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THE BOHNENBLUST–HILLE CYCLE OF IDEAS FROM A MODERN POINT OF VIEW

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: In 1931 H.F.Bohnenblust and E.Hille published a very important paper in which not only did they solve a long standing problem on convergence of Dirichlet series, but also gave a general version of a celebrated inequality of Littlewood. Although it is full of extremely valuable mathematical ideas, the paper has been overlooked for a long time and even today we feel that it does not get the credit it deserves. This may be caused by the not always accessible style that makes that the ideas are sometimes hidden. It is our intention to try to study the paper from a modern point of view and to bring to light the valuable aspects we believe it has.

Keywords: Dirichlet series, Bohnenblust–Hille inequality, polynomials, Bohr's problem.

1. The original paper

1.1. Introduction

In the beginning of the 20th century Harald Bohr devoted a big amount of efforts [11, 12, 13, 14] to study Dirichlet series, in the frame of the Riemmann's ζ function and (maybe) the Riemmann's hypothesis. His main goal was to study the different abscissas of convergence of a given Dirichlet series. For a Dirichlet series $\sum_n a_n/n^s$ he defined three numbers

$$\sigma_c = \inf\{r\colon \sum_n a_n/n^s \text{ converges in } [\operatorname{Re} s > r]\}$$

$$\sigma_u = \inf\{r\colon \sum_n a_n/n^s \text{ converges uniformly in } [\operatorname{Re} s > r+\varepsilon] \text{ for every } \varepsilon > 0\}$$

$$\sigma_a = \inf\{r\colon \sum_n a_n/n^s \text{ converges absolutely in } [\operatorname{Re} s > r]\}$$

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Clearly $\sigma_c \leqslant \sigma_u \leqslant \sigma_a$ for every Dirichlet series. Each one of these numbers determines a half-plane such that the Dirichlet series converges in the corresponding sense in $[\operatorname{Re} s \geqslant \sigma + \varepsilon]$ for every $\varepsilon > 0$. His aim was to be able to control σ_a for each Dirichlet series. He tried to do that by relating σ_a with the other two abscissas; that is by determining the maximal possible differences between these numbers; or in other words the width of the maximal strips on which a Dirichlet series can converge in one sense and not in the other.

He first focused in the difference between convergence and absolutely convergence. He showed that the strip on which a Dirichlet series can converge but not converge absolutely has width at most 1. Then he looked at the difference between uniform and absolute convergence. He defined

$$T = \sup\{\sigma_a - \sigma_u : \text{ Dirichlet series } \} \tag{1}$$

and he wanted to determine precisely the value of T. In the region where a Dirichlet series converges it defines a holomorphic function and Bohr's aim was to describe T (and then σ_a) by means of the analytic properties of this function. He proved what is nowadays sometimes called *Bohr's fundamental theorem* [13, Satz 1]:

$$\sigma_u = \inf\{r : f(s) \text{ is holomorphic and bounded on } [\operatorname{Re} s > r]\}. \tag{2}$$

His second main contribution to the subject was to realise that there is a close relation between Dirichlet series and power series in infinitely many variables. We denote by $\mathfrak{P}$ the set of formal power series and $\mathfrak{D}$ the set of Dirichlet series. Given $n \in \mathbb{N}$ we consider its decomposition into prime numbers $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = p^\alpha$ (where $(p_n)_n$ stands for the sequence of prime numbers). In this way each n has an associated α and vice-versa and we can define Bohr's transform ($\mathbb{N}_0^{(\mathbb{N})}$ is the set of multi-indices that eventually become 0):

$$\begin{gathered} \mathfrak{B} : \mathfrak{P} \longrightarrow \mathfrak{D} \\ \textstyle\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \rightsquigarrow \displaystyle\sum_{n=1}^\infty a_n n^{-s} \\ c_\alpha = a_{p^\alpha} \end{gathered}$$

This is an algebra homomorphism. He proved the following two theorems [12, Satz VII and Satz VIII]

Theorem. *Let σ_u be the abscissa of uniform convergence of the a Dirichlet series, then the associated power series is bounded in every domain $|x_n| \leqslant p_n^{-(\sigma_u + \delta)}$, i.e.*

$$\sup_{n \in \mathbb{N}} \sup_{z \in B_{\ell_\infty^n}} \Big| \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (p^{(\sigma_u + \delta)} z)^\alpha \Big| < \infty ,$$

where δ is any arbitrarily small positive number.

and the converse

Theorem. *If the power series is bounded in the domain $|x_n| \leqslant p_n^{-\sigma_0}$, then $\sigma_u \leqslant \sigma_0$.*

Let us briefly look at these results in modern terms. If $f : B_{c_0} \to \mathbb{C}$ is a holomorphic[1] function, we can consider its restriction f_n to the first n variables; this is a holomorphic function on $\mathbb{C}^n$ and has a Taylor expansion whose coefficients $c_\alpha^{(n)}(f)$ can be calculated with the Cauchy formula. It is not difficult to show that for every fixed α one has $c_\alpha^{(n)}(f) = c_\alpha^{(n+1)}(f)$ for all n. Hence every $f \in H_\infty(B_{c_0})$ (the space of bounded, holomorphic functions on B_{c_0}) defines a family of coefficients $(c_\alpha(f))_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ and $H_\infty(B_{c_0})$ can be seen as a subspace of $\mathfrak{P}$. On the other hand we consider the space of all Dirichlet series that define a bounded, holomorphic function on $\{s \in \mathbb{C}\colon \operatorname{Re} s > 0\}$ and denote it by $\mathcal{H}_\infty$. Then the ideas of Bohr in the proofs of the previous theorems can be used to prove that

$$\mathfrak{B} : H_\infty(B_{c_0}) \longrightarrow \mathcal{H}_\infty$$

is an isometric isomorphism. This was realised by Hedenmalm, Lindqvist and Seip in the deep paper [51].

By a translation argument this together with (2) imply that the abscissa of uniform convergence of every Dirichlet series can be described as [29, Lemma 4.5]

$$\sigma_u = \inf \left\{ \mu \in \mathbb{R}\colon \text{ there exists } f \in H_\infty(B_{c_0})\ ,\ c_\alpha(f) = \frac{a_{p^\alpha}}{p^{\alpha\mu}} \right\}, \tag{3}$$

In order to study T and taking into account this relation Bohr defined another number S as *'the least upper bound of all positive numbers α, such that every power series bounded in $|x_n| \leqslant G_n$ is absolutely convergent in $|x_n| \leqslant G_n\varepsilon_n$ whenever $0 < \varepsilon_n < 1$ and $\sum_n \varepsilon_n^\alpha$ converges'* [12, page 445]. This in modern terms is defined as

$$S = \inf\{s\colon \ell_s \cap B_{c_0} \subseteq \operatorname{mon} H_\infty(B_{c_0})\}\,, \tag{4}$$

where $\operatorname{mon} H_\infty(B_{c_0}) = \{x \in B_{c_0}\colon \sum_\alpha |c_\alpha(f)x^\alpha| < \infty \text{ for all } f \in H_\infty(B_{c_0})\}$ is the set of monomial convergence that is the set of points on which every bounded, holomorphic function has a power series expansion. Then Bohr proved [12, Satz IX]

Theorem.

$$T = \frac{1}{S} \tag{5}$$

Bohr showed that $S \geqslant 2$ [12, Satz III] (hence $T \leqslant 1/2$) but was not able to give an upper bound for S. Toeplitz gave an example [80] that implied $S \leqslant 4$. Then what was know on this particular subject at 1915 was that $1/4 \leqslant T \leqslant 1/2$.

The situation remained still for more than 15 years and no further advance in this direction was achieved. In 1930 Littlewood published a celebrated paper [60] in which he considered a problem proposed to him by Daniell that can reformulated in the following way (the first two citations in frames are reprinted from [60], all others come from [10]):

[1] A function f defined on an open subset of a complex Banach space is said holomorphic if it is complex Fréchet differentiable. The space $H_\infty(B_{c_0})$ is a Banach space with the norm $\|f\| = \sup_{x \in B_{c_0}} |f(x)|$.

The problem is equivalent to the following: to find a bilinear form

$$Q(x,y) = \sum\sum a_{mn} x_m y_n \tag{1}$$

in an infinite number of variables x, y, such that $\sum\sum |a_{mn}|$ is divergent, but Q is bounded for all x, y belonging to the range S defined by

$$|x_m| \leqslant 1 \ (m = 1, 2, \ldots), \qquad |y_n| \leqslant 1 \ (n = 1, 2, \ldots).$$

and proved the following

Let

$$b_n = \Big(\sum_{m=1}^{\infty} |a_{mn}|^2 \Big)^{\frac{1}{2}}, \qquad c_m = \Big(\sum_{n=1}^{\infty} |a_{mn}|^2 \Big)^{\frac{1}{2}},$$

$$\sum_{n=1}^{\infty} b_n = B, \qquad \sum_{m=1}^{\infty} c_m = C, \qquad (\sum\sum |a_{mn}|^{\frac{4}{3}})^{\frac{3}{4}} = D.$$

(B, C, D may of course be $+\infty$). Then I prove

THEOREM 1. (1) *In order that $|Q| \leqslant H$ in S it is necessary that B, C, D should be less than AH, where A is an absolute constant.*

(2) *Given any positive p_m, q_n, r_{mn} for which*

$$\lim_{m\to\infty} p_m = \lim_{n\to\infty} q_n = \lim_{m,n\to\infty} r_{mn} = \infty,$$

there exist bounded Q for which

$$\sum p_n b_n, \qquad \sum q_m c_m, \qquad \sum\sum r_{mn} |a_{mn}|^{\frac{4}{3}}$$

are divergent.

The fact that $D \leqslant AH$ can be in modern language stated as (note that by taking the infimum over all such H one gets the norm of Q)

Theorem. *There exist a constant $C > 0$ such that for every continuous and bilinear form $Q : c_0 \times c_0 \to \mathbb{C}$ with $Q(x,y) = \sum_{m,n} a_{mn} x_m y_n$ the following holds*

$$\Big(\sum_{m,n} |a_{mn}|^{4/3} \Big)^{3/4} \leqslant C \|Q\| \,, \tag{6}$$

equivalently $\big(\sum_{m,n} |Q(e_m, e_n)|^{4/3} \big)^{3/4} \leqslant C\|Q\|$ or $(a_{mn})_{m,n} \in \ell_{4/3}$. Moreover the exponent $4/3$ is optimal.

This is nowadays sometimes called *Littlewood's 4/3-inequality.* The main tool of the proof is

Khintchine Inequality. *For each* $1 \leqslant p < \infty$ *there are constants* $A_p > 0$ *and* $B_p > 0$ *so that for every finite choice* $\alpha_1, \dots, \alpha_n \in \mathbb{C}$

$$A_p \Big(\sum_{i=1}^n |\alpha_i|^2 \Big)^{1/2} \leqslant \Big(\int \big| \sum_{i=1}^n \alpha_i \varepsilon_i(\omega) \big|^p d\omega \Big)^{1/p} \leqslant B_p \Big(\sum_{i=1}^n |\alpha_i|^2 \Big)^{1/2}. \tag{7}$$

where $\varepsilon_1, \dots, \varepsilon_n$ *are independent Rademacher random variables (i.e. taking values* ± 1 *with probability* $1/2$*).*

The proof of the Khintchine inequality is far from being straightforward. Different approaches can be found in [59, Theorem 2.b.3], [41, Theorem 1.10] or [24, Section 8.5]. The best constants A_p and B_p in (7) were given by Haagerup [48] (see also [24, 41]). In the following we are going to need only the case $p = 1$, then the best constants are $A_1^{-1} = \sqrt{2}$.

The original proof of Littlewood is very hard to read, in part because in it he proves 'by hand' the Khintchine inequality (obtaining also the constant $\sqrt{2}$). R. Blei in [7] presents a simplified proof, that we reproduce here.

Proof. The proof begins by applying twice Hölder's inequality.

$$\begin{aligned}
\sum_{m,n} |a_{mn}|^{4/3} &= \sum_m \big(\sum_n |a_{mn}|^{2/3} |a_{mn}|^{2/3} \big) \qquad \text{Hölder, } p=3,\ q=3/2 \\
&\leqslant \sum_m \big(\sum_n |a_{mn}|^{\frac{2}{3}3} \big)^{1/3} \big(\sum_n |a_{mn}|^{\frac{2}{3}\frac{3}{2}} \big)^{2/3} \\
&= \sum_m \big(\sum_n |a_{mn}|^2 \big)^{1/3} \big(\sum_n |a_{mn}| \big)^{2/3} \qquad \text{Hölder, } p=3/2,\ q=3 \\
&\leqslant \Big(\sum_m \big(\sum_n |a_{mn}|^2 \big)^{\frac{1}{3}\frac{3}{2}} \Big)^{2/3} \Big(\sum_m \big(\sum_n |a_{mn}| \big)^{\frac{2}{3}3} \Big)^{1/3} \\
&= \Big[\sum_m \big(\sum_n |a_{mn}|^2 \big)^{1/2} \Big]^{2/3} \Big[\Big(\sum_m \big(\sum_n |a_{mn}| \big)^2 \Big)^{1/2} \Big]^{2/3}.
\end{aligned}$$

This is

$$\Big(\sum_{m,n} |a_{mn}|^{4/3} \Big)^{3/4} \leqslant \Big[\sum_m \big(\sum_n |a_{mn}|^2 \big)^{1/2} \Big]^{1/2} \Big[\Big(\sum_m \big(\sum_n |a_{mn}| \big)^2 \Big)^{1/2} \Big]^{1/2}.$$

By the Minkowski inequality

$$\Big(\sum_m \big(\sum_n |a_{mn}| \big)^2 \Big)^{1/2} \leqslant \sum_n \big(\sum_m |a_{mn}|^2 \big)^{1/2}.$$

Then

$$\Big(\sum_{m,n}|a_{mn}|^{4/3}\Big)^{3/4}\leqslant\Big(\sum_{m}\big(\sum_{n}|a_{mn}|^2\big)^{1/2}\Big)^{1/2}\Big(\sum_{n}\big(\sum_{m}|a_{mn}|^2\big)^{1/2}\Big)^{1/2}.$$

We bound now each one of these factors using Khintchine inequality.

$$\begin{aligned}
\sum_i\Big(\sum_j|a_{ij}|^2\Big)^{1/2} &\leqslant \sqrt{2}\sum_i\int\Big|\sum_j a_{ij}\varepsilon_j(\omega)\Big|d\omega\\
&=\sqrt{2}\int\sum_i\Big|\sum_j a_{ij}\varepsilon_j(\omega)\Big|d\omega\\
&\leqslant\sqrt{2}\int\sup_{\mu\in B_{\ell_\infty}}\sum_i\Big|\sum_j a_{ij}\mu_j\Big|d\omega\\
&=\sqrt{2}\sup_{\mu\in B_{\ell_\infty}}\sum_i\Big|\sum_j a_{ij}\mu_j\Big|\underbrace{\int d\omega}_{1}\\
&\leqslant\sqrt{2}\sup_{\mu,\eta\in B_{\ell_\infty}}\Big|\sum_{i,j}a_{ij}\eta_i\mu_j\Big|=\sqrt{2}\|Q\|.
\end{aligned}$$

Finally

$$\Big(\sum_{m,n}|a_{mn}|^{4/3}\Big)^{3/4}\leqslant\big(\sqrt{2}\|Q\|\big)^{1/2}\big(\sqrt{2}\|Q\|\big)^{1/2}=\sqrt{2}\|Q\|.\qquad\blacksquare$$

The optimality is proved by producing an example in some sense similar to that of Toeplitz. We reproduce it now in a slightly modified, up-to-date version. Let us suppose that $r\geqslant 1$ is an exponent satisfying an inequality like (6), i.e. there is a constant $C>0$ so that for every $Q\in\mathcal{L}(^2c_0)$[2] with matrix $(a_{ij})_{i,j}$ the following holds.

$$\Big(\sum_{i,j}|a_{ij}|^r\Big)^{1/r}\leqslant C\|Q\|. \tag{8}$$

Now, for each fixed n let $(a_{rs})_{r,s}$ be an $n\times n$ matrix such that $\sum_{k=1}^n a_{rk}\overline{a}_{sk}=\delta_{rs}\cdot n$ and $|a_{rs}|=1$ for every r and s (such a matrix can be obtained by doing $a_{rs}=e^{2\pi i(rs)/n}$, these are the so called Fourier matrices, see Section 1.3 for more detalis) and define

$$Q(x,y)=\sum_{r,s=1}^n a_{rs}x_ry_s.$$

[2] $\mathcal{L}(^mX)$ denotes the space of continuous, m-linear mappings on X with values in $\mathbb{C}$

Clearly $Q \in \mathcal{L}(^2\ell_\infty^n)$. Then it satisfies (8); on the left–hand side of the inequality we have $(\sum_{i,j=1}^n |a_{ij}|^r)^{1/r} = (n^2)^{1/r}$. We compute now the norm; to do so let us take $x, y \in B_{\ell_\infty^n}$ and by applying Cauchy-Schwarz inequality we have

$$\begin{aligned}
|Q(x,y)| &= \Big| \sum_{r,s=1}^n a_{rs} x_r y_s \Big| = \Big| \sum_s \Big(\sum_r a_{rs} x_r \Big) y_s \Big| \qquad \text{Cauchy-Schwarz} \\
&\leqslant \Big(\sum_s \Big| \sum_r a_{rs} x_r \Big|^2 \Big)^{1/2} \Big(\sum_s |y_s|^2 \Big)^{1/2} \\
&\leqslant n^{1/2} \Big(\sum_s \Big| \sum_r a_{rs} x_r \Big|^2 \Big)^{1/2} \\
&= n^{1/2} \left(\sum_s \Big(\sum_{r_1} a_{r_1 s} x_{r_1} \Big) \Big(\overline{\sum_{r_2} a_{r_2 s} x_{r_2}} \Big) \right)^{1/2} \\
&= n^{1/2} \left(\sum_s \sum_{r_1, r_2} a_{r_1 s} \overline{a_{r_2 s}} x_{r_1} x_{r_2} \right)^{1/2} \\
&= n^{1/2} \left(\sum_{r_1, r_2} x_{r_1} \overline{x_{r_2}} \underbrace{\Big(\sum_s a_{r_1 s} \overline{a_{r_2 s}} \Big)}_{n \delta_{r_1 r_2}} \right)^{1/2} \\
&= n^{1/2} n^{1/2} \Big(\sum_r x_r \overline{x_r} \Big)^{1/2} = n^{1/2} n^{1/2} \Big(\sum_r |x_r|^2 \Big)^{1/2} = n^{3/2}.
\end{aligned}$$

Then $n^{2/r} \leqslant C n^{3/2}$ holds for every n. This implies $2/r \leqslant 3/2$ and, from this $r \geqslant 4/3$. This shows the optimality.

