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EXISTENCE AND UNIQUENESS OF TRANSLATION INVARIANT MEASURES IN SEPARABLE BANACH SPACES

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: It is shown that for the vector space $\mathbb{R}^{\mathbb{N}}$ (equipped with the product topology and the Yamasaki-Kharazishvili measure) the group of linear measure preserving isomorphisms is quite rich. Using Kharazishvili's approach, we prove that every infinite-dimensional Polish linear space admits a σ-finite non-trivial Borel measure that is translation invariant with respect to a dense linear subspace. This extends a recent result of Gill, Pantsulaia and Zachary on the existence of such measures in Banach spaces with Schauder bases. It is shown that each σ-finite Borel measure defined on an infinite-dimensional Polish linear space, which assigns the value 1 to a fixed compact set and is translation invariant with respect to a linear subspace fails the uniqueness property. For Banach spaces with absolutely convergent Markushevich bases, a similar problem for the usual completion of the concrete σ-finite Borel measure is solved positively. The uniqueness problem for non-σ-finite semi-finite translation invariant Borel measures on a Banach space X which assign the value 1 to the standard rectangle (i.e., the rectangle generated by an absolutely convergent Markushevich basis) is solved negatively. In addition, it is constructed an example of such a measure μ_B^0 on X, which possesses a strict uniqueness property in the class of all translation invariant measures which are defined on the domain of μ_B^0 and whose values on non-degenerate rectangles coincide with their volumes.

Keywords: invariant Borel measure, admissible translation, Banach space, Polish linear space, Markushevich basis.

1. Introduction

The problem of the existence of a partial analog of the Lebesgue measure on a Polish group which is not locally compact, is interesting in itself and has been studied from many different standpoints. The pioneering result of Ulam is of special interest (see [14, Th. 1]). He showed that *if G is any Polish group which is not locally compact and m is any left-invariant Borel measure on G assuming at least one positive finite value, then every neighborhood contains uncountably many*

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mutually congruent disjoint sets with an equal finite positive measure. This result implies that (for non-zero Borel measures)

- On a non-locally compact Polish group, the joint properties of σ-finiteness and left-invariance are incompatible.

Oxtoby [14] showed that there exists a left-invariant quasi-finite Borel measure on any Polish group. In the same paper he also showed that a left-invariant quasi-finite Borel measure on a non-locally compact Polish group fails to have the uniqueness property [14, p. 224].

The purpose of this paper is to consider invariant measures in Polish linear spaces (i.e. in separable complete linear metric spaces) and in particular in separable Banach spaces. Our approach differs from that of Oxtoby [14]. We follow the methods developed in [23], [12], [1], [17], [13] and [8], to study invariant measures on topological vector spaces. We use essentially a method of transferring the properties, which are known for $\mathbb{R}^{\mathbb{N}}$, to other Polish linear spaces by linear Borel isomorphisms. For a detailed historical review and description of applications of non-locally compact invariant measures see [7].

In our approach, it is natural to consider measures (either σ-finite or non-σ-finite measures) which are invariant with respect to a linear subspace. In Banach spaces, it is natural to consider invariance with respect to linear subspaces spanned by bases. A typical example, in the case of $L_p[-\pi,\pi]$, $1 \leqslant p < \infty$, or $C[-\pi,\pi]$, is to use the Markushevich basis formed by trigonometric functions. For the description of probability measures, quasi-invariant with respect to the linear subspace spanned by an orthonormal basis of a Hilbert space, see [21, §23].

We recall the necessary definitions from measure theory.

Let X be a topological space and let μ be a non-negative countably additive (Borel) measure defined on the σ-algebra $\mathcal{B}(X)$ of Borel subsets of X. We say that μ

1. *lies* on a set $A \in \mathcal{B}(X)$ if $\mu(X \setminus A) = 0$ (then we call A the *carrier* of μ);
2. is *σ-finite* if X is a union of countably many sets of a finite measure;
3. is *quasi-finite* if there is a compact set $K \subset X$ for which $0 < \mu(K) < \infty$;
4. is *semi-finite* if for every Borel set A with $\mu(A) > 0$ there is a compact subset $K \subset A$ for which $0 < \mu(K) < \infty$;
5. is *inner regular* if $\mu(A) = \sup\{\mu(K) : \text{compact } K \subset A\}$ for all $A \in \mathcal{B}(X)$;
6. is *orthogonal* to a Borel measure μ' on X if there exists a Borel set $A \subset X$ such that $\mu(A) = 0$ and $\mu'(X \setminus A) = 0$.

Remark 1.1. Any *σ-finite* Borel measure μ on a Polish space X is inner regular. In particular, there exists a sequence of compact sets K_n such that μ lies on $\cup_n K_n$. This follows e.g. from [14, Lemma 1].

Let now X be a topological vector space and μ be a Borel measure in X. Put $\mu_x(A) = \mu(A + x)$, where $x \in X$ and $A \in \mathcal{B}(X)$. A Borel set $A \subset X$ is said to be *Haar null* (shy, by [9]) if there exists a quasi-finite Borel measure μ on X such that $\mu_x(A) = 0$ for all $x \in X$. Of course, in this definition one may take, instead of μ, a finite measure $\mu'(B) = \mu(B \cap K)$, $B \in \mathcal{B}(X)$, with K from the definition of quasi-finite measure.

An element $x \in X$ is called an *admissible translation in the sense of invariance* for a measure μ if μ and μ_x coincide. An element $x \in X$ is called an *admissible translation in the sense of quasi-invariance* for a measure μ if μ and μ_x are *equivalent*, i.e. $\mu_x(A) = 0$ if and only if $\mu(A) = 0$ for all $A \in \mathcal{B}(X)$.

If each element x of a set $L \subset X$ is an admissible translation in the sense of invariance (quasi-invariance) for a measure μ, then μ is said to be *L-invariant* (resp. *L-quasi-invariant*) or invariant (resp. quasi-invariant) with respect to L. The measure μ is *invariant* (resp. *quasi-invariant*) if $\mu = \mu_x$ (resp. μ is equivalent to μ_x) for all $x \in X$.

It is obvious that the set $\mathcal{I}_\mu$ of all admissible translations in the sense of invariance and the set $\mathcal{Q}_\mu$ of all admissible translations in the sense of quasi-invariance for the measure μ are groups under the addition operation of X and $\mathcal{I}_\mu \subseteq \mathcal{Q}_\mu$. In a very large class of infinite-dimensional separable topological vector spaces (which include linear metric spaces and Hausdorff locally convex spaces) there exists no nontrivial σ-finite quasi-invariant measure [5]. Measures, quasi-invariant with respect to a dense linear subspace, naturally appear in mathematical physics [6] and probability theory [21]. Though one can easily construct a probability measure in a separable Banach space which is quasi-invariant with respect to a dense linear subspace, we do not know whether such a measure exists in any separable Hausdorff topological vector space. To our best knowledge, a σ-finite measure in a separable Hilbert space which is invariant with respect to a dense linear subspace, originally was constructed by Kharazishvili [12]. We construct a similar measure in a Polish linear space (Corollary 4.1).

Definition 1.1. Let $\mathcal{M}$ be a class of measures defined on a measurable space. We say that $\mathcal{M}$ has the *uniqueness property* if for any μ, μ' in $\mathcal{M}$ there is $c > 0$ such that $\mu = c\mu'$. In this case we say that the measure μ possesses the *uniqueness property* in the class $\mathcal{M}$. We say that $\mathcal{M}$ has the *strict uniqueness property* if $\mu = \mu'$ for any μ, μ' in $\mathcal{M}$. In this case we say that μ possesses the *strict* uniqueness property in the class $\mathcal{M}$.

