let $((\alpha_i^\lambda)_i)_{\lambda\in\Lambda} \subset l^1$ be a $\sigma(l^1, c_0)$-nullnet, and $x' \in (X, \tau)'$. Then $\langle T((\alpha_i^\lambda)_i), x'\rangle = \sum_1^\infty \alpha_i^\lambda \langle x', x_i\rangle = \langle(\alpha_i^\lambda)_i, (\langle x', x_i\rangle)_i\rangle$ (duality $l^1 - c_0$) tends to zero over $\lambda \in \Lambda$ (as $(\langle x', x_i\rangle)_i \in c_0$.) Hence, $T(B_{l^1})$ is a $\sigma(X, X')$- compact disk, and thus, according to (a), $T(B_{l^1}) = B$.

(c) If C is a weakly compact disk in (X, τ), and $(x_n)_n \in C$, $B = clac\{x_n \mid n \in \mathbb{N}\} \subset C$ is a weakly compact, τ-metrizable (by assumption on X), and τ-separable disk. Thus, according to [16] and [11, Theorem 1], there exists a norm N on X_B such that the topologies induced on B by τ and N coincide. As this extends to the respective weak topologies (cf. [6, Ch. IV.3, Ex.3(a)]), B is thus a weakly compact disk in (X_B, N). According to Smulyan's Theorem ([15, 24.1 (3)]), B is N-weakly sequentially compact, and thus $(x_n)_n$ has a τ-weakly convergent subsequence. This completes the proof of Lemma 3.1. ∎

The heart of the matter for the proof of Theorem 2.1 is the equivalence of (a) with (b). It will be treated separately by the subsequent proposition.

Proposition 3.3. *If* (X, τ) *is a locally complete lcs with bounded sets metrizable, then a bounded sequence in* X *either has a weak Cauchy subsequence or a subsequence that spans an isomorphic copy of* l^1 *in* (X, τ).

Proof. For $(x_n)_n$ a bounded sequence in (X, τ), let $B := clac\{x_n \mid n \in \mathbb{N}\}$, and note that B is metrizable (by assumption) and separable. Thus, as in the proof of (c) of Lemma 3.1 above, by [16] and [11, Theorem 1], there exists a norm N on X_B such that the topologies induced on B by τ and N coincide. According to the Dor/Rosenthal l^1-Theorem [10, 20], the sequence $(x_n)_n$, viewed as a bounded sequence in the completion $(\tilde{X_B}, N)$ of (X_B, N) either has an N-weak, and thus a τ-weak Cauchy subsequence, or else a subsequence spanning an isomorph of l^1 in this completion. In the latter case, by relabeling if necessary, we can assume that $T : l^1 \to (\tilde{X_B}, N)$, $\{(\alpha_i)_i \mapsto \sum_1^\infty \alpha_i x_i\}$, is a topological isomorphism into $(\tilde{X_B}, N)$. It remains to show that T is a topological isomorphism into (X, τ) as well. First, $imT \subset X_B$, and $T(B_{l^1}) \subset B$ by proposition (a) of Lemma 3.1, so that T is continuous into (X, τ) (as N and τ coincide on B). It thus remains to prove τ- continuity of its inverse.

What we have is that there exists $c > 0$ such that $\sum_1^n |\alpha_i| \leqslant cN(\sum_1^n \alpha_i x_i)$ for all $n \in \mathbb{N}$, and all scalars $\alpha_1, ..., \alpha_n$. At this point, we invoke the seminorm approximation result of [21, Thm. 3.4]: as N restricted to B is τ-continuous, this result yields that, given any $\epsilon > 0$, there exists a τ-continuous seminorm q_ϵ on X_B such that $|N(b) - q_\epsilon(b)| < \epsilon q_B(b)$ for all $b \in X_B$. We thus have

$$\sum_1^\infty |\alpha_i| \leqslant cN\Big(\sum_1^\infty \alpha_i x_i\Big) \leqslant c\Big(q_\epsilon\Big(\sum_1^\infty \alpha_i x_i\Big) + \epsilon \sum_1^\infty |\alpha_i|\Big) \tag{3.1}$$

for all $(\alpha_i)_i \in l^1$.