With help of the Polarization Formula it is easily shown that an inequality like (6) holds for 2-homogeneous polynomials in c_0 (we will come back to this issue in more detail in Section 1.4). Then, if $z \in \ell_4$ and $P \in \mathcal{P}(^2 c_0)$ we have by Hölder inequality with $p = 4/3$ and $q = 4$

$$\begin{aligned}
\sum_{|\alpha|=2} |c_\alpha| |z|^\alpha = \sum_{|\alpha|=2} |c_\alpha| |z^\alpha| &\leqslant \Big(\sum_{|\alpha|=2} |c_\alpha|^{4/3} \Big)^{3/4} \Big(\sum_{|\alpha|=2} |z^\alpha|^4 \Big)^{1/4} \\
&\leqslant \Big(\sum_{|\alpha|=2} |c_\alpha|^{4/3} \Big)^{3/4} \Big(\sum_{|\alpha|=2} |z^4|^\alpha \Big)^{1/4}.
\end{aligned} \tag{9}$$

The first term is finite because of (6) and the second term is finite because $\sum_\alpha |w|^\alpha < \infty$ if and only if $w \in \ell_1$ (and this is the case since $z \in \ell_4$). This means $4 \geqslant S$.

ON THE ABSOLUTE CONVERGENCE OF DIRICHLET SERIES.[1]

By H. F. Bohnenblust and Einar Hille.

The problem of the absolute convergence for Dirichlet series $\sum a_n e^{-\lambda_n s}$ deals with the relative position of the abscissa σ_a of absolute convergence and the abscissae σ_u of uniform convergence and σ_b of boundedness and regularity. For ordinary Dirichlet series $\sum a_n n^{-s}$ we have $\sigma_u = \sigma_b$. In order to find an upper bound for the difference $\sigma_a - \sigma_u$, H. Bohr established in a paper published in 1913[2] a connection between the behavior of ordinary Dirichlet series and of power series in an infinite number of variables. Writing n as a product of prime numbers, the Dirichlet series

(1) $$\sum_{n=1}^{\infty} a_n n^{-s}$$

can be written as a power series

(2) $$c + \sum_{i_1=1}^{\infty} c_{i_1} x_{i_1} + \sum_{i_1 \leq i_2 = 1}^{\infty} c_{i_1 i_2} x_{i_1} x_{i_2} + \cdots$$

in the variables $x_n = p_n^{-s}$, where p_n is the n-th prime number. The coefficient $c_{i_1 i_2 \cdots i_m}$ of (2) is equal to the coefficient a_n of (1), whose index n is equal to $p_{i_1} p_{i_2} \cdots p_{i_m}$. Bohr showed that, though actually functions of a single variable s, the variables $x_n = p_n^{-s}$ behave in many ways as if they were independent of one another. This is due to the linear independence of the quantities $\log p_n$.

The power series (2) will be said to be *bounded* in the domain $(G) : |x_n| \leqq G_n$; where the G_n are non negative numbers, if

1° for every integer m, the m-truncated power series

(3) $$c + \sum_{i_1=1}^{m} c_{i_1} x_{i_1} + \sum_{i_1 \leq i_2 = 1}^{m} c_{i_1 i_2} x_{i_1} x_{i_2} + \cdots$$

obtained from (2) by putting $x_{m+1} = x_{m+2} = \cdots = 0$ is *absolutely convergent* in the domain (G), and

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[1] Received March 10, 1931.—Presented to the American Mathematical Society, Dec. 31, 1930, (abstract nos. 37-1-91, 95). A short account of the main results of this paper appeared in the Comptes Rendus, t. **192**, pp. 30–32, séance du 5 janvier 1931.

[2] H. Bohr, Über die Bedeutung der Potenzreihen unendlich vieler Variablen in der Theorie der Dirichletschen Reihen $\sum a_n n^{-s}$, Gött. Nachr. (1913), p. 441–488.

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With all this background Henry Frédéric Bohnenblust[3] presented his PhD dissertation, entitled *On the Absolute Convergence of Dirichlet Series*, at Princeton

[3] H.F.Bohnenblust (Boni, as he was known at Caltech) (Switzerland 1906 - USA 2000) started

University supervised by Einar Hille[4]. A part of the Thesis was published in [10], the paper we are just about to go through, where they finally solve the problem of determining the exact value of T.

Their idea, though apparently simple, is extremely clever. They work with numbers S_m defined like S only considering m-homogeneous polynomials[5] instead holomorphic functions; more precisely

$$S_m = \sup\{s\colon \ell_s \subseteq \operatorname{mon} \mathcal{P}(^m c_0)\} \tag{10}$$

Clearly $S \leqslant \inf_m S_m$. Then they show that this infimum is exactly 2. Since it was already known that $2 \leqslant S$ this gives $S = 2$ and hence $T = 1/2$. This has an immediate counterpart in terms of Dirichlet series. For each natural number n we consider its prime number decomposition $n = p^\alpha$ and define $\Omega(n) = |\alpha| = \alpha_1 + \cdots + \alpha_k$ (this is the number of primes in the decomposition, e.g. $\Omega(n) = 1$ if and only if n is prime and $\Omega(10) = \Omega(2 \times 5) = 2$). Then the Dirichlet series can be decomposed

$$\sum_{n=1}^\infty a_n \frac{1}{n^s} = \sum_{\Omega(n)=1} a_n \frac{1}{n^s} + \sum_{\Omega(n)=2} a_n \frac{1}{n^s} + \sum_{\Omega(n)=3} a_n \frac{1}{n^s} + \cdots = \sum_{m=1}^\infty \sum_{\Omega(n)=m} a_n \frac{1}{n^s}.$$

Each one of the terms $\sum_{\Omega(n)=m} a_n/n^s$ is called an m-homogeneous Dirichlet series. This is analog to the decomposition of a holomorphic function as a sum of m-homogeneous polynomials:

$$\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha = \sum_{|\alpha|=1} c_\alpha z^\alpha + \sum_{|\alpha|=2} c_\alpha z^\alpha + \sum_{|\alpha|=m} c_\alpha z^\alpha + \cdots = \sum_{m=1}^\infty \sum_{|\alpha|=m} c_\alpha z^\alpha.$$

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his studies at Princeton University in 1928; after his PhD he became part of the staff of the University, where he stayed until 1945. After one year at Indiana University he joined Caltech in 1946, where he remained until his retirement in 1974. He was head of the Mathematics faculty for 20 years and dean for graduate studies from 1956 to 1970. He was vicepresident of the AMS and co-editor of *Annals of Mathematics*. His mathematical interests, coming from Functional Analysis, went up to Game Theory and, in his last days, Computation. He is always referred as an gratest teacher (he had once his picture in the cover of the *Time* magazine as one of the ten outstanding teachers in USA (May 6th, 1966)) and as a wonderful person. The asteroid 15938, discovered by Paolo G. Comba (one of his students) on the 27th Dec. 1997, has been officially named Bohnenblust after him.

[4]E. Carl Hille (USA 28.06.1894-12.02.1980) lived in Stockholm (hometown of his mother) since he was 2 until the age of 26. He presented his PhD dissertation (*Some problems concerning spherical harmonics*, supervised by M.Riesz) at Stockholm University in 1918 and was awarded the Mittag-Leffler prize in 1919. In 1920 he got a job at Harvard and in 1922 he moved to Princeton University. In 1934 he was appointed full Professor at Yale and he held that position until his retirement in 1962. He was president of the AMS (1947/48) and editor of *Annals of Mathematics* (1929/33) and of *Transactions of AMS* (1937/43). He was member of the National Academy of Sciences (USA) since 1953 and of the Royal Swedish Academy of Sciences. After his death a memorial issue of *Integral Eq and Op. Theor.* (vol 4, no. 2/3) was delivered, where, among others, a personal account of Hille was done in [53]. Also, [44] and [70] contain thorough records of the life and work of E.Hille.

[5]A mapping $P : X \to \mathbb{C}$ defined on a Banach space is an m-homogeneous polynomial if there exists $L \in \mathcal{L}(^m X)$ such that $P(x) = L(x, \dots, x)$ for every x. The space of m-homogeneous polynomials is denoted $\mathcal{P}(^m X)$; it is a Banach space with the norm $\|P\| = \sup_{\|x\|\leqslant 1} |P(x)|$.

Each factor $\sum_{|\alpha|=m} c_\alpha z^\alpha$ is an m-homogeneous polynomial on c_0. Clearly with the Bohr transform to each m-homogeneous polynomial corresponds an m-homogeneous Dirichlet series. Then one can consider numbers

$$T_m = \sup\{\sigma_a - \sigma_u : \ m\text{-homogeneous Dirichlet polynomials}\}.$$

Clearly $T \geqslant \sup_m T_m$ and these tend to $1/2$ as n goes to ∞. Although Bohnenblust and Hille do not explicitly define these numbers S_m and T_m, the idea is around all the time.

So the whole aim was to find out the exact value of S_m; and this means some understanding of sets of monomial convergence of m-homogeneous polynomials. They realised that Littlewood's Theorem had consequences in sets of monomial convergence of polynomials (see (4)); namely that with this notation (9) implies $S_2 \geqslant 4$ (in fact the example of Toeplitz shows $S_2 = 4$). Then, in order to get general estimates for sets of monomial convergence of m-homogeneous polynomials it was necessary to find a multi-linear version of Littlewood's Theorem. This they do in the first section.

After this long introduction it is time now to go to the paper. We try to present the proofs and the ideas in a clear way, with a modern language and notation but as close to the originals as possible. Our aim is always to make as few changes as we can.

1.2. Bounded m-linear forms in an infinite number of variables

1. **Bounded m-linear forms in an infinite number of variables.** Let us consider the m-linear form

$$L(x^{(1)}, x^{(2)}, \cdots, x^{(m)}) \equiv \sum a_{i_1 i_2 \cdots i_m} x^{(1)}_{i_1} x^{(2)}_{i_2} \cdots x^{(m)}_{i_m}, \tag{1.1}$$

where the indices $i_1, i_2, \cdots, i_m$ independently assume all positive integral values. The coefficients $a_{i_1 \cdots i_m}$ and the variables $x^{(\nu)}_{i_\nu}$ are complex. The vector

$$x^{(\nu)} \equiv \{x^{(\nu)}_1, x^{(\nu)}_2, \cdots, x^{(\nu)}_n, \cdots\}$$

is said to belong to the domain (G_0), if for all values of n

$$|x^{(\nu)}_n| \leqq 1.$$

This is how the first section of the paper begins. Their domain (G_0) is what we would now call B_{c_0}, the (closed) unit ball of c_0. Then the m-linear form L defined in (1.1) is a continuous, m-linear form from c_0 to $\mathbb{C}$ (i.e. $L \in \mathcal{L}(^m c_0)$) and $a_{i_1 \dots i_m} = L(e_{i_1}, \dots, e_{i_m})$. Then they define

$$\rho = \frac{2m}{m+1}, \qquad S = \Big(\sum_{i_1,\dots,i_m=1}^{\infty} |a_{i_1,\dots,i_m}|^\rho \Big)^{1/\rho}.$$

At the end of the section the following theorem, an extension of Littlewood's Theorem, can be found.

> **Theorem I.** *In order that an m-linear form (1.1) be bounded by H in the domain (G_0), it is necessary that $T^{(1)}$, $T^{(2)}$, $\cdots$, $T^{(m)}$ and S should all be less than $A \cdot H$, where A is a constant depending only on m.*

The H that appears here can be seen as the supremum on $(G_0) = B_{c_0}$ of the values of L (i.e. $H = \sup_{x_j \in B_{c_0}} |L(x_1, \dots, x_m)|$); this is $\|L\|$, the norm in the Banach space $\mathcal{L}(^m c_0)$. Then this can be stated in modern terms as

Theorem I. *For each m there exists a constant $C_m > 0$ such that for all $L \in \mathcal{L}(^m c_0)$ with $L(x^{(1)}, x^{(2)}, \dots, x^{(m)}) = \sum_{i_1,\dots,i_m} a_{i_1,\dots,i_m} x^{(1)}_{i_1} \cdots x^{(m)}_{i_m}$ the following holds*

$$\Bigg(\sum_{i_1,\dots,i_m=1}^{\infty} |a_{i_1,\dots,i_m}|^{\frac{2m}{m+1}} \Bigg)^{\frac{m+1}{2m}} \leqslant C_m \|L\|.$$

The proof is again (as in Littlewood's case) hard to follow. We try now to give a slightly simplified version. First of all we introduce some notation

$$T^{(k)}_{i_k} = \Bigg(\sum_{\sim i_k} |a_{i_1,\dots,i_m}|^2 \Bigg)^{1/2}$$

where $\sim i_k$ means that the sum is taken fixing the k-th index at the value i_k and summing in all the others; and

$$T^{(k)} = \sum_{i_k=1}^{\infty} T^{(k)}_{i_k}.$$

That is, we have

$$\begin{array}{lll}
T^{(1)}_1 = \Big(\sum |a_{1,i_2,\dots,i_m}|^2 \Big)^{1/2}; & T^{(1)}_2 = \Big(\sum |a_{2,i_2,\dots,i_m}|^2 \Big)^{1/2}; & \cdots \rightsquigarrow T^{(1)} \\
T^{(2)}_1 = \Big(\sum |a_{i_1,1,i_3,\dots,i_m}|^2 \Big)^{1/2}; & T^{(2)}_2 = \Big(\sum |a_{i_1,2,i_3,\dots,i_m}|^2 \Big)^{1/2}; & \cdots \rightsquigarrow T^{(2)} \\
\vdots & \vdots & \\
T^{(m)}_1 = \Big(\sum |a_{i_1,\dots,i_{m-1},1}|^2 \Big)^{1/2}; & T^{(m)}_2 = \Big(\sum |a_{i_1,\dots,i_{m-1},2}|^2 \Big)^{1/2}; & \cdots \rightsquigarrow T^{(m)}
\end{array}$$

Lemma 1.1.

$$S^{\rho} \leqslant \sum_{k=1}^{m} (T^{(k)})^{\rho} \tag{11}$$

that is

$$\sum_{i_1,\dots,i_m=1}^{\infty} |a_{i_1,\dots,i_m}|^{\frac{2m}{m+1}} \leqslant \sum_{k=1}^{m} (T^{(k)})^{\frac{2m}{m+1}} = \sum_{k=1}^{m} \Bigg(\sum_{i_k=1}^{\infty} \Big(\sum_{\sim i_k} |a_{i_1,\dots,i_m}|^2 \Big)^{1/2} \Bigg)^{\frac{2m}{m+1}}.$$

Proof. First of all, if any of the $T^{(k)}$'s is not finite the inequality is trivially satisfied; hence we can assume that all $T_{i_k}^{(k)}$ and $T^{(k)}$ are finite and that $a_{i_1,\dots,i_m}$ are non-negative real numbers.