As for the results of Oxtoby [14] discussed earlier, the following two general problems are of interest.

Problem 1.1. Let X be a Banach space and let $\mathcal{M}$ be the class of all σ-finite Borel measures on X, which assume the value 1 on a fixed compact set $K \subset X$ and are invariant with respect to a fixed dense linear subspace L of X. Does $\mathcal{M}$ has the uniqueness property?

Problem 1.2. Let X be a Banach space and let $\mathcal{M}$ be the class of all semi-finite invariant Borel measures on X taking the value 1 on a fixed compact set $K \subset X$. Does $\mathcal{M}$ have the uniqueness property?

One of the objectives of this paper is to investigate Problems 1.1, 1.2 when the linear subspace L is spanned by a Markushevich basis and K is the rectangle generated by this basis (for the exact definitions see below).

The space $\mathbb{R}^{\mathbb{N}}$ (we prefer this notation for $\mathbb{R}^{\infty}$) plays the decisive role in our consideration. It is equipped with topology generated by Tychonoff's metric (cf. [4, p. 384]) and is, with this metric, a Polish linear space. We denote by $e_k = (0, \dots, 0, 1, 0, \dots)$, where 1 is in the k-th position, $k = 1, 2, \dots$, the unit vectors of $\mathbb{R}^{\mathbb{N}}$. The symbols c_{00}, ℓ_1, ℓ_2 and ℓ_∞ stand for linear spaces of all eventually null, absolutely summing, square summable and bounded sequences (all are linear subspaces of $\mathbb{R}^{\mathbb{N}}$) and they may carry two different topologies – the corresponding norm topologies and the topology of pointwise convergence inherited from $\mathbb{R}^{\mathbb{N}}$. The symbol $\prod$ denotes both the usual scalar and the cartesian product. To avoid cumbersome formulae, we write $\prod_k$ instead of $\prod_{k=1}^{\infty}$. We denote by $\mathrm{vol}\,(R) := \prod_k (b_k - a_k)$ the volume of an infinite-dimensional rectangle $R = \prod_k [a_k, b_k] \subset \mathbb{R}^{\mathbb{N}}$. We denote by $\mathcal{R}$ the family of all rectangles with finite positive volume. $Q = [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}$ stands for the standard cube of $\mathbb{R}^{\mathbb{N}}$. Following [1], we say that a measure λ defined on the σ-algebra of Borel subsets of $\mathbb{R}^{\mathbb{N}}$ is an *infinite-dimensional Lebesgue measure* if $\lambda(R) = \mathrm{vol}\,(R)$ for all rectangles.

Banach spaces are always real and equipped with the norm topology. All subspaces are linear but not necessarily closed. In the sequel we need the following auxiliary propositions.

Lemma 1.1 ([4, Th. 2.3.6]). *Let $(Y_i)_{i \in I}$ be a family of topological spaces and $Y = \prod_{i \in I} Y_i$ be their cartesian product endowed with product topology. Let $p_j : Y \to Y_j$ be the j-projection for $j \in I$, defined by*

$$p_j((y_i)_{i \in I}) = y_j \qquad (y_i)_{i \in I} \in Y.$$

If X is a topological space, then a mapping $f : X \to Y$ is continuous if and only if each composition $p_j \circ f$ is continuous for $j \in I$.

Lemma 1.2 (Lusin-Souslin [11, Th. 15.1]). *Let X, Y be Polish spaces and $f : X \to Y$ be continuous. If $A \subseteq X$ is Borel and $f|_A$ is injective then $f(A)$ is Borel.*

The rest of the paper is organized as follows. In Section 2 we present some auxiliary facts from measure theory and Banach spaces. In Section 3 we describe a special semi-finite inner regular invariant infinite-dimensional Lebesgue measure on $\mathbb{R}^{\mathbb{N}}$. In Section 4 we give solutions to Problems 1.1 and 1.2 and consider some related topics.

2. Auxiliary definitions and facts from measure theory and Banach spaces

a) Yamasaki-Kharazishvili measure

Let $[a_k, b_k]_{k \geqslant 1}$ be a sequence of nontrivial segments and $\mathbb{R}_k$ be the k-th copy of $\mathbb{R}$. Following Kharazishvili [12], put $R = \prod_k [a_k, b_k]$, for each n

$$\Pi_R^n := \prod_{k \leqslant n} \mathbb{R}_k \times \prod_{k > n} [a_k, b_k] = \prod_{k \leqslant n} \mathbb{R}_k \times (0, 0, \dots) + R,$$

and

$$\Pi_R := \cup_n \Pi_R^n = c_{00} + R.$$

Remark 2.1. Obviously, each $\Pi_R^n = \cup_m R_m^n$, where

$$R_m^n = [-m, m]^n \times \prod\nolimits_{k>n} [a_k, b_k], \qquad m, n = 1, 2, \dots .$$

In particular, Π_R is σ-compact.

For every k, denote by λ_k the usual Lebesgue measure on $\mathbb{R}_k = \mathbb{R}$, but normalized so that $\lambda_k([a_k, b_k]) = 1$. We will denote the restriction of λ_k to $[a_k, b_k]$ by the same symbol λ_k. For every n, let ν_R^n denote the measure on Π_R^n defined by

$$\nu_R^n = \prod\nolimits_{k \geqslant 1} \lambda_k.$$

Lemma 2.1 (see e.g. [15, Lemma 1.1]). *For any $B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$, the following limit exists*

$$\nu_R(B) = \lim_{n\to\infty} \nu_R^n(B \cap \Pi_R^n). \tag{2.1}$$

Moreover, ν_R is a nontrivial σ-finite Borel measure.

By definition, the measure ν_R lies on Π_R, $\nu_R(R) = 1$ and, by Remark 1.1, it automatically is inner regular and semi-finite. The next well known statement indicates an important property of ν_R. The proof is based on simple facts from mathematical analysis asserting that if $0 < d_k < 1$ and $\sum_k d_k = \infty$ then $\prod_k (1 - d_k) = 0$ and that if $\sum_k |d_k| < \infty$ then $\prod_{k>n}(1 - d_k) \to 1$ as $n \to \infty$.