Now, let $(\alpha_i)_i^\lambda$ be a net in l^1 such that $T((\alpha_i)_i^\lambda)$ tends to zero with respect to τ, and let $\eta > 0$. Choosing $0 < \epsilon < \frac{1}{2c}$, and the corresponding τ-continuous seminorm q_ϵ on X_B, let λ_0 such that $q_\epsilon(\sum_1^\infty \alpha_i^\lambda x_i) < \frac{\eta}{2c}$ for all $\lambda \succ \lambda_0$. We then conclude from (3.1) that $\sum_1^\infty \left|\alpha_i^\lambda\right| < \eta$ for all $\lambda \succ \lambda_0$. This shows that the inverse of T is in fact τ- continuous, and thus finishes the proof.

Remark 3.4. In the context of the above technique of proof based on [21, Thm. 3.4], it has been claimed in [5, Rem. 2.2] that the strict Mackey convergence condition is actually equivalent to: for each closed bounded disk D there exists a continuous seminorm q such that $q_{|D} = q_{D_{|D}}$. This is false: Let (X, τ) be an infinite-dimensional Fréchet space, and choose any $x_n \in U_n$ for $(U_n)_n$ a zero-neighbourhood base. Then $(x_n)_n$ is a nullsequence, and $D := clac\{x_n \mid n \in \mathbb{N}\}$ is a compact disk in X. If the above claim were true, there would exist a continuous seminorm q with $q_{|D} = q_{D_{|D}}$, implying that $q_{|X_D} = q_D$ (note that $X_D = \cup_1^\infty nD$). This, in turn, easily implies that D is actually compact in (X_D, D) as well, so that X_D is finite-dimensional. Since X is infinite-dimensional, there exists a countably-infinite linear independent sequence $(y_n)_n$ in X. Choose $\alpha_n > 0$ such that $x_n = \alpha_n y_n \in U_n$. Then the above argument shows that the linear span of the x_n's, and thus the one of the y_n's, is finite-dimensional. This contradiction proves the claim to be false.

Proof of Theorem 2.1. With the equivalence of (a) and (b) covered by Proposition 3.3, the remainder of the proof is a combination of this equivalence with classical work by A. Grothendieck [13], in tandem with Lemma 3.1: According to [13, Ch. V.3, Ex. suppl. 3], a subset H in the dual X' of an lcs X is limited if and only if it is precompact with respect to (the uniformity of) uniform convergence on all sets of the family $\mathcal{S}$ of subsets S of X with the property that every sequence in S has a weak Cauchy subsequence. So, the implication "(b) $\Rightarrow$ (c)" follows immediately, since (b) implies that $\mathcal{S}$ = family of all bounded subsets of X.

As for the equivalence of (c) and (d), we now show that, for (X, τ) locally complete with bounded sets metrizable, a subset H of X' is limited if and only if it is $\tau(X', X)$-precompact. In fact, assume that $H \subset X'$ is limited. Given any weakly compact disk C in X, proposition (c) of Lemma 3.1 above reveals that C is weakly sequentially compact. Thus, $\tau(X', X)$ is coarser than the $\mathcal{S}$-topology of the above result of Grothendieck. As, by that same result, H is $\mathcal{S}$-precompact, it is thus $\tau(X', X)$-precompact by the following result [15, Section 28.5.(2)] from general topology: given two Hausdorff linear topologies τ_1 coarser than τ_2 on a linear space, these topologies coincide on every τ_2- precompact set in case τ_2 has a zero-neighbourhood base consisting of τ_1-closed sets. (The latter is trivially true for $\tau(X', X)$ and $\beta(X', X)$.)