We can also assume that for each fixed k, $T_{i_k}^{(k)}$ can be reordered so that it is a non-increasing function of i_k; indeed, it is clear that any reordering of the $a_{i_1,\dots,i_m}$ does not alter any of the $T^{(k)}$ or S. Since $T_{i_k}^{(k)} \overset{i_k}{\to} 0$, the reordering can be made in a non-increasing way.

Hence, for each fixed $k = 1, \dots, m$ and every $i_k \in \mathbb{N}$

$$i_k T_{i_k}^{(k)} \leqslant T^{(k)}. \tag{12}$$

Indeed, since $T_{i_1}^{(k)} \geqslant T_{i_2}^{(k)} \geqslant \cdots \geqslant T_{i_k}^{(k)} \geqslant \cdots$

$$T^{(k)} = \sum_{i_k} T_{i_k}^{(k)} \geqslant T_{i_1}^{(k)} + T_{i_2}^{(k)} + \cdots + T_{i_k}^{(k)} \geqslant i_k T_{i_k}^{(k)}.$$

now, for each fixed k and i_k we denote

$$\sum_{(i_k)}^{(k)} \equiv \sum_{\substack{i_1 \leqslant i_k, \dots, i_{k-1} \leqslant i_k \\ i_{k+1} < i_k, \dots, i_m < i_k}} \qquad \text{e.g. } \sum_{(6)}^{(4)} \equiv \sum_{\substack{i_1 \leqslant 6, i_2 \leqslant 6, i_3 \leqslant 6 \\ i_5 < 6, \dots, i_m < 6}}$$

and we apply Hölder with

$$\begin{aligned} p &= \frac{2}{2-\rho} = \frac{2}{2 - \frac{2m}{m+1}} = \frac{2(m+1)}{2m+2-2m} = m+1 \\ q &= \frac{2}{\rho} = \frac{2}{\frac{2m}{m+1}} = \frac{m+1}{m} \\ \frac{1}{p} + \frac{1}{q} &= \frac{2-\rho}{2} + \frac{\rho}{2} = 1 \end{aligned}$$

then

$$\begin{aligned} \sum_{(i_k)}^{(k)} a_{i_1,\dots,i_m}^{\rho} &\leqslant \Big(\sum_{(i_k)}^{(k)} 1^{\frac{2}{2-\rho}} \Big)^{\frac{2-\rho}{2}} \Big(\sum_{(i_k)}^{(k)} a_{i_1,\dots,i_m}^{\rho \frac{2}{\rho}} \Big)^{\frac{\rho}{2}} \\ &= \Big(\sum_{(i_k)}^{(k)} 1 \Big)^{\frac{1}{m+1}} \Big(\sum_{(i_k)}^{(k)} a_{i_1,\dots,i_m}^{2} \Big)^{\frac{\rho}{2}} \quad \begin{pmatrix}\text{there are } m-1 \text{ sums and in each one} \\ \text{we sum up to at most } i_k \end{pmatrix} \\ &\leqslant \Big[\Big(\sum_{j=1}^{i_k} 1 \Big)^{m-1} \Big]^{\frac{1}{m+1}} \Big(\sum_{(i_k)}^{(k)} a_{i_1,\dots,i_m}^{2} \Big)^{\frac{\rho}{2}} = i_k^{\frac{m-1}{m+1}} \Big(\sum_{(i_k)}^{(k)} a_{i_1,\dots,i_m}^{2} \Big)^{\frac{\rho}{2}} \\ &\leqslant i_k^{\frac{m-1}{m+1}} \Big(\sum_{\substack{i_1,\dots,i_m=1 \\ \sim i_k}}^{\infty} a_{i_1,\dots,i_m}^{2} \Big)^{\frac{\rho}{2}} = i_k^{\frac{m-1}{m+1}} T_{i_k}^{(k)\rho}. \end{aligned}$$

Now, using (12)

$$\sum_{(i_k)}^{(k)} a^{\rho}_{i_1,\dots,i_m} \leqslant \big(i_k T^{(k)}_{i_k}\big)^{\frac{m-1}{m+1}} T^{(k)}_{i_k} \leqslant T^{(k)\,\frac{m-1}{m+1}}\, T^{(k)}_{i_k}.$$

And

$$\begin{aligned} S^{\rho} &= \sum_{i_1,\dots,i_m} a^{\frac{2m}{m+1}}_{i_1,\dots,i_m} \leqslant \sum_{k=1}^{m}\sum_{i_k=1}^{\infty}\Big(\sum_{(i_k)}^{(k)} a^{\rho}_{i_1,\dots,i_m}\Big) \\ &\leqslant \sum_{k=1}^{m}\sum_{i_k=1}^{\infty} T^{(k)\,\frac{m-1}{m+1}}\, T^{(k)}_{i_k} = \sum_{k=1}^{m} T^{(k)\,\frac{m-1}{m+1}} \sum_{i_k=1}^{\infty} T^{(k)}_{i_k} \\ &= \sum_{k=1}^{m} T^{(k)\,\frac{m-1}{m+1}}\, T^{(k)} = \sum_{k=1}^{m} T^{(k)\,\frac{2m}{m+1}} \end{aligned}$$

■

This is the first part of the proof. In the second part they prove a matrix version of Khintchine inequality from scratch. We give now a version using induction on the Khintchine inequality. We give first the $m = 2$ case.

Lemma 1.2 (Double Khintchine inequality). *For each $1 \leqslant p < \infty$ and every finite family $(a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,m}} \subseteq \mathbb{C}$ the following holds*

$$\begin{aligned} A_p^2\Big(\sum_{i,j}|a_{ij}|^2\Big)^{1/2} &\leqslant \Big(\iint \big|\sum_{i,j} a_{ij}\varepsilon_i(\omega_1)\varepsilon_j(\omega_2)\big|^p d\omega_2 d\omega_1\Big)^{1/p} \\ &\leqslant B_p^2\Big(\sum_{i,j}|\alpha_{ij}|^2\Big)^{1/2} \end{aligned}$$

where A_p and B_p are those from (7) *and $(\varepsilon_i)_i$, $(\varepsilon_j)_j$ are families of independent Rademacher random variables that take the values ± 1 with probability* $1/2$.

We are going to use the following integral version of Minkowski inequality: For every $r \geqslant 1$,

$$\Big(\int_Y\Big(\int_X |f(x,y)|dx\Big)^r dy\Big)^{1/r} \leqslant \int_X\Big(\int_Y |f(x,y)|^r dy\Big)^{1/r} dx.$$

Proof. Let us consider first the case $p \geqslant 2$. We apply the classical Khintchine inequality and the integral Minkowski inequality to get

$$\begin{aligned}
&\Big(\iint \Big| \sum_{i,j} a_{ij} \varepsilon_i(\omega_1)\varepsilon_j(\omega_2) \Big|^p d\omega_2 d\omega_1 \Big)^{1/p} \\
&\quad = \bigg(\int \bigg[\Big(\int \Big| \sum_j \underbrace{\sum_i a_{ij}\varepsilon_i(\omega_1)}_{\alpha_j} \varepsilon_j(\omega_2) \Big|^p d\omega_2 \Big)^{1/p} \bigg]^p d\omega_1 \bigg)^{1/p} \\
&\quad = \bigg(\int \bigg[\Big(\int \Big| \sum_j \alpha_j \varepsilon_j(\omega_2) \Big|^p d\omega_2 \Big)^{1/p} \bigg]^p d\omega_1 \bigg)^{1/p} \qquad \Big(\text{Khint. ineq.}\Big) \\
&\quad \leqslant B_p \bigg(\int \bigg[\Big(\sum_j |\alpha_j|^2 \Big)^{1/2} \bigg]^p d\omega_1 \bigg)^{1/p} \\
&\quad = B_p \bigg(\int \Big(\sum_j |\sum_i a_{ij}\varepsilon_i(\omega_1)|^2 \Big)^{p/2} d\omega_1 \bigg)^{\frac{2}{p}\frac{1}{2}} \qquad \Big(\text{cts. Mink. ineq. } (p/2 \geqslant 1)\Big) \\
&\quad \leqslant B_p \bigg(\sum_j \Big(\int \Big| \sum_i a_{ij}\varepsilon_i(\omega_1) \Big|^{2\frac{p}{2}} d\omega_1 \Big)^{2/p} \bigg)^{1/2} \\
&\quad = B_p \bigg(\sum_j \bigg[\Big(\int \Big| \sum_i a_{ij}\varepsilon_i(\omega_1) \Big|^p d\omega_1 \Big)^{1/p} \bigg]^2 \bigg)^{1/2} \qquad \Big(\text{Khint. ineq.}\Big) \\
&\quad \leqslant B_p^2 \bigg(\sum_j \bigg[\Big(\sum_i |a_{ij}|^2 \Big)^{1/2} \bigg]^2 \bigg)^{1/2} = B_p^2 \Big(\sum_{i,j} |\alpha_{ij}|^2 \Big)^{1/2}
\end{aligned}$$

Now, since $2 \leqslant p$ and we are in a probability space, $(\int |f|^2)^{1/2} \leqslant (\int |f|^p)^{1/p}$, then

$$\begin{aligned}
&\Big(\iint \Big| \sum_{i,j} a_{ij} \varepsilon_i(\omega_1)\varepsilon_j(\omega_2) \Big|^p d\omega_2 d\omega_1 \Big)^{1/p} \\
&\qquad \geqslant \Big(\iint \Big| \sum_{i,j} a_{ij} \varepsilon_i(\omega_1)\varepsilon_j(\omega_2) \Big|^2 d\omega_2 d\omega_1 \Big)^{1/2} \\
&\qquad = \bigg(\iint \Big(\sum_{i,j} a_{ij} \varepsilon_i(\omega_1)\varepsilon_j(\omega_2) \Big) \Big(\sum_{k,l} \bar{a}_{kl} \varepsilon_k(\omega_1)\varepsilon_l(\omega_2) \Big) d\omega_2 d\omega_1 \bigg)^{1/2} \\
&\qquad = \bigg(\sum_{i,j,k,l} a_{ij}\bar{a}_{kl} \underbrace{\int \varepsilon_i(\omega_1)\varepsilon_k(\omega_1) d\omega_1}_{\delta_{ik}} \underbrace{\int \varepsilon_j(\omega_2)\varepsilon_l(\omega_2) \Big) d\omega_2}_{\delta_{jl}} \bigg)^{1/2} \\
&\qquad = \Big(\sum_{ij} |a_{ij}|^2 \Big)^{1/2} .
\end{aligned}$$

Now, for $p \leqslant 2$ we again use the classical Khintchine and the integral Minkowski inequalities to get

$$
\begin{aligned}
\Big(\sum_{ij}|a_{ij}|^2\Big)^{1/2} &= \Bigg[\sum_i \Bigg(\Big(\sum_j |a_{ij}|^2\Big)^{1/2}\Bigg)^2\Bigg]^{1/2} \qquad \Big(\text{Khint. ineq.}\Big)\\
&\leqslant \frac{1}{A_p}\Bigg[\sum_i \Bigg(\Big(\int \Big|\sum_j a_{ij}\varepsilon_j(\omega_2)\Big|^p d\omega_2\Big)^{1/p}\Bigg)^2\Bigg]^{1/2}\\
&= \frac{1}{A_p}\Bigg[\sum_i \Bigg(\int \Big|\sum_j a_{ij}\varepsilon_j(\omega_2)\Big|^p d\omega_2\Bigg)^{2/p}\Bigg]^{\frac{p}{2}\frac{1}{p}} \qquad \Big(\text{cts. Mink. ineq. } (2/p \geqslant 1)\Big)\\
&\leqslant \frac{1}{A_p}\Bigg[\int \Bigg(\sum_i \Big(\Big|\underbrace{\sum_j a_{ij}\varepsilon_j(\omega_2)}_{\alpha_i}\Big|^p\Big)^{2/p}\Bigg)^{p/2} d\omega_2\Bigg]^{1/p}\\
&\leqslant \frac{1}{A_p}\Bigg[\int \Bigg(\Big(\sum_i |\alpha_i|^2\Big)^{1/2}\Bigg)^p d\omega_2\Bigg]^{1/p} \qquad \Big(\text{Khint. ineq.}\Big)\\
&\leqslant \frac{1}{A_p^2}\Bigg[\int \Bigg(\Big(\int \Big|\sum_i \alpha_i\varepsilon_i(\omega_1)\Big|^p d\omega_1\Big)^{1/p}\Bigg)^p d\omega_2\Bigg]^{1/p}\\
&= \frac{1}{A_p^2}\Bigg(\iint \Big|\sum_{i,j} a_{ij}\varepsilon_i(\omega_1)\varepsilon_j(\omega_2)\Big|^p d\omega_2 d\omega_1\Bigg)^{1/p}.
\end{aligned}
$$

The other inequality goes exactly in the same way as in the case $p \geqslant 2$. ■

With this one can easily prove by induction that for every m and every finite $(a_{i_1,\dots,i_m}) \subseteq \mathbb{C}$,

$$
\begin{aligned}
A_p^m&\Big(\sum_{i_1,\dots,i_m}|a_{i_1,\dots,i_m}|^2\Big)^{1/2}\\
&\leqslant \Big(\int\cdots\int\Big|\sum_{i_1,\dots,i_m} a_{i_1,\dots,i_m}\varepsilon_{i_1}(\omega_1)\cdots\varepsilon_{i_m}(\omega_m)\Big|^p d\omega_m\cdots d\omega_1\Big)^{1/p} \qquad (13)\\
&\leqslant B_p^m\Big(\sum_{i_1,\dots,i_m}|a_{i_1,\dots,i_m}|^2\Big)^{1/2}.
\end{aligned}
$$

We can now finish the proof of Theorem I. First of all, using (13) with $p = 1$ we have

$$
\begin{aligned}
&\sum_{i_1=1}^{N}\Big(\sum_{i_2,\dots i_m=1}^{N}|a_{i_1,\dots i_m}|^2\Big)^{1/2}\\
&\quad\leqslant(\sqrt{2})^{m-1}\sum_{i_1=1}^{N}\int\cdots\int\Big|\sum_{i_2,\dots,i_m=1}^{N}a_{i_1,\dots,i_m}\varepsilon_{i_2}(\omega_2)\cdots\varepsilon_{i_m}(\omega_m)\Big|d\omega_m\cdots d\omega_2\\
&\quad=(\sqrt{2})^{m-1}\int\cdots\int\sum_{i_1=1}^{N}\Big|\sum_{i_2,\dots,i_m=1}^{N}a_{i_1,\dots,i_m}\varepsilon_{i_2}(\omega_2)\cdots\varepsilon_{i_m}(\omega_m)\Big|d\omega_m\cdots d\omega_2\\
&\quad\leqslant 2^{\frac{m-1}{2}}\int\cdots\int\sup_{x^{(2)},\dots,x^{(m)}\in B_{\ell_\infty^N}}\sum_{i_1=1}^{N}\Big|\sum_{i_2,\dots,i_m=1}^{N}a_{i_1,\dots,i_m}x^{(2)}_{i_2}\cdots x^{(m)}_{i_m}\Big|d\omega_m\cdots d\omega_2\\
&\quad\leqslant 2^{\frac{m-1}{2}}\sup_{x^{(2)},\dots,x^{(m)}\in B_{\ell_\infty^N}}\sum_{i_1=1}^{N}\Big|\sum_{i_2,\dots,i_m=1}^{N}a_{i_1,\dots,i_m}x^{(2)}_{i_2}\cdots x^{(m)}_{i_m}\Big|\\
&\qquad\times\Big(\underbrace{\int d\omega}_{1}\Big)^{m-1}\qquad\left(\begin{array}{l}\text{this is a norm in }\ell_1^N\text{; we use}\\ \text{the duality with }\ell_\infty^N\end{array}\right)\\
&\quad=2^{\frac{m-1}{2}}\sup_{x^{(1)},\dots,x^{(m)}\in B_{\ell_\infty^N}}\Big|\sum_{i_1,\dots,i_m=1}^{N}a_{i_1,\dots,i_m}x^{(1)}_{i_1}x^{(2)}_{i_2}\cdots x^{(m)}_{i_m}\Big|\leqslant 2^{\frac{m-1}{2}}\|L\|.
\end{aligned}
$$

Since this is true for every N we have $T^{(1)}\leqslant 2^{\frac{m-1}{2}}\|L\|$. The same argument can be repeated for any index i_k; hence

$$
T^{(k)}\leqslant 2^{\frac{m-1}{2}}\|L\| \tag{14}
$$

for every $k=1,\dots,m$. Then

$$
S^{\rho}\leqslant\sum_{k=1}^{m}T^{(k)\rho}\leqslant\sum_{k=1}^{m}2^{\frac{m-1}{2}\rho}\|L\|^{\rho}.
$$

This is

$$
\Big(\sum_{i_1,\dots,i_m}|a_{i_1,\dots,i_m}|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}}\leqslant m^{\frac{m+1}{2m}}2^{\frac{m-1}{2}}\|L\|.
$$

This is essentially the proof given in [10]. It has only been slightly reorganised and the language has been a little bit updated. Also, the central part in which they re-prove Khintchine's inequality has been avoided.