Proposition 2.1. *For every $z = (c_k)_1^\infty$ satisfying $\sum_k \frac{|c_k|}{b_k - a_k} = \infty$ we have $\nu_R(\Pi_R + z) = 0$.*

Proof. Since Π_R is a union of countably many rectangles (see Remark 2.1), instead of all Π_R, we can consider only one rectangle $R^0 = \prod_k [a_k^0, b_k^0]$. Take any n and any rectangle $\prod_k [a_k^1, b_k^1] \in \{R_m^n : m = 1, 2, \cdots\}$, and show that

$$\prod\nolimits_k \lambda_k\{[a_k^0 + c_k, b_k^0 + c_k] \cap [a_k^1, b_k^1]\} = 0. \tag{2.2}$$

By Remark 2.1, there is m such that $a_k^0 = a_k^1 = a_k$ and $b_k^0 = b_k^1 = b_k$ for $k \geqslant m$. To prove that the product is 0, it is enough to show that the product which starts from m is 0. Moreover, the factors are non-zero if the corresponding intersections are nonempty, in particular if $|c_k| < b_k - a_k$ for $k > m$. So, we can consider (and obtain)

$$\prod_{k>m} \lambda_k\{[a_k + c_k, b_k + c_k] \cap [a_k, b_k]\} = \prod_{k>m} \left(1 - \frac{|c_k|}{b_k - a_k}\right) = 0.$$

Hence (2.2) is proved. This relation and Remark 2.1 give that for all n

$$\nu_R^n\{(R^0 + z) \cap \Pi_R^n\} = 0$$

which, together with (2.1), gives $\nu_R(R^0 + z) = 0$. ∎

Similarly, we can show that if $\sum_{k=1}^{\infty} |c_k|/(b_k - a_k) < \infty$ then $\nu_R(R^0 + z) = \nu_R(R^0)$ for every rectangle $R^0 \subset \Pi_R$ and obtain

Corollary 2.1 (a simple consequence of [15, Th. 1.4]). *For a rectangle $R = \prod_k [a_k, b_k]$ we have*

$$\mathcal{I}_{\nu_R} = \ell_1(R) := \left\{ (c_k)_1^{\infty} \in \mathbb{R}^{\mathbb{N}} \,:\, \sum\nolimits_{k=1}^{\infty} |c_k|/(b_k - a_k) < \infty \right\}.$$

Hence, if $b_k - a_k \to 1$ as $k \to \infty$ then $\mathcal{I}_{\nu_R} = \ell_1$.

Remark 2.2. Corollary 2.1 is similar to the well-known result of Kakutani [10] on the structure of admissible translations (in the sense of quasi-invariance) for products of linear Gaussian measures in $\mathbb{R}^{\mathbb{N}}$. Unlike these products, Proposition 2.1 and Corollary 2.1 immediately imply that $\mathcal{I}_{\nu_R} = \mathcal{Q}_{\nu_R}$.

Remark 2.3. By Corollary 2.1, $\mathcal{I}_{\nu_Q} = \ell_1$ (recall that $Q := [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}$). Moreover, ν_Q is invariant under the group G_Q of transformations of $\mathbb{R}^{\mathbb{N}}$ which is generated by:

- the group of ℓ_1-translations;
- the group of canonical permutations[1] of $\mathbb{R}^{\mathbb{N}}$;
- the group of symmetric transformation of $\mathbb{R}^{\mathbb{N}}$ with respect to the point $(0, 0, \dots)$;
- the group generated by multiplication using positive sequences $(a_k)_{k=1}^{\infty}$, such that $\sum_k |\ln(a_k)| < \infty$ and $\sum_k \ln(a_k) = 0$;
- the group generated by $\cup_{n=1}^{\infty}(D_n \times I_n)$, where D_n is the group of all rotations of the Euclidean space $\mathbb{R}^n$, and I_n is the identity operator on $\mathbb{R}^{\mathbb{N}\setminus\{1,\dots,n\}}$.

Remark 2.4. Using the method of [22], Yamasaki [23] constructed a σ-finite measure μ on $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ such that:

(i) $\mu(Q) = 1$, $\mathcal{I}_{\mu} = \ell_1$ and
(ii) μ lies on the set $\Pi_Q = c_{00} + Q$.

As mentioned above, Kharazishvili [12] constructed the measure ν_R as an inductive limit of a consistent family of suitable σ-finite measures which for $R = Q$ also satisfies properties (i),(ii). There naturally arises a question about the equality of these two measures. In order to give a positive answer to this question we recall the following definition:

Let $\mathcal{M}$ be a class of measures defined on a measurable space $(Y, \mathcal{F})$. A set $A \in \mathcal{F}$ is said to have the *property of essential uniqueness with respect to the class $\mathcal{M}$* if for all ν, ν' in $\mathcal{M}$ and $B \in \mathcal{F}$

$$\nu(A \cap B) = \nu'(A \cap B).$$

By the scheme presented in [17, Th. 8.3] it can easily established that the set Π_Q has the property of essential uniqueness with respect to the class $\mathcal{M}_0$ of all

[1] Let π be any permutation of the natural numbers $\mathbb{N}$. A mapping $T_\pi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by $T_\pi((c_k)_1^{\infty}) = (c_{\pi(k)})_1^{\infty}$ for $(c_k)_1^{\infty} \in \mathbb{R}^{\mathbb{N}}$ is called a canonical permutation of $\mathbb{R}^{\mathbb{N}}$.

c_{00}-invariant σ-finite Borel measures taking the value 1 on Q. Note that both measures μ and ν_Q belong to the class $\mathcal{M}_0$ (obviously, $c_{00} \subset \ell_1$), which implies

$$\mu(B \cap \Pi_Q) = \nu_Q(B \cap \Pi_Q), \qquad B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})\,.$$

Since both measures μ and ν_Q lie on the Borel set Π_Q, they coincide.

Since each ν_R can be obtained as a linear transformation of ν_Q (equivalently, μ), in the sequel that measure will be mentioned as the Yamasaki-Kharazishvili measure, and will be denoted by ν_R. Moreover, we will work with Kharazishvili's description of this measure.

By the scheme presented in [13, Th. 1], we get the following statement.

Lemma 2.2. *Let $\overline{\nu}_R$ be the completion of the measure ν_R. Then the measure $\overline{\nu}_R$ has the uniqueness property in the class of all ℓ_1-invariant σ-finite measures defined on the completion of $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ by the measure ν_R.*

b) Markushevich bases

Let X be an infinite-dimensional separable Banach space. A sequence $(x_k)_{k=1}^{\infty} \subset X$ is called *minimal* if each vector x_k is not contained in the closed linear span of $(x_l)_{l\neq k}$. A sequence in X is called *fundamental* if its closed linear span coincides with X. It is easy to verify that for a fundamental minimal sequence $(x_k)_{k=1}^{\infty}$ there exists a unique sequence $(x_k^*)_{k=1}^{\infty}$ of continuous linear functionals satisfying the condition $x_k^*(x_l) = \delta_{kl}$ $(k, l \in \mathbb{N})$. This sequence is called *biorthogonal* to $(x_k)_{k=1}^{\infty}$. Thus, if (x_k) is minimal and fundamental, then to each $x \in X$ there corresponds a formal generalized Fourier series

$$\sum\nolimits_{k=1}^{\infty} x_k^*(x) x_k.$$

The vector x is uniquely determined by this series if and only if the biorthogonal sequence $(x_k^*)_{k=1}^{\infty}$ is *total* (that is for each $x \neq 0$ there exists $k \in \mathbb{N}$ such that $x_k^*(x) \neq 0$). A fundamental minimal sequence with a total biorthogonal sequence is called the *Markushevich basis* (M-basis in short). By the Markushevich theorem, for every countably-dimensional dense subspace L of a separable Banach space X there is an M-basis $(x_k, x_k^*)_{k=1}^{\infty}$ of X such that the linear span $\operatorname{lin}(x_k)_1^{\infty} = L$ [20, p. 226]. Conversely, each Banach space with M-basis is separable. We call an M-basis *absolutely convergent* if $\sum_{k=1}^{\infty} \|x_k\| < \infty$. The following statement follows immediately from the Markushevich theorem mentioned above.