Conversely, assume that $H \subset X'$ is $\tau(X', X)$-precompact, and let $(x_n)_n$ be a weak nullsequence in X. Then, according to proposition (b) of Lemma 3.1 above, $C = clac\{x_n \mid n \in \mathbb{N}\}$ is a weakly compact disk in X. Thus, by τ-precompactness of H, given any $\epsilon > 0$, there exist $n \in \mathbb{N}$, and $h_i \in H$, $i \in \{1, ..., n\}$, such that $H \subset \bigcup_1^n (h_i + \epsilon C^\circ)$. Hence, if $k_0 \in \mathbb{N}$ is such that $|\langle h_i, x_k \rangle| < \epsilon$ for all $k \geqslant k_0$ and $i \in \{1, ..., n\}$, we have $|\langle h, x_k \rangle| < 2\epsilon$ for all such k as well and for all $h \in H$. Thus, $(x_n)_n$ converges to zero uniformly over H. This shows that H is limited, and completes the proof of equivalence of (c) and (d).

The implication "(d) $\Rightarrow$ (e)" is general topology: if $(x'_n)_n$ is a $\tau(X', X)$-nullsequence, then its union H with $0 \in X'$ is $\tau(X', X)$-compact. In case of (d), it is thus $\beta(X', X)$-precompact. Now, once again invoke [15, Section 28.5., (2)]) to conclude that $(x'_n)_n$ converges strongly to zero. This completes the proof of Theorem 2.1. ■

Remark 3.5. With regard to the above proof of equivalence of (c) and (d), the relationship of limitedness with τ-precompactness for subsets H in the dual of an lcs X dates back to [13, Ch. V.3, Ex. suppl. 3.3], where it is stated that (i) τ-precompactness implies limitedness for just any lcs X, while (ii) the converse is true any time Smulyan's Theorem holds in X. In the above proof, for the implication "τ-precompactness $\Rightarrow$ limitedness", I have not simply used (i) above, but have gone through proposition (b) of Lemma 3.1, because I am unable to verify this implication for general lcs. (In this respect, notice Remark 3.2 above.)

Proof of Proposition 2.3. The proof of this result follows the same lines as the corresponding one for [22, Prop. 2.2], based on the subsequent linearized Arzela-Ascoli theorem:

Given locally convex spaces X and Y, and a family $\mathcal{S}$ of bounded subsets of X that cover X, we consider the space $K_{\mathcal{S}}(X, Y)$ of all weak-to-weak continuous linear operators from X into Y that transform the sets $S \in \mathcal{S}$ into precompact subsets of Y, endowed with the topology of uniform convergence on the $S \in \mathcal{S}$. The space X' endowed with the topology of uniform convergence on the $S \in \mathcal{S}$ will be denoted by $X'_{\mathcal{S}}$. For subsets $H \subset K_{\mathcal{S}}(X, Y)$ and $A \subset Y'$, the subset $\bigcup\{h'(A) \mid h \in H\}$ of X' will be denoted by $H'(A)$. ■

Lemma 3.6 ([7, Corollary, section 3], [22, Lemma 3.1]). *A subset H of $K_{\mathcal{S}}(X, Y)$ is precompact if and only if*

(i) *$H(x)$ is precompact in Y for all $x \in X$, and*
(ii) *$H'(V^\circ)$ is precompact in $X'_{\mathcal{S}}$ for all zero-neighbourhoods V of Y.*

Condition (b) (ii) of Proposition 2.3 amounts to $\langle x', x_n \rangle \to 0$ uniformly over all $x' \in H'(V^\circ)$ for any zero-neighbourhood V in Y. Thus, letting $\mathcal{S} = \mathcal{B}_{\mathcal{X}}$ = all bounded subsets of X in Lemma 3.6, necessity of conditions (i) and (ii) in (b) holds for general X, as $H'(V^\circ)$ is precompact in X'_b by Lemma 3.6. In turn, in case X does not contain l^1, sufficiency of (i) and (ii) follows from combining Lemma 3.6 (for $\mathcal{S} = \mathcal{B}_{\mathcal{X}}$) and proposition (c) of Theorem 2.1, as (ii) amounts to $H'(V^\circ)$ being limited in X'. Finally, the special case of Y = scalars in (b), teamed with proposition (c) of Theorem 2.1, shows that (b) implies (a).