1.3. Theorem I in the first section is the best result of its kind

> **2. Theorem I in the first section is the best result of its kind.** As in the case $m = 2$, the proof is based upon the construction for every n of m-linear forms
>
> (2.1) $$\sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)},$$
>
> for which
>
> (2.2) $$\left\{ \sum_{i_1,\cdots,i_m=1}^{n} | a_{i_1\cdots i_m} |^{\frac{2m}{m+1}} \right\}^{\frac{m+1}{2m}} \geqq A_m \cdot H_n.$$

In this second section they prove that the exponent $\frac{2m}{m+1}$ is optimal. The case $m = 2$ is, obviously, Littlewood's theorem. They state the result in a similar way to that of Littlewood.

> We can express these results by means of the series S, $T^{(\nu)}$:
>
> THEOREM II. *Given any positive $t_{i_\nu}^{(\nu)}$ and $s_{i_1\cdots i_m}$, for which*
>
> $$\lim_{i_\nu\to\infty} t_{i_\nu}^{(\nu)} = \lim_{i_1,\cdots,i_m\to\infty} s_{i_1\cdots i_m} = \infty,$$
>
> *there exist bounded forms for which*
>
> $$\sum t_{i_\nu}^{(\nu)} T_{i_\nu}^{(\nu)} \quad \text{and} \quad \sum s_{i_1\cdots i_m} | a_{i_1\cdots i_m} |^{\frac{2m}{m+1}}$$
>
> *are divergent.*

We however focus on the optimality and state the following

Theorem II. *If $r > 1$ is such that there exists $C_m > 0$ for which*

$$\left(\sum_{i_1,\ldots,i_m} |a_{i_1,\ldots,i_m}|^r \right)^{1/r} \leqslant C_m \|L\|. \tag{15}$$

for every $L \in \mathcal{L}(^m c_0)$, then $r \geqslant \frac{2m}{m+1}$.

With a very clever modification of Littlewood's example [60, page 172] (see also [80, page 422]) that we have already mentioned they define a sequence of m-linear forms $L_n \in \mathcal{L}(^m \ell_\infty^n)$ that gives the result. They start with an $n \times n$ matrix $(a_{rs})_{rs}$ satisfying

$$\begin{cases} \displaystyle\sum_{t=1}^{n} a_{rt}\overline{a}_{st} = n\delta_{rs} \\ |a_{rs}| = 1 \end{cases} \tag{16}$$

Such a matrix is, for example $a_{rs} = e^{2\pi i \frac{rs}{n}}$. This matrix obvioulsy satisfies the second condition. For the first one we have

$$\sum_{t=1}^{n} a_{rt}\overline{a}_{st} = \sum_{t=1}^{n} e^{2\pi i \frac{rt}{n}} e^{-2\pi i \frac{st}{n}} = \sum_{t=1}^{n} e^{2\pi i \frac{(r-s)t}{n}}.$$

If $r=s$, then each factor equals 1 and $\sum_{t=1}^n a_{rt}\bar{a}_{rt}=n$. If $r\neq s$ then $r-s=k\neq 0$ and

$$\sum_{t=1}^n e^{2\pi i\frac{(r-s)t}{n}} = \sum_{t=1}^n \left(e^{2\pi i\frac{k}{n}}\right)^t = \frac{e^{2\pi ik\frac{1}{n}} - e^{2\pi ik\frac{n+1}{n}}}{1-e^{2\pi i\frac{k}{n}}} = \frac{e^{2\pi ik\frac{1}{n}} - \left(e^{\pi i}\right)^{2k} e^{2\pi ik\frac{1}{n}}}{1-e^{2\pi i\frac{k}{n}}} = 0.$$

Then they define L_n by

$$\begin{aligned} L_n(x^{(1)},\dots,x^{(m)}) &= \sum_{i_1,\dots,i_m=1}^n a_{i_1i_2}\cdots a_{i_{m-1}i_m} x_{i_1}^{(1)}\cdots x_{i_m}^{(m)} \\ &= \sum_{i_1,\dots,i_m=1}^n a_{i_1\dots i_m} x_{i_1}^{(1)}\cdots x_{i_m}^{(m)}. \end{aligned}$$

Since each $|a_{rs}|=1$, clearly $|a_{i_1\dots i_m}|=1$ and $\left(\sum_{i_1,\dots,i_m}|a_{i_1,\dots,i_m}|^r\right)^{1/r}=n^{m/r}$.

We now compute the norm of each of these forms. Let $x^{(1)},\dots,x^{(m)}\in B_{\ell_\infty^n}$, then

$$\begin{aligned} |L_n(x^{(1)},\dots,x^{(m)})| &= \Big| \sum_{i_1,\dots,i_m=1}^n a_{i_1i_2}\cdots a_{i_{m-1}i_m} x_{i_1}^{(1)}\cdots x_{i_m}^{(m)} \Big| \\ &\leqslant \sum_{i_m=1}^n \Big| \sum_{i_1,\dots,i_{m-1}=1}^n a_{i_1i_2}\cdots a_{i_{m-1}i_m} x_{i_1}^{(1)}\cdots x_{i_{m-1}}^{(m-1)} \Big| \cdot \underbrace{|x_{i_m}^{(m)}|}_{\leqslant 1} \\ &\leqslant \sum_{i_m=1}^n \Big| \sum_{i_1,\dots,i_{m-1}=1}^n a_{i_1i_2}\cdots a_{i_{m-1}i_m} x_{i_1}^{(1)}\cdots x_{i_{m-1}}^{(m-1)} \Big| \qquad \big(\text{Cauchy-Schwarz}\big) \\ &\leqslant n^{1/2}\Big(\sum_{i_m=1}^n \Big| \sum_{i_1,\dots,i_{m-1}=1}^n a_{i_1i_2}\cdots a_{i_{m-1}i_m} x_{i_1}^{(1)}\cdots x_{i_{m-1}}^{(m-1)} \Big|^2 \Big)^{1/2} \\ &= n^{1/2}\Big(\sum_{i_m=1}^n \sum_{\substack{i_1,\dots,i_{m-1}\\ j_1,\dots,j_{m-1}}} a_{i_1i_2}\bar{a}_{j_1j_2}\cdots a_{i_{m-1}i_m}\bar{a}_{j_{m-1}i_m} x_{i_1}^{(1)}\overline{x_{j_1}^{(1)}}\cdots x_{i_{m-1}}^{(m-1)}\overline{x_{j_{m-1}}^{(m-1)}} \Big)^{1/2} \\ &= n^{1/2}\Big(\sum_{\substack{i_1,\dots,i_{m-1}\\ j_1,\dots,j_{m-1}}} a_{i_1i_2}\bar{a}_{j_1j_2}\cdots a_{i_{m-2}i_{m-1}}\bar{a}_{j_{m-2}j_{m-1}} \\ &\qquad \times x_{i_1}^{(1)}\overline{x_{j_1}^{(1)}}\cdots x_{i_{m-1}}^{(m-1)}\overline{x_{j_{m-1}}^{(m-1)}} \underbrace{\sum_{i_m=1}^n a_{i_{m-1}i_m}\bar{a}_{j_{m-1}i_m}}_{n\delta_{i_{m-1}j_{m-1}}} \Big)^{1/2} \\ &= n^{1/2}n^{1/2}\Big(\sum_{i_{m-1}=1}^n \sum_{\substack{i_1,\dots,i_{m-2}\\ j_1,\dots,j_{m-2}}} a_{i_1i_2}\bar{a}_{j_1j_2}\cdots a_{i_{m-2}i_{m-1}}\bar{a}_{j_{m-2}i_{m-1}} \\ &\qquad \times x_{i_1}^{(1)}\overline{x_{j_1}^{(1)}}\cdots x_{i_{m-2}}^{(m-2)}\overline{x_{j_{m-2}}^{(m-2)}} |x_{i_{m-1}}^{(m-1)}|^2 \Big)^{1/2} \end{aligned}$$

Thus, we have

$$
\begin{aligned}
&|L_n(x^{(1)},\dots,x^{(m)})| \\
&\quad = n^{1/2}n^{1/2}\Big(\sum_{i_{m-1}=1}^{n} \underbrace{|x^{(m-1)}_{i_{m-1}}|^2}_{\leqslant 1} \Big|\sum_{i_1,\dots,i_{m-2}} a_{i_1i_2}\cdots a_{i_{m-2}i_{m-1}} x^{(1)}_{i_1}\cdots x^{(m-2)}_{i_{m-2}}\Big|^2\Big)^{1/2} \\
&\quad \leqslant n^{1/2}n^{1/2}\Big(\sum_{i_{m-1}=1}^{n} \sum_{\substack{i_1,\dots,i_{m-2}\\ j_1,\dots,j_{m-2}}} a_{i_1i_2}\bar{a}_{j_1j_2}\cdots a_{i_{m-2}i_{m-1}}\bar{a}_{j_{m-2}i_{m-1}} \\
&\qquad\qquad \times x^{(1)}_{i_1}\overline{x^{(1)}_{j_1}}\cdots x^{(m-2)}_{i_{m-2}}\overline{x^{(m-2)}_{j_{m-2}}}\Big)^{1/2} \\
&\quad \leqslant \cdots \leqslant n^{m/2}\Big(\sum_{i_1,j_1} x^{(1)}_{i_1}\overline{x^{(1)}_{j_1}}\Big)^{1/2} = n^{m/2}\Big(\sum_{i_1}|x^{(1)}_{i_1}|^2\Big)^{1/2} \leqslant n^{m/2}n^{1/2}. \qquad (17)
\end{aligned}
$$

We have then constructed for each n a form $L_n \in \mathcal{L}(^m\ell_\infty^n)$ for which $\|L_n\| \leqslant n^{\frac{m+1}{2}}$ and $\big(\sum_{i_1,\dots,i_m}|a_{i_1,\dots,i_m}|^r\big)^{1/r} = n^{m/r}$. Hence, if (15) holds, then we have $n^{\frac{m}{r}} \leqslant C_m n^{\frac{m+1}{2}}$ for every n. This implies $\frac{m}{r} \leqslant \frac{m+1}{2}$ and gives $r \geqslant \frac{2m}{m+1}$.

1.4. Symmetric m-linear forms and m-ic forms

> **3. Symmetric m-linear forms and m-ic forms.** In the application to Dirichlet series, we shall deal with forms
>
> $$Q(x) \equiv \sum a_{i_1\cdots i_m} x_{i_1}\cdots x_{i_m}$$
>
> of the m-th degree instead of m-linear forms, where the coefficients $a_{i_1\cdots i_m}$ are supposed to be symmetrical. To every such form we associate the m-linear form
>
> $$L(x^{(1)}, x^{(2)}, \cdots, x^{(m)}) \equiv \sum a_{i_1\cdots i_m} x^{(1)}_{i_1}\cdots x^{(m)}_{i_m}$$
>
> and conversely, to every symmetrical linear form there corresponds an m-ic form, obtained by putting all $x^{(\nu)} = x$. Denoting by H, resp. $\mathfrak{H}$,

What they call m-ic forms (the Q in the text) are what we would now call m-homogeneous polynomial on c_0 and $Q \in \mathcal{P}(^m c_0)$. Then the L in the text is simply the associated symmetric m-linear form[6] (sometimes called $\check{Q}$). A modern approach to the relation between polynomials and multilinear mappings on infinite dimensional spaces can be found in [42, Chapter 1]. Bohnenblust and Hille immediately go to the spaces of continuous m-homogeneous polynomials and m-linear mappings.

[6] An m-linear mapping is symmetric if $L(x_{\sigma(1)},\dots,x_{\sigma(m)}) = L(x_1,\dots,x_m)$ for every permutation σ of $\{1,\dots,m\}$. The space of continuous, symmetric m linear forms is denoted $\mathcal{L}_s(^m X)$.

> m-ic form, obtained by putting all $x^{(\nu)} = x$. Denoting by H, resp. $\mathfrak{H}$, the maxima of $Q(x)$, resp. $L(x^{(1)}, \cdots, x^{(m)})$ for the domain (G_0), we obviously have $H \leqq \mathfrak{H}$. On the other hand the identity
>
> $$L(x^{(1)}, \cdots, x^{(m)}) \equiv \frac{1}{2^{m-1} \cdot m!} \sum_{\varepsilon} (-1)^{\varepsilon_2+\varepsilon_3+\cdots+\varepsilon_m} Q(x^{(1)} + (-1)^{\varepsilon_2} x^{(2)} + \cdots + (-1)^{\varepsilon_m} x^{(m)}), \tag{3.1}$$
>
> where the sum is to be extended over $\varepsilon_\nu = 0, 1;\ \nu = 2, 3, \cdots, m$, shows that
>
> $$\mathfrak{H} \leqq \frac{m^m}{m!} H.$$

Clearly $H = \|Q\|$, the norm of the polynomial, and $\mathfrak{H} = \|L\|$, the norm of the associated symmetric m-linear form. Then they prove (3.1) (that is the Polarization Formula [42, Corollary 1.6]) and what they obtain is nothing less than the Polarization Inequality (see [42, Proposition 1.8])

$$\|Q\| \leqslant \|L\| \leqslant \frac{m^m}{m!} \|Q\| \tag{18}$$

for every $Q \in \mathcal{P}(^m c_0)$ and every $L \in \mathcal{L}_s(^m c_0)$ associated one another. This seems to be one of the first written records (if not the first one) of a proof of the Polarization Formula [42, page 76]. They then state

> Hence:
>
> *A symmetrical m-linear form is bounded in the domain* (G_0) *if and only if the corresponding m-ic form is bounded in* (G_0).

This, in modern terms, means that $\mathcal{P}(^m c_0) = \mathcal{L}_s(^m c_0)$ holds topologically. And they conclude

> Theorem I therefore holds for m-ic forms; but as the examples we have constructed for Theorem II are not symmetrical, we must refine them in order to show that Theorem I is also the best result of its kind for the symmetrical case. We can still vary the matrix $\|a_{rs}\|$, subject only to

We know now [38, Lemma 5] that the exponent is optimal for m-linear forms if and only if it is optimal for m-homogeneous polynomials. The proof uses tensor product techniques (we will come back to this later in Section 2.3. Bohnenblust and Hille obviously did not have this at their disposal and they had to construct the following example. However, this example is used several times after in the paper to produce Dirichlet series with certain properties in a way that the tensor product approach does not provide.

Before we present the example, let us go a little bit further into the relationship between m-linear forms and m-homogeneous polynomials. More concretely, if $Q \in \mathcal{P}({}^m\ell_\infty^n)$, then it has a monomial expansion $\sum_{|\alpha|=m} c_\alpha x^\alpha$. Let us show how the coefficients of Q and those of the associated m-linear form L are related. First of all, there is a one-to-one relation between the sets of indices $\{(i_1,\dots,i_m)\colon i_1 \leqslant \dots \leqslant i_m\}$ and $\{\alpha \in \mathbb{N}_0^n\colon |\alpha| = m\}$. On the one hand, given $(i_1,\dots,i_m)$, one can define α by doing $\alpha_r = |\{k\colon i_m = r\}|$ (i.e α_1 counts how many times 1 occurs in $(i_1,\dots,i_m)$, α_2 how many times 2 occurs and so on ...); on the other hand, for each α, we consider $(i_1,\dots,i_m) = (1,\overset{\alpha_1}{\dots},1,2,\overset{\alpha_2}{\dots},2,\dots,n\overset{\alpha_n}{\dots},n)$. Then the coefficient c_α is obtained by symmetrising the coefficients of the associated index $(i_1,\dots,i_m)$, more precisely

$$c_\alpha = \frac{1}{\alpha!}\sum_{\sigma\in\Sigma_m} a_{\sigma i_1\dots\sigma i_m}$$

where Σ_m stands for the group of permutations of $\{1,\dots,m\}$. If we choose L to be symmetric (which we know we can) then obviously $c_\alpha = \frac{m!}{\alpha!}a_{i_1\dots i_m}$ since $|\Sigma_m| = m!$.