Lemma 2.3. *Every infinite-dimensional separable Banach space has an absolutely convergent M-basis.*

Lemma 2.4. *Let (x_k, x_k^*) be an absolutely convergent M-basis in a Banach space X. Then for every bounded scalar sequence (a_k) the series $\sum_k a_k x_k$ is absolutely convergent to some element $x \in X$ and moreover $x_k^*(x) = a_k$ for all k.*

Proof. Since X is a Banach space, the first part of the lemma will be proved if we show that $\sum_{k=1}^{\infty}\|a_k x_k\| < \infty$. But this is obvious. Indeed, for $b = \sup_{k\in\mathbb{N}}|a_k|$

$$\sum_{k=1}^{\infty}\|a_k x_k\| \leqslant \sum_{k=1}^{\infty} b\|x_k\| = b\sum_{1}^{\infty}\|x_k\| < \infty.$$

Moreover, $x_k^*(x) = x_k^*(a_k x_k) = a_k$ for all k. ■

Let $(x_k, x_k^*)_{k=1}^{\infty}$ be an absolutely convergent M-basis of a Banach space X and a rectangle P be defined by

$$P = \{x \in X \ : \ |x_k^*(x)| \leqslant 1/2 \quad \text{for all} \quad k \in \mathbb{N}\}. \tag{2.3}$$

Obviously, P is a compact subset in X.

Let L_1 be a (dense) linear subspace of X defined by

$$L_1 = \left\{\sum_{k=1}^{\infty} a_k x_k : (a_k)_1^{\infty} \in \ell_1\right\}.$$

The operator $T : X \to \mathbb{R}^{\mathbb{N}}$ defined by

$$Tx = (x_k^*(x))_{k=1}^{\infty} \tag{2.4}$$

is clearly linear, injective (because the M-basis is total), continuous (by Lemma 1.1) and $Tx_k = e_k$ for all k. Furthermore, $T(L_1) = \ell_1$, $T(P) = Q$ and $T(X) \supset \ell_\infty$.

Lemma 2.5. *The subspace $S := T(X)$ is a Borel subset of $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$, and the operator $T : X \to S$ is a Borel isomorphism provided that $T : X \to S$ is one-to-one linear operator and $A \in X \cap \mathcal{B}(X)$ if and only if $T(A) \in S \cap \mathcal{B}(\mathbb{R}^{\mathbb{N}})$.*

Proof. Since T is injective and continuous, by Lemma 1.2, for each Borel subset $A \subseteq X$ the image $T(A)$ is Borel in $\mathbb{R}^{\mathbb{N}}$. In particular, $S = T(X)$ is Borel. Therefore T is a Borel isomorphism between X and S. ■

3. An infinite-dimensional Lebesgue measure

This section presents a special sum of Yamasaki-Kharazishvili measures [17, p. 208], [18, p. 249]. We show that some properties of this sum can be checked simpler than in the original proof. We use simple and probably well known facts about sums of measures.

a) Sums of measures

Let $(\mu_i)_{i\in I}$ be a family of Borel measures on a topological space X. A direct sum of Borel measures $(\mu_i)_{i\in I}$ denoted by $\sum_{i\in I}\mu_i$ is defined by

$$\sum_{i\in I}\mu_i(Y) = \sup\left\{\sum_{j\in J}\mu_j(Y) : J \subseteq I \ \& \ \mathrm{card}(J) < \aleph_0\right\}$$

for every Borel subset $Y \subseteq X$, where $\aleph_0$ denotes the cardinality of all natural numbers.

Proposition 3.1. *Let $(\mu_i)_{i\in I}$ be a family of Borel measures on a topological space X and $\mu := \sum_{i\in I}\mu_i$. Then:*

(1) *μ is a Borel measure;*
(2) *if each μ_i is inner regular then μ is inner regular as well.*

Proof. (1) Let $(A_n)_{n\geqslant 1}$ be pairwise disjoint Borel subsets in X. Then

$$\mu\left(\bigcup\nolimits_{n\geqslant 1} A_n\right) := \sum\nolimits_{i\in I}\mu_i\left(\bigcup\nolimits_{n\geqslant 1} A_n\right) = \sum\nolimits_{i\in I}\sum\nolimits_{n\geqslant 1}\mu_i(A_n)$$
$$= \sum\nolimits_{n\geqslant 1}\sum\nolimits_{i\in I}\mu_i(A_n) = \sum\nolimits_{n\geqslant 1}\mu(A_n).$$

Hence, μ is σ-additive.

(2) Let first $0 < \mu(A) < \infty$[2] and $\epsilon > 0$. By definition, there is a finite sequence $(i_k)_1^n$ such that

$$\mu(A) < \sum\nolimits_{k=1}^n \mu_{i_k}(A) + \epsilon.$$

Of course, each $\mu_{i_k}(A) < \infty$ and it can be assumed that $\mu_{i_k}(A) > 0$. Since every μ_i is inner regular, there are compact subsets $K_k \subset A$ such that for every k

$$\mu_{i_k}(A\setminus K_k) < \epsilon/n.$$

The set $K := \bigcup_1^n K_k \subset A$ is compact and

$$\mu(A\setminus K) = \mu(A) - \mu(K) < \sum\nolimits_{k=1}^n \mu_{i_k}(A) + \epsilon - \sum\nolimits_{k=1}^n \mu_{i_k}(K_k)$$
$$= \sum\nolimits_{k=1}^n \mu_{i_k}(A\setminus K_k) + \epsilon < \sum\nolimits_1^n \epsilon/n + \epsilon = 2\epsilon.$$

Let now $\mu(A) = \infty$ and $c > 0$. Then there exist indices $(i_k)_{k=1}^n$ such that $\sum_{k=1}^n \mu_{i_k}(A) > c$. If $\mu_{i_k}(A) = \infty$ for some k then, by the inner regularity of μ_{i_k}, there is a compact $K \subset A$ such that $\mu(K) \geqslant \mu_i(K) > c$. If all $\mu_{i_k}(A) < \infty$ then, by the inner regularity of each μ_{i_k}, for any $\epsilon > 0$ there is a compact subset $K_k \subset A$ such that $\mu_{i_k}(K_k) > \mu_{i_k}(A) - \epsilon/n$. The set $K := \bigcup_{k=1}^n K_k \subset A$ is compact and

$$\mu(K) \geqslant \sum\nolimits_{k=1}^n \mu_{i_k}(K) \geqslant \sum\nolimits_{k=1}^n \mu_{i_k}(K_k) > \sum\nolimits_{k=1}^n \mu_{i_k}(A) - \epsilon > c - \epsilon.$$

Since c and ϵ are arbitrary, this proves the proposition. ∎

Proposition 3.2. *Let X be a topological vector space, L be its linear subspace and Z be a linear complement of L in X. Let $(\mu_i)_{i\in I}$ be a family of Borel measures on X and $\mu := \sum_{i\in I}\mu_i$. Then:*

(1) *If each μ_i is L-invariant then so is μ;*
(2) *If μ is L-invariant then the measure $\lambda(A) := \sum_{z\in Z}\mu(A+z)$, $A \subset \mathcal{B}(X)$, is X-invariant.*

[2] We realize that for σ-finite measures on Polish spaces one may not consider this case.