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References

[1] K.D. Bierstedt and J. Bonet, *Stefan Heinrich's density condition for Fréchet spaces and the characterization of the distinguished Köthe echelon spaces*, Math. Nachr. **135** (1985), 149–180
[2] K.D. Bierstedt and J. Bonet, *Dual density conditions in (DF)-spaces I*, Results Math. **14** (1988), 242–274
[3] K.D. Bierstedt and J. Bonet, *Density conditions in Fréchet and (DF)-spaces*, Rev. Mat. Complut. **2** (1989), 59–75
[4] J. Bonet and M. Lindström, *Convergent sequences in Fréchet spaces*. In: Functional Analysis (Essen, 1991; K.D. Bierstedt, A. Pietsch, W.M. Ruess and D. Vogt, Eds.), 391-404; Lecture Notes Pure Appl. Math. **150**, Dekker, New York, 1994
[5] C. Bosch, T. Gilsdorf, C. Gomez and R. Vera, *Local completeness of $l_p(E)$, $1 \leqslant p < \infty$*, IJMMS **31**(11) (2002), 651–657
[6] N. Bourbaki, *Espaces vectoriels topologiques, Ch. III-IV*, Hermann, Paris, 1964
[7] A. Defant and K. Floret, *The precompactness-lemma for sets of operators*; in: Funct. Analysis, Holomorphy and Approximation Theory II (G. Zapata, Ed.), North-Holland Math. Studies **86** (1984), 39–55
[8] J.C. Diaz, *Montel subspaces in the countable projective limits of $L_p(\mu)$-spaces*, Canad. Math. Bull. **32** (1989), 169–176
[9] P. Domanski and L. Drewnowski, *Fréchet spaces of continuous vector-valued functions: Complementability in dual Fréchet spaces and injectivity*, Studia Math. **102** (1992), 257–267
[10] L.E. Dor, *On sequences spanning a complex l_1-space*, Proc. Amer. Math. Soc. **47**(2) (1975), 515–516
[11] L. Drewnowski, *The metrizable linear extensions of metrizable sets in topological linear spaces*, Proc. Amer. Math. Soc. **51**(2) (1975), 323–329

[12] G. Emmanuele, *A dual characterization of Banach spaces not containing l^1*, Bull. Acad. Polon. Sci. **34** (1986), 155–160

[13] A. Grothendieck, *Espaces Vectoriels Topologiques*, Sociedade Mat. S. Paulo, Sao Paulo, 1964

[14] S. Heinrich, *Ultrapowers of locally convex spaces and applications I*, Math. Nachr. **118** (1984), 285–315

[15] G. Köthe, *Topological Vector Spaces I*, Springer, Berlin, 1969

[16] D.G. Larman and C.A. Rogers, *The normability of metrizable sets*, Bull. London Math. Soc. **5** (1973), 39–48

[17] R.H. Lohman, *On the selection of basic sequences in Fréchet spaces*, Bull. Inst. Math. Acad. Sinica **2** (1974), 145–152

[18] F. Mayoral, *Compact sets of compact operators in absence of l^1*, Proc. Amer. Math. Soc. **129** (2000), 79–82

[19] P. Perez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, Math. Studies **131** (1987), North-Holland, Amsterdam

[20] H.P. Rosenthal, *A characterization of Banach spaces containing l^1*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 2411–2413

[21] W.M. Ruess, *A Grothendieck representation for the completion of cones of continuous seminorms*, Math. Ann. **208** (1974), 71–90

[22] W.M. Ruess, *An operator space characterization of Fréchet spaces not containing l^1*, Funct. Approx. Comment. Math. **44**(2) (2011), 259–264

[23] M. Valdivia, *On totally reflexive Fréchet spaces*, Atti del Congreso Int. Analisi Mat. e sue Applicazioni, Bari (1991), 39–55

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