With this notation, *'Theorem I therefore holds for m-ic forms'* can be understood as

Proposition 1.3. *There exists a constant $K_m > 0$ such that for every m-homogeneous polynomial $Q \in \mathcal{P}({}^m c_0)$,*

$$\Big(\sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leqslant K_m\|Q\|. \tag{19}$$

Proof. Given $Q \in \mathcal{P}({}^m c_0)$, let us choose L the associated symmetric m-linear form, then

$$\begin{aligned}
\Big(\sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} &= \Big(\frac{m!}{m!}\Big)^{\frac{m-1}{2m}}\Big(\sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \\
&= (m!)^{\frac{m-1}{2m}}\Big(\sum_{|\alpha|=m} \frac{|c_\alpha|^{\frac{2m}{m+1}}}{(m!)^{\frac{m-1}{m+1}}}\Big)^{\frac{m+1}{2m}} = (m!)^{\frac{m-1}{2m}}\Big(\sum_{|\alpha|=m} \frac{|c_\alpha|^{\frac{2m}{m+1}}}{(\alpha!\frac{m!}{\alpha!})^{\frac{m-1}{m+1}}}\Big)^{\frac{m+1}{2m}} \\
&= (m!)^{\frac{m-1}{2m}}\Big(\sum_{|\alpha|=m} \frac{m!}{\alpha!}\Big(\frac{|c_\alpha|}{m!/\alpha!}\Big)^{\frac{2m}{m+1}}\Big(\frac{1}{\alpha!}\Big)^{\frac{m-1}{m+1}}\Big)^{\frac{m+1}{2m}} \\
&\leqslant (m!)^{\frac{m-1}{2m}}\Big(\sum_{|\alpha|=m} \frac{m!}{\alpha!}\Big|\frac{\alpha!}{m!}c_\alpha\Big|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \\
&= (m!)^{\frac{m-1}{2m}}\Big(\sum_{i_1,\dots,i_m=1}^{n} |a_{i_1,\dots,i_m}|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \\
&\leqslant (m!)^{\frac{m-1}{2m}} m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}\|L\| \leqslant (m!)^{\frac{m-1}{2m}} m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}\frac{m^m}{m!}\|Q\|.
\end{aligned}$$ ∎

We give now Bohnenblust and Hille's example that shows that the exponent $\frac{2m}{m+1}$ is also optimal in (19). Let us suppose that $r \geqslant 1$ is such that the inequality

$$\Big(\sum_{|\alpha|=m} |c_\alpha|^r \Big)^{\frac{1}{r}} \leqslant C\|Q\|. \tag{20}$$

holds for every m-homogeneous polynomial Q and let us see that $r \geqslant \frac{2m}{m+1}$.

Let $p > m$ be a prime number. We begin with $M_1 = (m_{rs})$ a $p \times p$ matrix and define the following

$$\begin{aligned}
M_1 &= \big(e^{2\pi i \frac{rs}{p}}\big)_{r,s} \equiv p \times p \\
M_2 &= \begin{pmatrix} m_{11}M_1 & \dots & m_{1p}M_1 \\ \vdots & & \vdots \\ m_{p1}M_1 & \dots & m_{pp}M_1 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} m_{11}m_{11} & \dots & m_{11}m_{1p} & \dots\dots & m_{1p}m_{11} & \dots & m_{1p}m_{1p} \\ m_{11}m_{21} & \dots & m_{11}m_{2p} & \dots\dots & m_{1p}m_{21} & \dots & m_{1p}m_{2p} \\ \vdots & & & & & & \vdots \\ m_{11}m_{p1} & \dots & m_{11}m_{pp} & \dots\dots & m_{1p}m_{p1} & \dots & m_{1p}m_{pp} \\ \vdots & & & & & & \vdots \\ \vdots & & & & & & \vdots \\ m_{p1}m_{p1} & \dots & m_{p1}m_{pp} & \dots\dots & m_{pp}m_{p1} & \dots & m_{pp}m_{pp} \end{pmatrix}}_{p^2 \times p^2} \\
M_3 &= \begin{pmatrix} m_{11}M_2 & \dots & m_{1p}M_2 \\ \vdots & & \vdots \\ m_{p1}M_2 & \dots & m_{pp}M_2 \end{pmatrix} \equiv p^3 \times p^3 \\
&\vdots \\
M_n &= \begin{pmatrix} m_{11}M_{n-1} & \dots & m_{1p}M_{n-1} \\ \vdots & & \vdots \\ m_{p1}M_{n-1} & \dots & m_{pp}M_{n-1} \end{pmatrix} \equiv p^n \times p^n
\end{aligned}$$

This is the so called matrix Kronecker product of M_1 with M_{n-1}. Let us call $M_n = (a_{rs}^{(n)})_{r,s=1,\dots,p^n} = (a_{rs})_{r,s=1,\dots,p^n}$. Each a_{rs} is a product of n elements of M_1; that is $a_{rs} = e^{2\pi i \frac{r_1 s_1 + \dots + r_n s_n}{p}}$. Hence each a_{rs} is a p-th root of unity (i.e. $a_{rs}^p = 1$). Also $|a_{rs}| = 1$ for all r, s. Moreover, $\sum_{t=1}^{p^n} a_{rt}\bar{a}_{st} = p^n \delta_{rs}$; let us see this by induction. The $n = 1$ case was shown in Section 1.3. Let us now look at the case $n = 2$. In $\sum_{t=1}^{p^2} a_{rt}\bar{a}_{st}$ we are considering all the elements in the r-th and the

s-th row of M_2. These rows have the following shape

$$\begin{aligned} r &\equiv m_{r'1}m_{r''1}\dots m_{r'1}m_{r''p}\; m_{r'2}m_{r''1}\dots m_{r'2}m_{r''p}\dots\dots m_{r'p}m_{r''1}\dots m_{r'p}m_{r''p} \\ s &\equiv m_{s'1}m_{s''1}\dots m_{s'1}m_{s''p}\; m_{s'2}m_{s''1}\dots m_{s'2}m_{s''p}\dots\dots m_{s'p}m_{s''1}\dots m_{s'p}m_{s''p} \end{aligned}$$

Then

$$\begin{aligned} \sum_{t=1}^{p^2} a_{rt}^{(2)}\bar{a}_{st}^{(2)} &= m_{r'1}m_{r''1}\overline{m_{s'1}m_{s''1}} + \cdots + m_{r'1}m_{r''p}\overline{m_{s'1}m_{s''p}} \\ &\quad + m_{r'2}m_{r''1}\overline{m_{s'2}m_{s''1}} + \cdots + m_{r'2}m_{r''p}\overline{m_{s'2}m_{s''p}} \\ &\quad + \cdots + m_{r'p}m_{r''1}\overline{m_{s'p}m_{s''1}} + \cdots + m_{r'p}m_{r''p}\overline{m_{s'p}m_{s''p}} \\ &= m_{r'1}\overline{m}_{s'1}(m_{r''1}\overline{m}_{s''1} + \cdots + m_{r''p}\overline{m}_{s''p}) + \cdots \\ &\quad + m_{r'p}\overline{m}_{s'p}(m_{r''1}\overline{m}_{s''1} + \cdots + m_{r''p}\overline{m}_{s''p}) \\ &= (m_{r'1}\overline{m}_{s'1} + \cdots + m_{r'p}\overline{m}_{s'p})(m_{r''1}\overline{m}_{s''1} + \cdots + m_{r''p}\overline{m}_{s''p}) \\ &= \Big(\sum_{t=1}^{p} a_{r't}^{(1)}\bar{a}_{s't}^{(1)}\Big)\Big(\sum_{t=1}^{p} a_{r''t}^{(1)}\bar{a}_{s''t}^{(1)}\Big). \end{aligned}$$

In the same way, in the general case the rows have the following shape

$$\begin{aligned} r &\equiv m_{r'1}a_{r''1}^{(n-1)}\dots m_{r'1}a_{r''p}^{(n-1)}\; m_{r'2}a_{r''1}^{(n-1)}\dots m_{r'2}a_{r''p}^{(n-1)} \\ &\quad \times \dots m_{r'p}a_{r''1}^{(n-1)}\dots m_{r'p}a_{r''p}^{(n-1)} \\ s &\equiv m_{s'1}a_{s''1}^{(n-1)}\dots m_{s'1}a_{s''p}^{(n-1)}\; m_{s'2}a_{s''1}^{(n-1)}\dots m_{s'2}a_{s''p}^{(n-1)} \\ &\quad \times \dots m_{s'p}a_{s''1}^{(n-1)}\dots m_{s'p}a_{s''p}^{(n-1)} \end{aligned}$$

and, using the induction hypothesis

$$\sum_{t=1}^{p^n} a_{rt}^{(n)}\bar{a}_{st}^{(n)} = \Big(\sum_{t=1}^{p} a_{r't}^{(1)}\bar{a}_{s't}^{(1)}\Big)\Big(\sum_{t=1}^{p^{n-1}} a_{r''t}^{(n-1)}\bar{a}_{s''t}^{(n-1)}\Big) = p\delta_{r's'}p^{n-1}\delta_{r''s''}.$$

But the pair (r', r'') is uniquely determined by r (same for (s, s'') and s); this means that $r = s$ if and only if $(r', r'') = (s', s'')$ or in other words $\delta_{rs} = \delta_{r's'} \cdot \delta_{r''s''}$. This finally gives $\sum_{t=1}^{p^n} a_{rt}\bar{a}_{st} = p^n\delta_{rs}$. Then the matrix M_n satisfies both conditions in (16) and we can consider the corresponding m-linear form on $\mathbb{C}^n$

$$L_n(x^{(1)}, \dots, x^{(m)}) = \sum_{i_1,\dots,i_m} a_{i_1 i_2}\cdots a_{i_{m-1}i_m} x_{i_1}^{(1)}\cdots x_{i_m}^{(m)}.$$

We obtain the associated polynomial by symmetrising the coefficients

$$Q_n(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha, \qquad c_\alpha = \frac{1}{\alpha!}\sum_{\sigma\in\Sigma_m} a_{i_{\sigma 1}i_{\sigma 2}}\cdots a_{i_{\sigma m-1}i_{\sigma m}}. \tag{21}$$

Then by (18) and (17)

$$\|Q_n\| \leqslant \|L_n\| \leqslant \left(p^n\right)^{\frac{m+1}{2}}. \tag{22}$$

Let us show now that $\inf\{|c_\alpha|\colon \alpha \in \mathbb{N}_0^{(\mathbb{N})}\ |\alpha| = m\} = \eta > 0$ (i.e. all the coefficients of all the polynomials Q_n are bounded from below by $\eta > 0$). Since $\sup\{\alpha!\colon \alpha \in \mathbb{N}_0^{(\mathbb{N})}, |\alpha| = m\} \leqslant (m!)^m$, it is enough to focus on elements of the form

$$\alpha! c_\alpha = \sum_{\sigma \in \Sigma_m} a_{i_{\sigma 1} i_{\sigma 2}} \cdots a_{i_{\sigma m-1} i_{\sigma m}}. \tag{23}$$

Let $1 \neq \zeta$ be a p-th root of 1 (e.g. $\zeta = e^{2\pi i/p}$). Each $a_{j_1 j_2} \cdots a_{j_{m-1} j_m}$ is a p-th root of unity, hence there exists some $0 \leqslant k \leqslant p-1$ for which it equals ζ^k. In (23) we have a sum of such elements, then

$$\alpha! c_\alpha = \sum_{k=0}^{p-1} \lambda_k \zeta^k$$

where $\sum_{k=0}^{p-1} \lambda_k = |\Sigma_m| = m!$ (each λ_k is the number of times that the term ζ^k appears in the sum (23)).

Let us suppose that $\sum_{k=0}^{p-1} \lambda_k \zeta^k = 0$. This means that the polynomial $\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{p-1} x^{p-1}$ has the same roots as $1 + x + x^2 + \cdots + x^{p-1}$. Then they have to divide each other, but the second polynomial is irreducible (prime) in $\mathbb{Q}[x]$ (note that $\lambda_k \in \mathbb{N}$) therefore there exists some $\lambda \in \mathbb{N}$ such that

$$\lambda(1 + x + x^2 + \cdots + x^{p-1}) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{p-1} x^{p-1}.$$

This gives $\lambda_0 = \lambda_1 = \cdots = \lambda_{p-1} = \lambda$. We see now a slightly different way to conclude this (see [73, Example 2.5]). First of all, we know that $\sum_{k=0}^{p-1} \zeta^k = \sum_{k=0}^{p-1} \left(e^{2\pi i/p}\right)^k = 0$, hence $\sum_{k=1}^{p-1} \zeta^k = -1$. Then

$$0 = \sum_{k=0}^{p-1} \lambda_k \zeta^k = \lambda_0 + \sum_{k=1}^{p-1} \lambda_k \zeta^k = -\lambda_0 \sum_{k=1}^{p-1} \zeta^k + \sum_{k=1}^{p-1} \lambda_k \zeta^k = \sum_{k=1}^{p-1} (\lambda_k - \lambda_0) \zeta^k.$$

But the system $\{\zeta, \zeta^2, \ldots, \zeta^{p-1}\}$ is linearly independent over $\mathbb{Q}$ thus $\lambda_0 - \lambda_k = 0$ for every k and we again have $\lambda_0 = \lambda_1 = \cdots = \lambda_{p-1} = \lambda$. But if this is true then

$$m! = \sum_{k=0}^{p-1} \lambda_k = \sum_{k=0}^{p-1} \lambda = \lambda p.$$

and this gives $\frac{m!}{p} = \lambda \in \mathbb{N}$. This is impossible, since p is a prime number bigger that m, hence it does not divide neither m nor $m-1$ or $m-2, \ldots$. This shows that no sum as (23) can be 0 and then $c_\alpha \neq 0$ for all α.

On the other hand since $\sum_{k=0}^{p-1} \lambda_k = m!$ we have $\lambda_k \leqslant m!$ for every k. This implies that the number of different values that c_α can take (independently of n) is not bigger than $|\{1, \dots, m!\}|^p$, and this is a finite number. That is, we only have a finite number for possible values for c_α and $\inf\{|c_\alpha| \colon \alpha \in \mathbb{N}_0^{p^n}, n \in \mathbb{N}, |\alpha| = m\} = \eta > 0$. Then, if (20) holds we have that

$$\eta K \big(p^n\big)^{\frac{m}{r}} \leqslant \Big(\sum_{|\alpha|=m} |c_\alpha|^r \Big)^{\frac{1}{r}} \leqslant C_m \|Q_n\| \leqslant C_m \big(p^n\big)^{\frac{m+1}{2}}$$

(K appears because we do not sum exactly $(p^n)^m$ elements, but a smaller quantity that is smaller than $K(p^n)^m$). Since this holds for every n, we have $\frac{m}{r} \leqslant \frac{m+1}{2}$ and then $r \geqslant \frac{2m}{m+1}$. This gives the optimality.

These kind of polynomials with unimodular coefficients built from Walsh matrices (i.e. $n \times n$ matrices A such that $A^* A = nI$) were re-descovered more than 20 years later by Shapiro and Rudin [79, 75] and are now called Rudin-Shapiro polynomials (see also [62, Section 3]). They are used to obtain polynomials with unimodular coefficients and with small norm. Rudin and Shapiro construct a sequence of pairs of polynomials (P_n, Q_n) of degree n by taking $P_0 = Q_0 = 1$ and defining

$$P_{n+1}(z) = P_n(z) + z^{2^n} Q_n(z), \qquad Q_{n+1}(z) = P_n(z) + z^{2^n} Q_n(z).$$

These polynomials have all coefficients ± 1 and its construction can be seen as an iterative action of shifts and action of $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (the real Walsh matrix of order 2×2):

$$\begin{aligned} (P_0, Q_0) &\overset{\text{shift}}{\to} (P_0, zQ_0) \overset{A}{\to} (P_1, Q_1) \to \cdots \to (P_n, Q_n) \\ &\overset{\text{shift}}{\to} (P_n, z^{2^n} Q_n) \overset{A}{\to} (P_{n+1}, Q_{n+1}) \to \cdots \end{aligned}$$

1.5. Application to power series in an infinite number of variables

4. Application to power series in an infinite number of variables. In this section we take the first steps toward the application of the preceding results to ordinary Dirichlet series. Bohr proved the following theorem.[17]

Let G_n be a sequence of positive numbers and

$$P(x_1, x_2, \cdots, x_n, \cdots) \equiv c + \sum_{i=1}^{\infty} c_i x_i + \sum_{i_1, i_2 = 1}^{\infty} c_{i_1 i_2} x_{i_1} x_{i_2} + \cdots$$

a power series in an infinite number of variables, bounded in the domain $|x_n| \leqq G_n$. Let further ε_n be a sequence of positive number such that $1°$ $0 < \varepsilon_n < 1$ and $2°$ $\sum \varepsilon_n^2$ is convergent; then the power series P is absolutely convergent in the domain $|x_n| \leqq \varepsilon_n G_n$.