Proof. Item (1) is obvious. To prove the item (2) note that every $x \in X$ has the form $x = y_1 + z_1$, where $y_1 \in L$, $z_1 \in Z$ and that $Z = Z + z_1$. So, for every $A \in \mathcal{B}(X)$

$$\begin{aligned}\lambda(A+x) &= \sum_{z\in Z} \mu(A + y_1 + z_1 + z) = (\text{put } \ z' = z + z_1)\\ &= \sum_{z'\in Z} \mu(A + z') = \lambda(A).\end{aligned}$$ ∎

b) Main results of the section

We need a modification of the Yamasaki-Kharazishvili measure which is simpler and more natural than the original one and is defined for $R \in \mathcal{R}$. For all n put $\widetilde{\nu}_R^n = \mathrm{vol}\,(R)\,\nu_R^n$ and $\widetilde{\nu}_R = \mathrm{vol}\,(R)\,\nu_R$. We obtain $\widetilde{\nu}_R^n$ (and $\widetilde{\nu}_R$) if in the definition of Kharazishvili's measure we take for every k the usual Lebesgue measure on $\mathbb{R}_k$ instead of the normalized measure λ_k. We denote this Lebesgue measure by λ. Obviously, $\widetilde{\nu}_R(R') = \mathrm{vol}\,(R')$ for every $R' \subset \Pi_R$. Denote by $\mathcal{R}_0$ the subset of $\mathcal{R}$ consisting of rectangles of the form $R = \prod_k [0, a_k]$. In the lemma, two propositions and the corollary below, $R = \prod_k [0, a_k]$ and $R' = \prod_k [0, b_k]$ are from $\mathcal{R}_0$.

Lemma 2.1, the reasoning after that lemma and Corollary 2.1 imply

Lemma 3.1. *The set function, $\widetilde{\nu}_R$ is an ℓ_1-invariant inner regular semi-finite σ-finite Borel measure.*

Proposition 3.3. *For every $z = (c_1, c_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ we have $\widetilde{\nu}_R(R' + z) \leqslant \widetilde{\nu}_R(R')$.*

Proof. Since the condition

$$\lambda\{[c_k, b_k + c_k] \cap [0, a_k]\} \leqslant \lambda\{[0, b_k] \cap [0, a_k]\}$$

holds for each k, we have

$$\begin{aligned}\widetilde{\nu}_R^n\{(R' + z) \cap \Pi_R^n\} &= \prod\nolimits_{k\leqslant n} \lambda\{[c_k, b_k + c_k]\} \cdot \prod\nolimits_{k>n} \lambda\{[c_k, b_k + c_k] \cap [0, a_k]\}\\ &\leqslant \prod\nolimits_{k\leqslant n} \lambda\{[0, b_k]\} \cdot \prod\nolimits_{k>n} \lambda\{[0, b_k] \cap [0, a_k]\} = \widetilde{\nu}_R^n\{R' \cap \Pi_R^n\}\end{aligned}$$

for every n. Passing to the limit on n, we obtain the proposition. ∎

To prove the next proposition note that for every $R \in \mathcal{R}_0$, $\lim_n \prod_{k>n} a_k = 1$, and that if $\alpha_k > 0$ and $\prod_k \alpha_k = 0$ then $\prod_{k>n} \alpha_k = 0$ for all n.

Proposition 3.4 (see [17, Lemma 15.3.4, p. 207]). *Either $\widetilde{\nu}_R(\Pi_{R'}) = 0$ or $\widetilde{\nu}_R = \widetilde{\nu}_{R'}$.*

Proof. Denote $\alpha_k = \min(a_k, b_k)$. By Lemma 3.1, the product $\prod_k \alpha_k$ exists, moreover, since $\alpha_k \leqslant a_k$, it is finite. Putting $R'' = R \cap R'$ we get

$$R, R', R'' = \prod\nolimits_k [c_k, d_k], \qquad \text{where } \ c_k = 0 \ \text{ and } \ d_k = \begin{cases} a_k & \text{for } R,\\ b_k & \text{for } R',\\ \alpha_k & \text{for } R''.\end{cases}$$

Let us consider two cases.

Case 1: $\prod_k \alpha_k = 0$. Then for $n > 1$

$$\begin{aligned}\widetilde{\nu}_R^n(R' \cap \Pi_R^n) &= \prod_{k \leqslant n} \lambda\{[c_k, d_k]\} \cdot \prod_{k>n} \lambda\{[0, b_k] \cap [0, a_k]\} \\ &= \prod\nolimits_{k \leqslant n}(d_k - c_k) \prod\nolimits_{k>n} \alpha_k = 0.\end{aligned}$$

Passing to the limit on n and taking into account that one can choose eventually zero sequences (h_m) (under action of which the measure $\widetilde{\nu}_R$ is invariant), so that $\cup_m R' + h_m = \Pi_{R'}$, we have $\widetilde{\nu}_R(\Pi_{R'}) = 0$.

Case 2: $\prod_k \alpha_k \neq 0$. Hence, $\lim_n \prod_{k>n} \alpha_k = 1$. For $n > 1$ we have

$$\begin{aligned}\widetilde{\nu}_R^n(R \cap \Pi_{R''} \cap \Pi_R^n) \geqslant \widetilde{\nu}_R^n(R \cap \Pi_{R''}^n \cap \Pi_R^n) &= \widetilde{\nu}_R^n(R \cap \Pi_{R''}^n) \\ &= \prod\nolimits_{k \leqslant n}(d_k - c_k) \cdot \prod\nolimits_{k>n} \alpha_k.\end{aligned}$$

Passing to the limit on n, we obtain $\widetilde{\nu}_R(R \cap \Pi_{R''}) \geqslant \mathrm{vol}\,(R)$. Since

$$\widetilde{\nu}_R(R \cap \Pi_{R''}) \leqslant \widetilde{\nu}_R(R) = \mathrm{vol}\,(R),$$

we have

$$\widetilde{\nu}_R(R \cap \Pi_{R''}) = \widetilde{\nu}_R(R).$$

Similarly,

$$\widetilde{\nu}_{R'}(R' \cap \Pi_{R''}) = \widetilde{\nu}_{R'}(R').$$

Hence, both measures $\widetilde{\nu}_R$ and $\widetilde{\nu}_{R'}$ lie on $\Pi_{R''}$.

Moreover, for $n > 1$ we have

$$\widetilde{\nu}_R^n(R'' \cap \Pi_R^n) = \prod_{k \leqslant n} \lambda\{[c_k, d_k]\} \cdot \prod_{k>n} \lambda\{[0, \alpha_k]\} = \widetilde{\nu}_{R'}^n(R'' \cap \Pi_{R'}^n).$$

Passing to the limit on n, we obtain $\widetilde{\nu}_R(R'') = \widetilde{\nu}_{R'}(R'')$.

Since eventually zero shifts of R'' generate the σ-algebra $\mathcal{B}(\Pi_{R''})$ and both measures $\widetilde{\nu}_R$ and $\widetilde{\nu}_{R'}$ are invariant under the action of the group of all eventually zero sequences, we deduce that these measures coincide on $\mathcal{B}(\Pi_{R''})$. The latter relation implies that both measures $\widetilde{\nu}_R$ and $\widetilde{\nu}_{R'}$ also coincide on $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ because they lie on $\Pi_{R''}$. ∎

Corollary 3.1. *If $\widetilde{\nu}_R \neq \widetilde{\nu}_{R'}$ then $\widetilde{\nu}_R(\Pi_{R'} + z) = 0$ for every $z \in \mathbb{R}^{\mathbb{N}}$.*

Construction. We write that $R \simeq R'$, where $R, R' \in \mathcal{R}_0$, if and only if $\widetilde{\nu}_R = \widetilde{\nu}_{R'}$. This clearly induces an equivalence relation on $\mathcal{R}_0$. Let us consider the equivalence classes $(\mathcal{R}_i)_{i \in I}$ of $\mathcal{R}_0$ generated by the relation $\simeq$. For every $i \in I$, take a representative $R_i \in \mathcal{R}_i$, and denote $\Pi_i = \Pi_{R_i}$, $\nu_i = \widetilde{\nu}_{R_i}$. Put

$$\lambda_P(B) = \sum_{i \in I,\, z \in Z} \nu_i(B + z), \qquad B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}),$$

where Z is a linear complement of the vector subspace ℓ_1 in $\mathbb{R}^{\mathbb{N}}$.