This theorem [12, page 462] can be rewritten as (see (4))

Theorem. $\ell_2 \cap B_{c_0} \subseteq \operatorname{mon} H_\infty(B_{c_0})$

This gives $S \geqslant 2$ (and $T \leqslant 1/2$). As we have already mentioned in (10) the approach of Bohnenblust and Hille was similar of that of Bohr but considering m-homogeneous Dirichlet polynomials (and, naturally, m-homogeneous polynomials). As in the approach of Bohr they first get results for holomorphic functions in infinitely many variables in order to apply them later to Dirichlet series.

> **We now prove the following results, which complete Bohr's theorem.**
> **THEOREM III.** *If the power series P is bounded in $|x_n| \leqq G_n$: then its m-th polynomial P_m*
>
> $$P_m \equiv c + \sum_{i=1}^{\infty} c_i x_i + \cdots + \sum_{i_1, \cdots, i_m = 1}^{\infty} c_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$
>
> *is absolutely convergent in $|x_n| \leqq \varepsilon_n G_n$, when $\sum \varepsilon_n^{\sigma_m}$ converges, $\sigma_m = \dfrac{2m}{m-1}$.*

In a now natural way we can define the set of monomial convergence of a single m-homogeneous polynomial P as $\operatorname{mon} P = \{x \in c_0 \colon \sum_\alpha |c_\alpha(P) x^\alpha| < \infty\}$. Then we can rewrite

Theorem III. *Let $f \in H_\infty(B_{c_0})$ and let us write its Taylor series expansion around 0, $f = \sum_m P_m$[7]. Then, for every m,*

$$\ell_{\frac{2m}{m-1}} \subseteq \operatorname{mon} P_m.$$

In particular,

$$\ell_{\frac{2m}{m-1}} \subseteq \operatorname{mon} \mathcal{P}(^m c_0) := \bigcap_{P \in \mathcal{P}(^m c_0)} \operatorname{mon} P.$$

It could be argued that 'our' version is not exactly the same as that of Bohnenblust and Hille since we state it for m-homogeneous polynomials and in the original version it is stated for non homogeneous polynomials of degree m. This however is not the case since, as they also point out *'It suffices therefore to prove Theorem III for m-ic forms'*: the statement for homogeneous polynomials implies that for general polynomials. Indeed, if $P = \sum_{k=0}^m P_k$ is a polynomial of degree m, $\operatorname{mon} P$ is the set of absolute convergence of the power expansion $\sum_{|\alpha| \leqslant m} c_\alpha(P) z^\alpha = c_{(0,0,\dots)} + \sum_{|\alpha|=1} c_\alpha(P) z^\alpha + \cdots + \sum_{|\alpha|=m} c_\alpha(P) z^\alpha$ and clearly this converges absolutely if and only if each one of the homogeneous parts does; hence $\operatorname{mon} P = \cap_{k=1}^m \operatorname{mon} P_k$. But if the result holds for homogeneous polynomials we have $\ell_{\frac{2k}{k-1}} \subseteq \operatorname{mon} P_k$ for $k = 1, \dots, m$ and $\cap_{k=1}^m \ell_{\frac{2k}{k-1}} \subseteq \operatorname{mon} P$. Since $\frac{2k}{k-1}$ is a decreasing sequence, $\ell_{\frac{2m}{m-1}} = \cap_{k=1}^m \ell_{\frac{2k}{k-1}}$ and this gives the result for general polynomials.

[7] A function f is holomorphic if and only if it can be expanded locally uniformly as a sum of m-homogeneous polynomials [42, Proposition 3.2]

Their main ingredient for the proof is the following

> LEMMA 2. *If the power series $P(x_1, \cdots)$ is bounded by H in (G_0); then H is also an upper bound for the form*
>
> (4.1) $$\sum_{i_1, \cdots i_m = 1}^{\infty} c_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.$$

This is Cauchy inequality: if $f : B_{c_0} \to \mathbb{C}$ is holomorphic and bounded and $\sum_m P_m$ is its Taylor series expansion around 0, then $\|P_m\| \leqslant \|f\|$. This is now a standard fact of the theory of infinite dimensional holomorphy (see [42, Proposition 3.2] for a general version). We include now the proof they give of this fact in this particular case, it follows the lines of the analog result of Bohr for the case $m = 1$ [12, Theorem V].

Proof. Let us recall that in the language of Bohnenblust and Hille, 'bounded' means that the truncations are uniformly bounded; we take then

$$\sum_{i_1, \dots i_m = 1}^{N} c_{i_1, \dots i_m} x_{i_1} \cdots x_{i_m} \tag{24}$$

and we want to see that these are all bounded by H. Let f_N be the restriction of f to $B_{\ell_\infty^N}$; we take $x^* \in B_{\ell_\infty^N}$ and define $F(t) = f_N(tx^*)$ for $t \in (1+\delta)\mathbb{D}$ for some $0 < \delta$ properly chosen. This defines a holomorphic function in one variable whose power series expansion is

$$F(t) = c + t \sum_{i=1}^{N} c_i x_i^* + t^2 \sum_{i_1, i_2 = 1}^{N} c_{i_1 i_2} x_{i_1}^* x_{i_2}^* + \cdots + t^m \sum_{i_1, \dots, i_m = 1}^{N} c_{i_1 \dots i_m} x_{i_1}^* \cdots x_{i_m}^* + \cdots$$

Now, by the Cauchy Formula in $\mathbb{C}$ for the m-th coefficient we have

$$\sum_{i_1, \dots, i_m = 1}^{N} c_{i_1 \dots i_m} x_{i_1}^* \cdots x_{i_m}^* = \frac{1}{2\pi i} \int_{C(0,1)} \frac{F(\omega)}{\omega^{m+1}} d\omega.$$

Since f is bounded in B_{c_0} its truncations f_N are uniformly bounded by H, therefore $|F(\omega)| \leqslant H$ for every $|\omega| = 1$ and

$$\Big| \sum_{i_1, \dots, i_m = 1}^{N} c_{i_1 \dots i_m} x_{i_1}^* \cdots x_{i_m}^* \Big| \leqslant \frac{1}{2\pi} \int_{C(0,1)} \frac{H}{|\omega|^{m+1}} d\omega = H.$$

Since this holds for every x^*, this completes the proof. ■

Proof of Theorem III. Now, given $x \in \ell_{\frac{2m}{m-1}}$, we simply have to apply Hölder's inequality with $p = \frac{2m}{m-1}$ and $q = \frac{2m}{m+1}$ and Theorem II to get

$$\begin{aligned}\sum_{|\alpha|=m} |c_\alpha(f)| x^\alpha &\leqslant \Big(\sum_{|\alpha|=m} |c_\alpha(f)|^{\frac{2m}{m-1}} \Big)^{\frac{m-1}{2m}} \Big(\sum_{|\alpha|=m} |x^\alpha|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \\ &\leqslant C_m \|P_m\| \Big(\sum_{|\alpha|=m} |x^{\frac{2m}{m+1}}|^\alpha \Big)^{\frac{m+1}{2m}} \leqslant C_m \|f\| \Big(\sum_{|\alpha|=m} |x^{\frac{2m}{m+1}}|^\alpha \Big)^{\frac{m+1}{2m}}\end{aligned}$$

and this is finite since $z \in \ell_1$ if and only if $\sum_\alpha |z^\alpha| < \infty$. This gives $x \in \operatorname{mon} P_m$ and completes the proof. ■

If we consider numbers S_m defined for m-homogeneous polynomials in (10) in an analogous way as S for holomorphic functions, then Theorem III gives $S_m \geqslant \frac{2m}{m-1}$. This is not enough for our purposes of giving an upper bound for S since $S \leqslant \inf S_m$. We then need to show that $S_m = \frac{2m}{m-1}$. This means that $\ell_r \not\subseteq \operatorname{mon} \mathcal{P}(^m c_0)$ for every $r > \frac{2m}{m-1}$ or, in other words, that the exponent in Theorem III is optimal.

> **This exponent σ_m is the best possible one:**
>
> THEOREM IV. *There exist polynomials of the m-th degree in an infinite number of variables bounded in $|x_n| \leqq 1$, such that for every $\delta > 0$, the polynomial is non-absolutely convergent for $x_n = \varepsilon_n$ although the series $\sum \varepsilon_n^{\sigma_m + \delta}$ converges.*

Theorem IV. *For every $m \geqslant 2$ there exists $Q \in \mathcal{P}(^m c_0)$ such that for every $\varepsilon > 0$ the monomial expansion of Q does not converge in some $x \in \ell_{\frac{2m}{m-1}+\varepsilon}$.*

Proof. In Section 1.4 we defined polynomials $Q_n \in \mathcal{P}(^m \ell_\infty^{p^n})$ such that $\|Q_n\| \leqslant p^{n\frac{m+1}{2}}$ (where p is some fixed prime number, see (21) and (22)). We are going to use these polynomials to construct the one we look for. First of all we identify $c_0 = c_0(\ell_\infty^{p^n})$, this means that we divide the elements of c_0 into blocks of length p^n: if $x \in c_0$ we do

$$x = (\underbrace{x_1^{(1)}, \overset{p}{\ldots}, x_p^{(1)}}_{x^{(1)}}, \underbrace{x_1^{(2)}, \ldots \overset{p^2}{\ldots}, x_{p^2}^{(2)}}_{x^{(2)}}, \underbrace{x_1^{(3)}, \ldots \overset{p^3}{\ldots}, x_{p^3}^{(3)}}_{x^{(3)}}, \ldots)$$

and define

$$Q(x) = \sum_{n=1}^\infty \frac{1}{n^2} p^{-n\frac{m+1}{2}} Q_n(x^{(n)}). \tag{25}$$

Clearly $Q \in \mathcal{P}(^m c_0)$ since for every $N \in \mathbb{N}$

$$\sum_{n=1}^N \frac{1}{n^2} p^{-n\frac{m+1}{2}} \|Q_n\| \leqslant \sum_{n=1}^N \frac{1}{n^2} p^{-n\frac{m+1}{2}} p^{n\frac{m+1}{2}} < \sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

We know that there exists some $\eta > 0$ for which $|c_\alpha(Q_n)| > \eta$ for every α and every n. Then for each x we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} Q_n(x^{(n)}) = \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \sum_{|\alpha|=m} c_\alpha(Q_n) x^\alpha$$

and

$$\begin{aligned}\sum_{|\alpha|=m} |c_\alpha(Q) x^\alpha| &= \sum_{n=1}^{\infty} \sum_{|\alpha|=m} \Big|\frac{1}{n^2} p^{-n\frac{m+1}{2}} c_\alpha(Q_n)(x^{(n)})^\alpha\Big| \\ &\geqslant \eta \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \sum_{|\alpha|=m} |(x^{(n)})^\alpha| \qquad (26) \\ &\geqslant \frac{\eta}{m!} \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \Big(\sum_{i=1}^{p^n} |x_i^{(n)}| \Big)^m\end{aligned}$$

Hence it is enough to find x so that the last sum is not convergent. Let us note first that for every ε there is some δ so that $\frac{2m}{m-1}\frac{1}{1-\delta} = \frac{2m}{m-1} + \varepsilon$. Now, $p^\delta > 1$ then we can choose $b < 1$ so that $p^\delta b^{1-\delta} > 1$. We take $h = (p^\delta b^{1-\delta})^{\frac{m-1}{2}} > 1$ and define x blockwise by

$$x_k^{(n)} = \Big(\frac{b}{p}\Big)^{n\frac{m-1}{2m}(1-\delta)} \qquad \text{for } k = 1, \dots, p^n. \qquad (27)$$

Let us see that $x \in \ell_{\frac{2m}{m-1}\frac{1}{1-\delta}}$.

$$\begin{aligned}\sum_{k=1}^{\infty} x_k^{\frac{2m}{m-1}\frac{1}{1-\delta}} &= \sum_{n=1}^{\infty} \sum_{k=1}^{p^n} \Big(\frac{b}{p}\Big)^{n\frac{m-1}{2m}(1-\delta)\frac{2m}{m-1}\frac{1}{1-\delta}} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{p^n} \Big(\frac{b}{p}\Big)^n = \sum_{n=1}^{\infty} \frac{b^n}{p^n} p^n = \sum_{n=1}^{\infty} b^n,\end{aligned}$$

and the last sum is finite since $b < 1$. On the other hand

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \Big(\sum_{i=1}^{p^n} |x_i^{(n)}| \Big)^m &= \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \Big(\sum_{i=1}^{p^n} \big(\frac{b}{p}\big)^{n\frac{m-1}{2m}(1-\delta)} \Big)^m \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-n\frac{m+1}{2}} \big(\frac{b}{p}\big)^{n\frac{m-1}{2m}(1-\delta)m} p^{nm} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} b^{n\frac{m-1}{2}(1-\delta)} p^{-n\frac{m+1}{2}+nm-n\frac{m-1}{2}(1-\delta)} \\ &=^{8} \sum_{n=1}^{\infty} \frac{1}{n^2} b^{n\frac{m-1}{2}(1-\delta)} p^{n\frac{m-1}{2}\delta} = \sum_{n=1}^{\infty} \frac{1}{n^2} \big(p^\delta b^{1-\delta}\big)^{\frac{m-1}{2}n} = \sum_{n=1}^{\infty} \frac{1}{n^2} h^n.\end{aligned}$$

This cannot converge since $h > 1$ and thus $x \notin \operatorname{mon} Q$. ■

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[8] $-\frac{m+1}{2} + m - \frac{m-1}{2}(1-\delta) = \frac{2m-m-1}{2} - \frac{m-1}{2}(1-\delta) = \frac{m-1}{2} - \frac{m-1}{2}(1-\delta) = \frac{m-1}{2}\delta$

The previous result shows that $\ell_{\frac{2m}{m-1} \frac{1}{1-\delta}} \not\subset \operatorname{mon} \mathcal{P}({}^m c_0)$. Then $S_m = \frac{2m}{m-1}$ and

$$2 \leqslant S \leqslant \inf_m S_m = \inf_m \frac{2m}{m-1} = 2.$$

Finally

$$T = \frac{1}{2}.$$

This fact is noted in the paper

> **Since $\sigma_m \to 2$, when $m \to \infty$ it follows immediately from this theorem that the exponent 2 of Bohr's theorem cannot be increased.**

With this the main goal stated at the beginning of the paper (determining the maximal width of the band of uniform but not absolute convergence of a Dirichlet series) is accomplished. The paper could then finish here but it does not. They go further on to find an example of a Dirichlet series whose width is exactly 1/2. To that aim the next two sections are devoted.

1.6. Applications to ordinary Dirichlet series

> 5. **Applications to ordinary Dirichlet series.** We now apply the results of the preceeding section to ordinary Dirichlet series, with the help

They begin by noting that the previous results give $T = 1/2$. But since this is defined as a supremum in principle this only gives

> *There exist ordinary Dirichlet series for which the widths of their strips of uniform, non-absolute convergence are arbitrarily close to $\frac{1}{2}$.*

Their aim now is to show that the supremum is actually a maximum. More precisely they want to produce a Dirichlet series that attains this maximal width. In order to do so they first show

> THEOREM V. *Let σ_u be the abscissa of uniform convergence of the Dirichlet series*
>
> (5.1) $$\sum_{n=1}^{\infty} a_n n^{-s}.$$
>
> *If we replace by zero those terms for which n, decomposed into a product of prime numbers, contains more than m factors, then the new series is absolutely convergent in the half plane*
>
> $$\sigma > \sigma_u + \frac{m-1}{2m},$$

This is

Theorem V. *If a Dirichlet series D has abscissa of uniform convergence σ_u, then its m-homogeneous Dirichlet part satisfies $\sigma_a \geqslant \sigma_u + \frac{m-1}{2m}$.*

We give here a proof both in old and in modern terms. We first give a proof closer to that given in the paper. First of all, if σ_u is the abscissa of uniform convergence of D, then by the first one of the Bohr's theorems stated in the introduction, the associated power series $P = \sum_\alpha c_\alpha x^\alpha$ is bounded in the domain $|x_n| \leqslant p_n^{-\sigma_u - \delta}$ for every $\delta > 0$ (where (p_n) stands for the sequence of prime numbers). We fix $\delta > 0$ and by Theorem III the power series P_m of degree m converges absolutely in the domain $|x_n| \leqslant \varepsilon_n p_n^{-\sigma_u - \delta}$ whenever $\sum_n \varepsilon_n^{\frac{2m}{m-1}} < \infty$. Let $\varepsilon_n = p_n^{-\frac{m-1}{2m} - \delta}$ we have

$$\sum_n \varepsilon_n^{\frac{2m}{m-1}} = \sum_n p_n^{(-\frac{m-1}{2m} - \delta)\frac{2m}{m-1}} = \sum_n \frac{1}{p_n^{1+\delta\frac{2m}{m-1}}} < \infty.$$

Then P_m converges absolutely in $|x_n| \leqslant p_n^{-\sigma_u - \frac{m-1}{2m} - 2\delta}$. Let us see that the Dirichlet m-homogeneous part of D converges absolutely for $\sigma > \sigma_u + \frac{m-1}{2m} + 2\delta$. Using Bohr's tranform we get

$$\begin{aligned}\sum_{\Omega(n)=m} |a_n \frac{1}{n^s}| &\leqslant \sum_n |a_n| \frac{1}{n^{\sigma_u + \frac{m-1}{2m} + 2\delta}} = \sum_{|\alpha|=m} |a_{p^\alpha}| \frac{1}{(p^\alpha)^{\sigma_u + \frac{m-1}{2m} + 2\delta}} \\ &= \sum_\alpha |c_\alpha| x^\alpha < \infty.\end{aligned}$$

Hence $\sigma_a > \sigma_u + \frac{m-1}{2m} + 2\delta$ for every δ and this finally gives the conclusion. ■

We rewrite this proof. First of all we fix $\delta > 0$ and doing $\mu = \sigma_u + \delta$ we know from (3) that there exists $f \in H_\infty(B_{c_0})$ so that $c_\alpha(f) = \frac{a_{p^\alpha}}{(p^\alpha)^{\sigma_u+\delta}}$; in other words $\sum_\alpha \frac{a_{p^\alpha}}{(p^\alpha)^{\sigma_u+\delta}} x^\alpha \in H_\infty(B_{c_0})$. We consider the m-homogeneous polynomial $P_m = \sum_{|\alpha|=m} \frac{a_{p^\alpha}}{(p^\alpha)^{\sigma_u+\delta}} x^\alpha$. By Theorem III $\ell_{\frac{2m}{m-1}} \subseteq \operatorname{mon} P_m$. We define $x_n = p_n^{-\frac{m-1}{2m} - \delta}$. We have $x \in \ell_{\frac{2m}{m-1}}$ since

$$\sum_n x_n^{\frac{2m}{m-1}} = \sum_n p_n^{(-\frac{m-1}{2m} - \delta)\frac{2m}{m-1}} = \sum_n \frac{1}{p_n^{1+\delta\frac{2m}{m-1}}} < \infty.$$

Then $\sum_{|\alpha|=m} \frac{a_{p^\alpha}}{(p^\alpha)^{\sigma_u+\delta}} x^\alpha$ converges absolutely and this gives

$$\sum_{\Omega(n)=m} |a_n| \frac{1}{n^{\sigma_u + \frac{m-1}{2m} + 2\delta}} = \sum_{|\alpha|=m} \frac{|a_{p^\alpha}|}{(p^\alpha)^{\sigma_u+\delta}} \frac{1}{(p^\alpha)^{\frac{m-1}{2m}+\delta}} = \sum_{|\alpha|=m} \frac{a_{p^\alpha}}{(p^\alpha)^{\sigma_u+\delta}} x^\alpha < \infty.$$

From this we have $\sigma_a \geqslant \sigma_u + \frac{m-1}{2m} + 2\delta$ and, since this is true for every $\delta > 0$, $\sigma_a \geqslant \sigma_u + \frac{m-1}{2m}$. ■

In the same way that we have considered numbers S_m that are m-homogeneous versions of Bohr's S we can consider

$$T_m = \sup\{\sigma_a - \sigma_u : \ m\text{-homogeneous Dirichlet series}\}.$$

If we knew that $T_m = 1/S_m$ we would automatically have $T_m = \frac{m-1}{2m}$, but this is not proved in the paper. Let us note that Theorem V gives in particular (just by taking an m-homogeneous Dirichlet polynomial from the very beginning) that $\sigma_a \geqslant \sigma_u + \frac{m-1}{2m}$ for every m-homogeneous Dirichlet polynomial. This means $T_m \leqslant \frac{m-1}{2m}$. They then prove the converse inequality by showing that there are m-homogeneous Dirichlet polynomials that attain this width.

> **Theorem VI.** *There exist ordinary Dirichlet series with $a_n = 0$, when n contains more than m prime factors, which converge uniformly, but non-absolutely in strips whose widths are exactly equal to $\frac{m-1}{2m}$:*
>
> $$\sigma_a - \sigma_u = \frac{m-1}{2m}.$$ [18]

Theorem VI. *There exist m-homogeneous Dirichlet polynomials for which $\sigma_a - \sigma_u = \frac{m-1}{2m}$.*

The case $m = 1$ was proved by Bohr in [12, page 468]. We modify slightly the proof of Bohnenblust and Hille.

Lemma 1.4. *If there exists $P \in \mathcal{P}(^m c_0)$ such that $\ell_{r+\varepsilon} \not\subseteq \operatorname{mon} P$ for every $\varepsilon > 0$, then there is $\tilde{P} \in \mathcal{P}(^m c_0)$ whose associated Dirichlet series satisfies $\sigma_a - \sigma_u \geqslant 1/r$.*

Proof. We consider $\sum_{|\alpha|=m} c_\alpha x^\alpha$ the monomial expansion of P. By doing $a_{p^\alpha} = c_\alpha$ and $\mu = 0$ we can rewrite it as $\sum_{|\alpha|=m} \frac{a_{p^\alpha}}{(p^\alpha)^\mu} x^\alpha$ and this defines a holomorphic function on B_{c_0}. Then (3) gives $\sigma_u \leqslant 0$.

Now, for each $\varepsilon > 0$ there is some $x \in \ell_{r+\varepsilon}$ such that $\sum_{|\alpha|=m} c_\alpha x^\alpha$ does not converge absolutely. We can assume that x is non increasing (if it were not we would simply consider its increasing rearrangement and then rearrange P to obtain $\tilde{P}$). The sequence $(x_n^{r+\varepsilon})_n$ is non increasing, since $(x_n)_n$ is so; then $n x_n^{r+\varepsilon} \leqslant \sum_{k=1}^n x_k^{r+\varepsilon}$ for every n. This implies that $x_n n^{\frac{1}{r+\varepsilon}} \leqslant \|x\|_{r+\varepsilon}$ for every n and

$$\sup_n x_n n^{\frac{1}{r+\varepsilon}} = C < \infty.$$

Hence

$$x_n \leqslant C \frac{1}{n^{\frac{1}{r+\varepsilon}}}$$

for every n. Now, by the Prime Number Theorem we have that for all $\varepsilon' > 0$

$$p_n \leqslant Dn \log n \leqslant D_{\varepsilon'} n^{1+\varepsilon'}$$

then

$$\frac{1}{n^{1+\varepsilon'}} \leqslant M \frac{1}{p_n}$$

and

$$\frac{1}{n} \leqslant K \frac{1}{p_n^{\frac{1}{1+\varepsilon'}}}.$$

These altogether for some $\delta > 0$ and all n gives

$$x_n \leqslant C \frac{1}{n^{\frac{1}{r+\varepsilon}}} \leqslant CK \frac{1}{p_n^{\frac{1}{r+\varepsilon}\frac{1}{1+\varepsilon'}}} = CK \frac{1}{p_n^{\frac{1}{r}-\delta}}.$$

Then the m-homogeneous Dirichlet series does not converge absolutely in $[\operatorname{Re} s > \frac{1}{r} - \delta]$, indeed

$$\sum_n |a_n| \frac{1}{n^{\frac{1}{r}-\delta}} = \sum_\alpha \left| c_\alpha \frac{1}{(p^{\frac{1}{r}-\delta})^\alpha} \right| \geqslant CK \sum_\alpha |c_\alpha x^\alpha|$$

and this last is not convergent. Since this holds for every $\delta > 0$ we have $\sigma_a \geqslant 1/r$. Finally $\sigma_a - \sigma_u \geqslant 1/r - 0 = 1/r$. ■

Proof of Theorem VI. In (25) we defined polynomials $Q \in \mathcal{P}(^m c_0)$ so that $\ell_{\frac{2m}{m-1}+\varepsilon} \not\subseteq \operatorname{mon} Q$ for every $\varepsilon > 0$. Moreover we can always choose a non increasing $x \in \ell_{\frac{2m}{m-1}+\varepsilon}$ (the one defined in (27)) for which $\sum_\alpha c_\alpha(Q) x^\alpha$ does not converge absolutely (this means that in the proof of Lemma 1.4 we do not need to rearrange the polynomial and $\tilde{Q} = Q$). Then by Lemma 1.4 the Dirichlet polynomial defined by Q satisfies $\sigma_a - \sigma_u \geqslant \frac{2m}{m-1}$. On the other hand, we know from Theorem V that $\sigma_a - \sigma_u \leqslant \frac{2m}{m-1}$ then

$$\sigma_a - \sigma_u = \frac{2m}{m-1}.$$

Looking at the proof of Lemma 1.4 we have $\sigma_a \geqslant \frac{2m}{m-1}$ and $\sigma_u \leqslant 0$. This finally even gives $\sigma_a = \frac{2m}{m-1}$ and $\sigma_u = 0$. ■

1.7. Solution of the main problem

> **6. Solution of the main problem.** We give first an example of an ordinary Dirichlet series, for which $\sigma_a - \sigma_u = \frac{1}{2}$.

The Dirichlet series is constructed using the polynomials defined in (25). We recall that for each m we defined $P_m \in \mathcal{P}(^m c_0)$ (we change the notation to avoid confusions) by

$$P_m(x) = \sum_{n=1}^\infty \frac{1}{n^2} p^{-n\frac{m+1}{2}} Q_n(x^{(n)}).$$

Let now $(q_m)_m$ be any sequence of positive numbers such that $\sum_m q_m$ converges and define for $x \in c_0$

$$f(x) = \sum_{m=1}^{\infty} q_m \frac{P_m(x)}{\|P_m\|}. \tag{28}$$

If $x \in B_{c_0}$ we have $|f(x)| \leqslant \sum_{m=1}^{\infty} q_m$; hence (28) defines a holomorphic function $f \in H_\infty(B_{c_0})$. By (3) the Dirichlet series $\mathfrak{B}(f)$ has abscissa of uniform convergence $\sigma_u \leqslant 0$.

On the other hand, if we consider the monomial expansion of f

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha(f) x^\alpha = \sum_{m=1}^{\infty} \sum_{|\alpha|=m} \frac{q_m}{\|P_m\|} c_\alpha(P_m) x^\alpha.$$

This means that $c_\alpha(f) = \frac{q_m}{\|P_m\|} c_\alpha(P_m)$ for each $|\alpha| = m$. Then if we separate the Dirichlet series defined by f into its m-homogeneous parts

$$\sum_{n=1}^{\infty} a_n \frac{1}{n^s} = \sum_{m=1}^{\infty} \sum_{\Omega(n)=m} a_n \frac{1}{n^s} \tag{29}$$

we have that each one of the m-homogeneous parts is precisely the m-homogeneous Dirichlet polynomial defined by P_m and that we already considered in the proof of Theorem VI. Each one of these has abscissa of absolute convergence $\sigma_a^{(m)} = \frac{m-1}{2m}$; this implies that the Dirichlet series (29) has abscissa $\sigma_a \geqslant 1/2$. Indeed, if $\sigma_0 < 1/2$, let us choose m so that $\sigma_0 < \frac{m-1}{2m} < 1/2$. Then there exists s_0 with $\operatorname{Re} s_0 = \sigma_0$ such that $\sum_{\Omega(n)=m} a_n/n^{s_0}$ does not converge absolutely. But this immediately gives that the Dirichlet series (29) does not converge absolutely for s_0 and $\sigma_a \geqslant 1/2$.

We then have that $\sigma_a - \sigma_u \geqslant 1/2 - 0 = 1/2$. But we know from the classical result of Bohr that $T \leqslant 1/2$; then necessarily $\sigma_a - \sigma_u \leqslant 1/2$. This gives $\sigma_a - \sigma_u = 1/2$. Moreover $\sigma_a = 1/2$ and $\sigma_u = 0$.

They have shown then that the supremum that defines T is actually a maximum, since there is a Dirichlet series that attains the maximal width. ∎

This solves the problem of Bohr, but they now want to prove more: that a Dirichlet series can be produced attaining any given width.

Theorem VII. *For any given σ, in the interval $0 \leqq \sigma \leqq \frac{1}{2}$, there exist ordinary Dirichlet series for which the width of the strip of uniform, but non-absolute convergence is exactly equal to σ.*[21]

They devote the rest of the section to construct the Dirichlet series. To do that they show that not only they can produce Dirichlet series attaining any given width, but also m-homogeneous Dirichlet series.

> and Theorem VII will be proved, if we can show
>
> THEOREM VIII. *There exist ordinary Dirichlet series associated with m-ic forms, for which* $\sigma_a - \sigma_u = \sigma$, *for every* σ *in the interval* $0 \leqq \sigma \leqq \frac{m-1}{2m}$.

They put some effort to produce such Dirichlet series, and dedicate about two pages to do that. But all this can be avoided, since (quoting Boas, [8, page 1435]) 'Bohr cut through this problem with a knife' with a remark at the end of the paper

> REMARK (added in proof, May, 1931). As Prof. Bohr pointed out to us, Theorem VII can be proved more simply as follows. Let $f(s)$ be a Dirichlet series whose $\sigma_u = 0$ and whose $\sigma_a = \frac{1}{2}$. If $\zeta(s)$ denotes the Riemann Zeta-function ($\sigma_a = \sigma_u = 1$) and σ any real number $0 \leqq \sigma \leqq \frac{1}{2}$, then $f(s) + \zeta(s+1-\sigma)$ is a Dirichlet series for which $\sigma_a - \sigma_u = \sigma$. Similar examples prove Theorem VIII. It may be interesting however, to see that the method used to obtain best possible examples is flexible enough to give the whole range of possible values for the difference $\sigma_a - \sigma_u$.

We present now a slight modification of the idea given by Bohr. First of all, let $\sum_n a_n \frac{1}{n^s}$ be a Dirichlet series with $\sigma_a = 1/2$ and $\sigma_u = 0$. We take the Riemann's ζ function $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ and define for $0 \leqslant \sigma \leqslant 1/2$

$$\sum_{n=1}^\infty b_n \frac{1}{n^s} = \sum_{n=1}^\infty a_n \frac{1}{n^s} + \zeta(s+1/2+\sigma) = \sum_{n=1}^\infty a_n \frac{1}{n^s} + \sum_{n=1}^\infty \frac{1}{n^{s+1/2+\sigma}}. \tag{30}$$

The key point now is that the ζ function has abscissas $\sigma_a(\zeta) = \sigma_u(\zeta) = \sigma_c(\zeta) = 1$. We have $\sigma_c(\zeta) \geqslant 1$ since $\sum_n 1/n$ does not converge. On the other hand, if $\operatorname{Re} s < 1$ we have

$$\sum_{n=1}^\infty \left| \frac{1}{n^s} \right| = \sum_{n=1}^\infty \frac{1}{n^{\operatorname{Re} s}} < \infty$$

Hence $\sigma_a \leqslant 1$. Then we have $1 \leqslant \sigma_c(\zeta) \leqslant \sigma_u(\zeta) \leqslant \sigma_a(\zeta) \leqslant 1$ and they are all equal to 1.

Let us see now that $\sigma_a(\sum b_n/n^s) = 1/2$ and $\sigma_u(\sum b_n/n^s) = 1/2-\sigma$. For a given $\varepsilon > 0$ obviously $\sum a_n/n^{1/2+\varepsilon}$ and $\sum_{n=1}^\infty \frac{1}{n^{1/2+\varepsilon+1/2+\sigma}}$ converge absolutely. Then the series in (30) converges absolutely for $\operatorname{Re} s = 1/2+\varepsilon$ and $\sigma_a(\sum b_n/n^s) \geqslant 1/2$. On the other hand if $\varepsilon < \sigma$ we have $1/2-\sigma < 1/2-\varepsilon < 1/2$ and $\sum_{n=1}^\infty \frac{1}{n^{1/2-\varepsilon+1/2+\sigma}}$ converges absolutely. If we assume that $\sum b_n \frac{1}{n^{1/2-\varepsilon}}$ converges absolutely we have

$$\begin{aligned}
\sum_n |a_n| \frac{1}{n^{1/2-\varepsilon}} &= \sum_n \Big| a_n + \frac{1}{n^{1/2+\sigma}} - \frac{1}{n^{1/2+\sigma}} \Big| \frac{1}{n^{1/2-\varepsilon}} \\
&\leqslant \sum_n \Big| a_n + \frac{1}{n^{1/2+\sigma}} \Big| \frac{1}{n^{1/2-\varepsilon}} + \sum_n \Big| \frac{1}{n^{1/2+\sigma}} \Big| \frac{1}{n^{1/2-\varepsilon}} \\
&= \sum_n |b_n| \frac{1}{n^{1/2-\varepsilon}} + \sum_n \frac{1}{n^{1/2+\sigma+1/2-\varepsilon}}
\end{aligned}$$

This is finite since both sums converge absolutely; but this contradicts the fact that $\sigma_a(\sum a_n/n^s) = 1/2$. Hence $\sigma_a(\sum b_n/n^s) \leqslant 1/2$ and we have the first equality.