Corollary 3.2 (of Corollary 3.1). *For each i and a Borel subset $B \subset \Pi_i$ we have $\lambda_P(B) = \nu_i(B)$.*

Theorem 3.1. *The set function λ_P is an inner regular semi-finite invariant infinite-dimensional Lebesgue measure.*

Proof. By Lemma 3.1, and Proposition 3.1, λ_P is σ-additive and inner regular. By Lemma 3.1 and Proposition 3.2, λ_P is translation invariant.

By Corollary 3.2, $\lambda_P(R) = \mathrm{vol}\,(R)$ for every $R \in \mathcal{R}_0$. Since both λ_P and vol are invariant, this equality is valid for every $R \in \mathcal{R}$. So, λ_P is an infinite-dimensional Lebesgue measure in $\mathbb{R}^{\mathbb{N}}$.

Show that λ_P is semi-finite. Indeed, if $\lambda_P(B) > 0$, then there exist $i \in I$ and $z \in Z$ such that $\nu_i(B + z) > 0$. Put $B' = (B + z) \cap \Pi_i$. By our reasoning after Lemma 2.1, the measure ν_i lies on Π_i. So, $\nu_i(B') = \nu_i(B + z) > 0$. Since, by Lemma 3.1, ν_i is σ-finite there is a compact set $K \subset B'$ such that $0 < \nu_i(K) < \infty$. Finally, by Corollary 3.2, $\lambda_P(K) = \nu_i(K) < \infty$. ∎

Remark 3.1. The set function λ_B (the Baker measure) defined in [1] is a quasi-finite invariant infinite-dimensional Lebesgue measure on $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$. However, by [18, p. 253], λ_B is not semi-finite.

Let $\mathcal{F}$ be the σ-algebra generated by the union of this family $\{B \in \mathcal{B}(R) : R \in \mathcal{R}\}$. It is not difficult to prove that $\mathcal{F} \neq \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ and that for every $B \in \mathcal{F}$ there is a sequence $(R_n)_{n\geqslant 1}$ in $\mathcal{R}$ such that

(i) either $B \in \mathcal{B}(\cup_{n\geqslant 1} R_n)$ or
(ii) $B = \mathbb{R}^{\mathbb{N}} \setminus C$, where $C \in \mathcal{B}(\cup_{n\geqslant 1} R_n)$.

Denote by $\mathcal{M}$ the class of all invariant measures on $\mathcal{F}$ which get the value $\mathrm{vol}\,(R)$ on every $R \in \mathcal{R}$.

Lemma 3.2 (see the proof of [17, Th. 8.3]). *For every $\nu \in \mathcal{M}$ and $R \in \mathcal{R}$ we have $\nu = \widetilde{\nu}_R$ on $\mathcal{B}(R)$.*

Theorem 3.2. *The class $\mathcal{M}$ has the strict uniqueness property, and the restriction λ_P^0 of the measure λ_P to the σ-algebra $\mathcal{F}$ belongs to $\mathcal{M}$.*

Proof. Let $\nu, \nu' \in \mathcal{M}$. By Lemma 3.2, $\nu = \nu'$ on $\mathcal{B}(R)$. It is obvious that if A, B are Borel subsets of $\mathbb{R}^{\mathbb{N}}$ such that $A \subset B$ and $\nu = \nu'$ on $\mathcal{B}(B)$ then $\nu = \nu'$ on $\mathcal{B}(A)$, and if Borel sets (A_n) are pairwise disjoint and $\nu = \nu'$ on each $\mathcal{B}(A_n)$ then $\nu = \nu'$ on $\mathcal{B}(\cup_n A_n)$. Hence for all sequences of rectangles $R_n \in \mathcal{R}$ we have $\nu = \nu'$ on $\mathcal{B}(\cup_n R_n)$.

Let now $B = \mathbb{R}^{\mathbb{N}} \setminus C$, where $C \in \mathcal{B}(\cup_{n\geqslant 1} R_n)$. By the standard diagonal process one may choose a cube $R = \prod_n [a_n, a_n + 2] \subset \mathbb{R}^{\mathbb{N}} \setminus \cup_{n\geqslant 1} R_n$. Evidently, $\mathrm{vol}\,(R) = \infty$, so $\nu(R) = \nu'(R) = \infty$.

Since $\mathcal{F}$ is an invariant sub-σ-algebra of $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$, $\lambda_P^0 \in \mathcal{M}$. ∎

Remark 3.1. It is obvious that the measure λ_P^0 is non-σ-finite on $\mathcal{F}$.

Lemma 3.3 ([17, Th. 15.3.2, p. 209]). *There exists an inner regular semi-finite translation invariant Borel measure λ_B^0 in $\mathbb{R}^{\mathbb{N}}$ for which $\lambda_B^0(Q) = 1$ and there exists a rectangle $R \in \mathcal{R}$ such that $\lambda_B^0(R) = 0$.*

4. Existence and uniqueness of invariant measures

a) Invariant σ-finite measures on Polish linear spaces

In this subsection we show that on every infinite-dimensional Polish linear space X there exists a σ-finite measure which is invariant with respect to a dense linear subspace and that this measure cannot be unique. The next proposition follows from a result of Peck [19, Th. 2.1] (see also [2, Th. 1]) and from Lemma 1.2.

Proposition 4.1. *If X is an infinite-dimensional Polish linear space then there exists a continuous one-to-one linear operator $T : \ell_2 \to X$ such that $T(\ell_2)$ is a dense Borel subset of X and T^{-1} is a Borel map.*

Corollary 4.1. *On every infinite-dimensional Polish linear space X there is a non-zero σ-finite Borel measure which is invariant with respect to some dense linear subspace.*

Proof. By a result of [12] on ℓ_2 there is a σ-finite Borel measure ν which is L-invariant with respect to a dense linear subspace $L \subset \ell_2$. One can transfer ν to X using T; or more exactly, take the measure

$$\mu(A) = \nu\{T^{-1}(T(\ell_2) \cap A)\}, \qquad A \in \mathcal{B}(X).$$

The measure μ is σ-finite and $T(L)$-invariant.

Since Kharazishvili measure ν [12] takes a numerical value 1 on the infinite-dimensional Hilbert cube $R = \prod_k [0, 1/k]$, we deduce that μ is non-zero because

$$\mu\{T(R)\} = \nu\{T^{-1}(T(\ell_2) \cap T(R))\} = \nu\{R\} = 1. \qquad \blacksquare$$

Proposition 4.2. *Let X be an infinite-dimensional Polish linear space, $K \subset X$ be a compact set and μ be a σ-finite Borel measure on X. Then μ lies on a meager linear subspace M such that $K \subset M \subset X$.*

Proof. By Remark 1.1, one can indicate a sequence of compact subsets K_n in X such that μ lies on $\cup_{n=1}^{\infty} K_n$. One can also suppose $K_1 = K$ and the sequence (K_n) to be increasing. For each n, put

$$K'_n = \left\{ \sum_{i=1}^{n} a_i x_i : \sum_{1}^{n} |a_i| \leqslant n \ , \ \ x_i \in K'_{n-1} \cup K_n \right\}.$$

Put $M = \cup_n K'_n$. Then μ lies on M. Since each K'_n is compact, M is a meager linear subspace in X. $\blacksquare$

Corollary 4.2. *An infinite-dimensional Polish linear space admits no σ-finite quasi-invariant Borel measure μ.*

Proof. Let M be the subspace from Proposition 4.3. A Polish linear space cannot be meager, so there is $x_0 \in X \setminus M$. Since M is a linear subspace, $M \cap (M + x_0) = \emptyset$, hence $\mu(M + x_0) = 0$. $\blacksquare$

Of course, this corollary is well known [14], [5]. Proposition 4.2 and the above corollary use the methods of [6, Ch. IV, §5.3, Th. 4]. As known, the Gelfand-Vilenkin theorem is formally contained in the Feldman result [5]. However, the Gelfand-Vilenkin theorem provides also new information: the measure lies on a meager linear subspace. But the proof uses an additional assumption that in the space X the closed convex hull of a compact set is compact. This assumption is valid in a Banach space (the Mazur theorem) but fails in ℓ_p , $p < 1$. The Corollary 4.2 shows that the condition of compactness of a convex hull of a compact set is not necessary.

Corollary 4.3. *Let X be an infinite-dimensional Polish linear space, $K \subset X$ be compact, $L \subset X$ be a linear subspace and μ be an L-invariant σ-finite Borel measure on X with $\mu(K) = 1$. Then there exists an L-invariant σ-finite Borel measure μ' on X with $\mu'(K) = 1$ and such that μ and μ' are not equivalent.*

Proof. Let M and x_0 be from Corollary 4.2. Since M is a linear subspace, $M \cap (M + x_0) = M \cap (M - x_0) = \emptyset$, hence $\mu(M + x_0) = \mu(M - x_0) = 0$. We define a new measure μ_{x_0} by $\mu_{x_0}(A) = \mu(A + x_0)$ for $A \in \mathcal{B}(X)$. Then for all $A \in \mathcal{B}(X)$ and $x \in L$

$$\mu_{x_0}(A + x) = \mu(A + x + x_0) = \mu(A + x_0) = \mu_{x_0}(A),$$

and

$$\mu_{x_0}(K) \leqslant \mu_{x_0}(M) = \mu(M + x_0) = 0.$$

Hence $\mu' := \mu + \mu_{x_0}$ is a σ-finite L-invariant Borel measures in X, and $\mu'(K) = \mu(K)$. Since $\mu(M - x_0) = 0$ and

$$\mu'(M - x_0) = \mu_{x_0}(M - x_0) = \mu(M) \geqslant \mu(K) = 1,$$

these measures are non-equivalent. ■

This corollary implies that the set of all L-invariant σ-finite Borel measures on X with $\mu(K) = 1$ fails to have the uniqueness property.

b) σ-finite measures, invariant with respect to M-bases

Let (x_k, x_k^*) be an absolutely convergent M-basis of a separable Banach space X. The standard rectangle denoted by P is defined by

$$P = \{x \in X : |x_k^*(x)| \leqslant ||x_k|| \text{ for all } k \in \mathbb{N}\}.$$

In the subsections below we fix an absolutely convergent M-basis (x_k, x_k^*) of a Banach space X, the standard rectangle P, the subspace L_1 and the operator T; see (2.3), (2.4).

We look for L_1-invariant σ-finite Borel measures μ in X which are normalized so that $\mu(P) = 1$. The next theorem generalizes the construction of [8], [7].

Theorem 4.1. *Let X be a Banach space with an absolutely convergent M-basis (x_k, x_k^*). Then the set function*

$$\mu_Q(A) := \nu_Q\{T(A)\}, \qquad A \in \mathcal{B}(X), \tag{4.1}$$

is a σ-finite Borel measure in X for which $\mu_Q(P) = 1$ and $\mathcal{I}_{\mu_Q} = L_1$.

Proof. As mentioned, $T(P) = Q$. By virtue of the paragraph after Lemma 2.1,

$$\mu_Q(P) = \nu_Q\{T(P)\} = \nu_Q(Q) = 1.$$

Let us show that $\mathcal{I}_{\mu_Q} = L_1$. If $x \in L_1$ and $A \in \mathcal{B}(X)$ then

$$\begin{aligned}\mu_Q(A + x) &= \nu_Q\{T(A + x)\} = \nu_Q\{T(A) + Tx\}\\ &= (\text{since } Tx \in \ell_1 = \mathcal{I}_{\nu_Q}) = \nu_Q\{T(A)\} = \mu_Q(A).\end{aligned}$$

If $x \notin L_1$ then $Tx \notin \ell_1$. As mentioned, $\mu_Q\{T^{-1}(Q)\} = \nu_Q(Q) = 1$. On the other hand, by Proposition 2.1,

$$\mu_Q\{T^{-1}(Q) + x)\} = \nu_Q(Q + Tx) = 0.$$

Thus $\mathcal{I}_{\mu_Q} = L_1$. ∎

Remark 4.1. If an M-basis (x_k, x_k^*) is absolutely convergent then $\|x_k^*\| \to \infty$ as $k \to \infty$ and $x_k^*(x) \to 0$ cannot be satisfied for all $x \in X$. Now let us show that the requirement that M-basis (x_k, x_k^*) be absolutely convergent in Theorem 4.1 is essential. Indeed, if an M-basis (x_k, x_k^*) is chosen such that $x_k^*(x) \to 0$ for every element $x \in X$, then a direct application of such an M-basis for the construction of μ_Q by the scheme presented in Theorem 4.1 implies that a set function μ_Q is trivial. Indeed, we have $S = \cup_{n=1}^{\infty} S_n$, where

$$S_n = \{(x_k^*(x))_{k=1}^{\infty} : x \in X \text{ and } |x_k^*(x)| < 1/4 \ \text{ for } k \geqslant n\}.$$

Since $\nu_Q(S_n) = 0$ for every n, we get $\mu_Q(S) = 0$.

The following corollary is a direct consequence of the paragraph after Corollary 4.3 if we recall that for an absolutely convergent M-basis the corresponding rectangle P is compact.

Corollary 4.4. *Let X be a Banach space with an absolutely convergent M-basis (x_k, x_k^*) and let μ_Q be defined by (4.1). The measure μ_Q fails to have the uniqueness property in the class of all L_1-invariant σ-finite Borel measures in X taking the value 1 on P.*

Theorem 4.2. *Let X be a Banach space with an absolutely convergent M-basis (x_k, x_k^*) and let μ_Q be defined by (4.1). Then the completion $\overline{\mu}_Q$ of the measure μ_Q has the uniqueness property in the class of all L_1-invariant σ-finite measures in X with domain $\operatorname{dom}\overline{\mu}_Q$.*

Proof. Assume the contrary. Then there will be an L_1-invariant σ-finite measure μ in X with domain $\operatorname{dom}\overline{\mu}_Q$ such that $\mu \neq c\overline{\mu}_Q$ for any $c > 0$. Put

$$\nu = \mu\{T^{-1}(B)\}, \qquad B \in \operatorname{dom}\overline{\nu}_Q.$$

It is obvious that ν is an ℓ_1-invariant σ-finite measure on $\operatorname{dom}\overline{\nu}_Q$ such that $\nu \neq c\nu_Q$ for any $c > 0$. This contradicts Lemma 2.2 and therefore Theorem 4.2 is proved. ∎

c) Non-σ-finite measures.

The next theorem is a version of [14, Th. 3] and gives a solution of Problem 1.2.

Theorem 4.3. *Let X be a Banach space with an absolutely convergent M-basis (x_k, x_k^*). Then there exists an inner regular semi-finite invariant Borel measure μ_P in X with $\mu_P(P) = 1$.*

Proof. Put

$$\mu_P(A) = \lambda_P\{T(A)\}, \qquad A \in \mathcal{B}(X),$$

where the measure λ_P is defined by Theorem 3.1 and the operator T by (2.4). Obviously,

$$\mu_P(P) = \lambda_P\{T(P)\} = \lambda_P(Q) = 1.$$

By the invariance of λ_P, for all $x \in X$ and $A \in \mathcal{B}(X)$ we have

$$\mu_P(A + x) = \lambda_P\{T(A + x)\} = \lambda_P\{T(A)\} = \mu_P(A).$$

Since λ_P is inner regular for a set $A \in \mathcal{B}(X)$ and $\epsilon > 0$ there is a compact set $F_\epsilon \subseteq T(A)$ in $\mathbb{R}^{\mathbb{N}}$ such that $\lambda_P\{T(A) \setminus F_\epsilon\} < \epsilon$. Since T is linear, injective (because the M-basis is total) and continuous (by Lemma 1.1), we claim that $T^{-1}(F_\epsilon)$ is compact in $\mathcal{B}(X)$. Finally we get

$$\mu_P(A \setminus T^{-1}(F_\epsilon) = \lambda_P\{T(A \setminus T^{-1}(F_\epsilon)) = \lambda_P\{T(A) \setminus F_\epsilon\} < \epsilon,$$

which means that μ_P is inner regular. Now it is obvious to see that the measure μ_P is semi-finite. ∎

Note that since the properties of σ-finiteness and invariance for the measure μ_P are not compatible, μ_P is non-σ-finite.

Remark 4.2. Let μ and μ' be two invariant Borel measures in the Euclidean space $\mathbb{R}^n$ with the following properties:

1. the measure μ gets the value 1 on the unit ball and
2. the measure μ' gets the value 1 on some non-degenerate rectangle.

Then one can easily prove the existence of a $c > 0$ for which $\mu' = c\mu$.

Note that an analogous result is not valid in an infinite-dimensional separable Banach space X. Indeed, on the one hand, there exists no invariant Borel measure in X which gets the value 1 on the unit ball; see e.g. [9, p. 218]. On the other hand, Theorem 4.3 implies that there are invariant Borel measures in X which take the value 1 on some non-degenerated rectangle. For this reason, such measures can be adopted as direct analogs of the Lebesgue measure in X.

The next theorem gives a solution of Problem 1.1.

Theorem 4.4. *Let X be a Banach space with an absolutely convergent M-basis (x_k, x_k^*). Then the class of invariant inner regular semi-finite non-σ-finite Borel measures in X which take the value 1 on the set P fails to have the uniqueness property.*

Proof. Let λ_B^0 be defined by Lemma 3.3. Put

$$\mu_B^0(A) = \lambda_B^0\{T(A)\}, \qquad A \in \mathcal{B}(X).$$

Let μ_P be the measure from Theorem 4.3. Then the equality $\mu_B^0 = c\mu_P$ fails to hold for all $c > 0$ because

$$\mu_B^0(P) = \mu_P(P) = 1,$$

but for the rectangle R from Lemma 3.3

$$\mu_B^0\{T^{-1}(R)\} = 0 \qquad \text{and} \qquad \mu_P\{T^{-1}(R)\} = \mathrm{vol}\,(R) > 0. \qquad \blacksquare$$

Remark 4.3. By Theorem 4.4, there exists no Borel measure in a Banach space X with an absolutely convergent M-basis $(x_k, x_k^*)_{k=1}^{\infty}$, which has the uniqueness property in the class of all invariant semi-finite non-σ-finite Borel measures in X taking the value 1 on the set P. It is natural to ask whether there exist two orthogonal invariant semi-finite Borel measures in X. Note that the answer to this question is no. Indeed, if we assume that such measures μ and μ' exist, then there will be a Borel set A in X such that $\mu(A) = 0$ and $\mu'(X \setminus A) = 0$. Since every null set w.r.t. invariant semi-finite Borel measures in X is Haar null (=shy; see [18, Corollary 2.1, p.241]), this implies that A and $X \setminus A$ are Haar null. By [3], the union X is Haar null, which of course is impossible.

Corollary 4.5 (of Theorem 3.2). *Let X be a Banach space with an absolutely convergent M-basis (x_k, x_k^*). Put $\Sigma = \{T^{-1}(A) : A \in \mathcal{F}\}$, where $\mathcal{F}$ is defined after Remark 3.1, and for every $A \in \Sigma$ put*

$$\mu_P^0(A) = \lambda_P^0\{T(A)\},$$

where λ_P^0 is defined by Theorem 3.2.

Let $\mathcal{M}_X$ be the class of all invariant measures on Σ which get the value $\mathrm{vol}\,(R)$ *on the set $T^{-1}(R)$ for which $R \cap T(X) \in \mathcal{R}$. Then the measure $\mu_P^0 \in \mathcal{M}_X$ possesses the strict uniqueness property in the class $\mathcal{M}_X$.*

Finally, we state the following problems

Problem 4.1. Let X be an infinite-dimensional separable Banach space and μ be an invariant semi-finite non-σ-finite Borel measure in X. Does $\overline{\mu}$ have the uniqueness property in the class of all invariant semi-finite non-σ-finite measures in X defined on $\operatorname{dom}(\overline{\mu})$, where $\overline{\mu}$ denotes the usual completion of μ?

Problem 4.2. Let X be a Polish linear space and K be a closed convex symmetric compact subset of X whose linear span L is dense in X. Does there exist an L-invariant σ-finite Borel measure μ on X with $\mu(K) = 1$?

Problem 4.3. Let X be a Polish linear space and A be a Haar null set. Does there exist an invariant quasi-finite Borel measure μ on X with $\mu(A) = 0$?

Remark 4.4. Let us add some additional information to the historical review in [7]. By using an additional set-theoretical axiom asserting that each subset of $\mathbb{R}$ is Lebesgue measurable, an example of an invariant measure has been constructed in [17, Th. 7.3] on the powerset of a Banach space with absolutely convergent Schauder basis such that the constructed measure takes the value 1 on the standard rectangle. A version of the Lebesgue measure on every separable Banach space that has a Schauder basis was originally constructed in [8, Th. 12]. Later, it was shown that the completion $\overline{\mu}_Q$ of the Yamasaki-Kharazishvili measure μ_Q has the uniqueness property in the class of all L_1-invariant σ-finite measures in X with domain $\operatorname{dom}(\overline{\mu}_Q)$; see the comment in [7, p. 121]. The latter result is extended by Theorem 4.2 to all infinite-dimensional separable Banach spaces.

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