Proceeding in the same way we can prove that $\sigma_u(\sum b_n/n^s) = 1/2 - \sigma$. Then obviously

$$\sigma_a\big(\sum b_n/n^s\big) - \sigma_u\big(\sum b_n/n^s\big) = \tfrac{1}{2} - \left(\tfrac{1}{2} - \sigma\right) = \sigma.$$

For Theorem VIII we proceed in a similar way. Starting from an m-homogeneous Dirichlet polynomial for which $\sigma_a = \frac{m-1}{2m}$ and $\sigma_u = 0$ (we know it exists from Theorem V) and considering the m-homogeneous part of the ζ function we define an m-homogeneous Dirichlet polynomial as in (30). Again this satisfies $\sigma_a = \frac{m-1}{2m}$ and $\sigma_u = \frac{m-1}{2m} - \sigma$ and we have what we wanted. The only key point left is to show that the m-homogeneous parts of the ζ function, $\sum_{\Omega(n)=m} 1/n^s$ has also abscissas equal to 1. We denote $\sigma_.^{(m)}$ for the abscissas of the m-homogeneous part and $\sigma_.$ for the abscissas of the full series. First of all we have

$$\sum_{\Omega(n)=m} |a_n| \frac{1}{n^s} \leqslant \sum_{n=1}^{\infty} |a_n| \frac{1}{n^s}$$

we have $\sigma_a^{(m)} \leqslant \sigma_a$ for every Dirichlet series. In our particular case we have $\sigma_c^{(m)} \leqslant \sigma_u^{(m)} \leqslant \sigma_a^{(m)} \leqslant \sigma_a = 1$. Let us see now that $\sigma_c^{(m)} \geqslant 1$. To do so it suffices to show that $\sum_{\Omega(n)=m} \frac{1}{n} = \sum_{|\alpha|=m} \frac{1}{p^\alpha}$ does not converge. But we have

$$\left(\sum_{n=1}^{N} \frac{1}{p_n}\right)^m = \sum_{\substack{|\alpha|=m \\ \leqslant N}} \frac{1}{p^\alpha}$$

$$\sup_N \sum_{\substack{|\alpha|=m \\ \leqslant N}} \frac{1}{p^\alpha} = \sum_{|\alpha|=m} \frac{1}{p^\alpha}.$$

On the other hand, by the Prime Number Theorem

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \geqslant C \sum_{n=1}^{\infty} \frac{1}{n \log n} = C \sum_{n=1}^{\infty} 2^n \frac{1}{2^n \log 2^n} = C \sum_{n=1}^{\infty} \frac{1}{n \log 2} = \infty,$$

Hence, the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ does not converge and obviously neither does $\left(\sum_{n=1}^{N} \frac{1}{p_n}\right)^m$. This implies that $\sum_{|\alpha|=m} \frac{1}{p^\alpha}$ does not converge and $\sigma_c^{(m)} \geqslant 1$. Thus $\sigma_c^{(m)} = \sigma_u^{(m)} = \sigma_a^{(m)} = 1$. Now, proceeding as in the previous example we define as in (30) an m-homogeneous Dirichlet polynomial for which

$$\sigma_a\big(\sum b_n/n^s\big) - \sigma_u\big(\sum b_n/n^s\big) = \sigma.$$

This piece of fine work is finished with a last, seventh section of about two pages dedicated to obtain some analogous results for generalised Dirichlet series. But this is another story ...

2. Further developments

Some of the things contained in the original paper [10] can be now proved in a different way using modern techniques. Also, their ideas have allowed new developments in more general settings. We dedicate the rest of the paper to show some of these.

2.1. New proofs of Theorem I

Theorem I was overlooked for long time, maybe because the statement was somehow not too clear and it was in some sense 'hidden' at the end of the section. More than 40 years latter Davie [21] and Kaijser [55] re-discovered it in the frame of and using techniques of tensor products. The result is often attributed to them. We sketch now their proof; we use now tensor products, as presented in [24].

2.1.1. The new proof: Kaijser

In the 1950's Grothendieck developed the metric theory of tensor products in Banach spaces in his famous Résumé [47]. Littlewood and Bohnenblust-Hille inequalities can be reformulated in this language as follows.

$$\sup_n \| \mathrm{id} : \ell_1^n \otimes_\varepsilon \ell_1^n \to \ell_{4/3}^{n^2} \| \leqslant \sqrt{2} \quad (\textbf{Littlewood}) \tag{31}$$

$$\sup_n \| \mathrm{id} : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \to \ell_{(2m)/(m+1)}^{n^m} \| \leqslant C_m \quad (\textbf{Bohnenblust-Hille}) \tag{32}$$

Here for M a finite dimensional Banach space, $M \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon M$ stands for the mth full tensor product of M with itself (we use the notation as presented in [24]). Then it is a well known fact that $M' \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon M' = \mathcal{L}(^m M)$. With this in mind the mappings in (31) and (32) are $L \rightsquigarrow (a_{i_1 \dots i_m})_{i_1, \dots, i_m}$.

S. Kaijser in 1978 re-discovered the Bohnenblust-Hille inequality in its tensor reformulation in [55]. We sketch now his proof and we will come back to it in more detail in a slightly more systematic way in Section 2.1.2. The proof of Bohnenblust and Hille gives $C_m = m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$; we will see that with the proof of Kaijser the constant improves to $C_m = (\sqrt{2})^{m-1}$.

Kaijser's proof for Littlewood's inequality (31) runs as follows.

Take D_1 and D_2 two arbitrary index sets and fix $1 \leqslant p \leqslant 2$. First we use an inequality of Hardy and Littlewood [49]:

$$\ell_p(D_1) \otimes_\varepsilon \ell_1(D_2) \hookrightarrow \ell_p(D_2, \ell_2(D_1)). \tag{33}$$

On the other hand we have an inequality of Littlewood [60] and of Orlicz [68] (this will be the case $m = 1$ in Proposition 2.5)

$$\ell_p(D_1) \otimes_\varepsilon \ell_1(D_2) \hookrightarrow \ell_p(D_1, \ell_2(D_2))$$

which, with Minkowski's inequality

$$\ell_p(D_1, \ell_2(D_2)) \hookrightarrow \ell_2(D_2, \ell_p(D_1))$$

gives

$$\ell_p(D_1) \otimes_\varepsilon \ell_1(D_2) \hookrightarrow \ell_p(D_1, \ell_2(D_2)) \hookrightarrow \ell_2(D_2, \ell_p(D_1)) \tag{34}$$

With (33), (34) and Hölder inequality we get

$$\ell_p(D_1) \otimes_\varepsilon \ell_1(D_2) \hookrightarrow \ell_p(D_2, \ell_2(D_1)) \cap \ell_2(D_2, \ell_p(D_1)) \overset{\text{Hölder}}{\hookrightarrow} \ell_r(D_1 \times D_2)$$

where $r = \frac{4p}{2+p}$. In particular, doing $p = 1$ we have

$$\ell_1(D_1) \otimes_\varepsilon \ell_1(D_2) \hookrightarrow \ell_{4/3}(D_1 \times D_2).$$

Kaijser proofs (32) by induction. The starting point is Littlewood's inequality as we have just presented. For the induction procedure he uses the following result, that will be reformulated and proved in Lemma 2.5. Its proof follows by induction from the inequality of Littlewood and Orlicz with the Multiple Khintchine inequality (which Kaijser attributes to Davie [21]).

Theorem 2.1 ([55], Theorem 1.2). *Let (Ω, μ) be a measure space and $(D_i)_{i=1,\dots,s}$ index sets and $D = D_1 \times \cdots \times D_s$, then*

$$L_p(\Omega, \mu) \otimes_\varepsilon \ell_1(D_1) \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1(D_s) \hookrightarrow L_p(\Omega, \ell_2(D))$$

for every $1 \leqslant p \leqslant \infty$ and the inclusion has norm $\leqslant 2^{s/2}$.

Then he gets the following

Corollary 2.2 ([55], Corollary 1.3). *Let $\{D_i\}_{i=1,\dots,s}$ be discrete spaces; let $D = D_1 \times \cdots \times D_s$ and let $r = \frac{2s}{s+1}$; then*

$$\ell_1(D_1) \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1(D_s) \hookrightarrow \ell_r(D).$$

Proof. By Theorem 2.1

$$\ell_1(D_1) \otimes_\varepsilon \ell_1(D_2) \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1(D_s) \hookrightarrow \ell_1\big(D_1; \ell_2(D_2 \times \cdots \times D_s)\big).$$

We now proceed by induction; let us suppose

$$\ell_1(D_1) \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1(D_{s-1}) \hookrightarrow \ell_{\frac{2(s-1)}{s}}(D_1 \times \cdots \times D_{s-1}).$$

Then

$$\begin{aligned} \ell_1(D_1) \otimes_\varepsilon \big(\ell_1(D_2) \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1(D_s)\big) &\to \big(\ell_1(D_2) \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1(D_s)\big) \otimes_\varepsilon \ell_1(D_1) \\ &\overset{\text{induction}}{\hookrightarrow} \ell_{\frac{2(s-1)}{s}}(D_2 \times \cdots \times D_s) \otimes_\varepsilon \ell_1(D_1) \\ &\overset{\text{Thm}}{\hookrightarrow} \ell_{\frac{2(s-1)}{s}}\big(D_2 \times \cdots \times D_s; \ell_2(D_1)\big) \\ &\overset{\text{cts Mink}}{\hookrightarrow} \ell_2\big(D_1; \ell_{\frac{2(s-1)}{s}}(D_2 \times \cdots \times D_s)\big). \end{aligned}$$

Hence, by Hölder inequality

$$\begin{aligned} &\ell_1(D_1) \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1(D_s) \\ &\qquad \hookrightarrow \ell_1\big(D_1; \ell_2(D_2 \times \cdots \times D_s)\big) \cap \ell_2\big(D_1; \ell_{\frac{2(s-1)}{s}}(D_2 \times \cdots \times D_s)\big) \overset{\text{Hölder}}{\hookrightarrow} \ell_r(D). \quad \blacksquare \end{aligned}$$

2.1.2. The new new proof: summing operators

The proof of Kaijser is certainly much more compact than the original of Bohnenblust and Hille. A modification of Kaijser's proof, looking at it from a slightly different point of view introducing summing operators, allows us to give a very compact proof of both Littlewood and Bohnenblust-Hille Theorems. We then get a compact, systematic proof that allows serveral further developments [38, 34] (see Sections 2.1.5, 2.1.6, 2.1.7, 2.3).

We begin by recalling the definition and some elementary facts of absolutely p-summing operators. For details, see [24, Section 11], [41, Chapter 2], [71, Chapter 17] or [81, Section 9].

Definition 2.3. An operator between Banach spaces, $v : X \to Y$ is absolutely p-summing ($1 \leqslant p < \infty$) if there exists a constant $c > 0$ such that for every finite family $x_1, \ldots, x_N \in X$

$$\Big(\sum_{k=1}^{N} \|v(x_k)\|_Y^p \Big)^{1/p} \leqslant c \sup_{x' \in B_{X'}} \Big(\sum_{k=1}^{N} |x_k|^p \Big)^{1/p} \tag{35}$$

The best constant c in (35) is denoted by $\pi_p(v)$ (this defines a norm) and the Banach space of all absolutely p-summing between X and Y is denoted by $\Pi_p(X,Y)$.

The right-hand side of the inequality can be rewritten as

$$\begin{aligned}
\sup_{x' \in B_{X'}} \Big(\sum_{k=1}^{N} |x_k|^p \Big)^{1/p} &= \sup_{x' \in B_{X'}} \|(x'(x_k))_k\|_{\ell_p} \\
&= \sup_{x' \in B_{X'}} \sup_{\lambda \in B_{\ell_{p'}}} \Big| \sum_{k=1}^{N} x'(x_k)\lambda_k \Big| \\
&= \sup_{\lambda \in B_{\ell_{p'}}} \sup_{x' \in B_{X'}} \Big| x'\big(\sum_{k=1}^{N} x_k \lambda_k \big) \Big| \\
&= \sup_{\lambda \in B_{\ell_{p'}}} \Big\| \sum_{k=1}^{N} x_k \lambda_k \Big\|_X .
\end{aligned}$$

It is a well known fact (see [41, Theorem 2.8], [24, 11.3] or [81, Proposition 9.6]) that if $p < q$, then $\Pi_p(X,Y) \subseteq \Pi_q(x,Y)$ and $\pi_q(v) \leqslant \pi_p(v)$ for every v (i.e., every absolutely p summing operator is also absolutely q-summing for very $q > p$).

Absolutely p-summing operators can be characterised in terms of tensor products. First, the space $\ell_p(Y)$ of absolutely p-summing sequences in a Banach space Y defines a norm in the tensor product:

$$\ell_p \otimes_p Y \hookrightarrow \ell_p(Y), \qquad \Big\| \sum_{k=1}^{N} e_k \otimes x_k \Big\|_{\ell_p \otimes_p Y} = \Big(\sum_{k=1}^{N} \|x_k\|_Y^p \Big)^{1/p} .$$

Also, $\ell_p \hat{\otimes}_\varepsilon Y \hookrightarrow \ell_p^w(Y)$ by means of

$$\Big\| \sum_{k=1}^N e_k \otimes x_k \Big\|_{\ell_p \otimes_\varepsilon Y} = \sup_{\substack{x' \in B_{X'} \\ \eta \in B_{\ell_{p'}}}} \Big| \sum_{k=1}^N \eta(e_k) x'(x_k) \Big| = \sup_{x' \in B_{X'}} \sup_{\eta \in B_{\ell_{p'}}} \Big| \sum_{k=1}^N \eta_k x'(x_k) \Big|$$

$$= \sup_{x' \in B_{X'}} \|(x'(x_k))_k\|_{\ell_p} = \sup_{x' \in B_{X'}} \Big(\sum_{k=1}^N |x'(x_k)|^p \Big)^{1/p}$$

With this notation we have (see [24, 11.1])

Theorem 2.4. *Let $v : X \to Y$ be a non-zero operator; then the following are equivalent*

1. $v \in \Pi_p(X, Y)$.
2. $\operatorname{id} \otimes v : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_p Y$ *is continuous.*
3. $\sup_n \| \operatorname{id} \otimes v : \ell_p^n \otimes_\varepsilon X \to \ell_p^n \otimes_p Y \| < \infty$.
4. $\operatorname{id} \otimes v : L_p(\mu) \otimes_\varepsilon X \to L_p(\mu, Y)$ *is continuous for every* $L_p(\mu)$.

In this case

$$\pi_p(v) = \| \operatorname{id} \otimes v : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_p Y \| = \sup_n \| \operatorname{id} \otimes v : \ell_p^n \otimes_\varepsilon X \to \ell_p^n \otimes_p Y \|$$
$$= \| \operatorname{id} \otimes v : L_p(\mu) \otimes_\varepsilon X \to L_p(\mu, Y) \|.$$

In view of Theorem 2.4 we can then reformulate and prove [55, Theorem 1.2] (Theorem 2.1) in terms of summing operators.

Proposition 2.5. $\operatorname{id} : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell 1^n \to \ell_2^{n^m}$ *is absolutely* 1*-summing and*

$$\pi_1(\operatorname{id} : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \longrightarrow \ell_2^{n^m}) \leqslant \big(\sqrt{2}\big)^m.$$

In particular for all $1 \leqslant p < \infty$ *we have*

$$\pi_p(\operatorname{id} : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \longrightarrow \ell_2^{n^m}) \leqslant \big(\sqrt{2}\big)^m.$$

Proof. Let $x_1, \ldots, x_N \in \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n$. Each one of them has a representation

$$x_k = \sum_{i_1, \ldots, i_m = 1}^n a_k(i_1, \ldots, i_m) e_{i_1} \otimes \cdots \otimes e_{i_m}.$$

We want

$$\sum_{k=1}^N \|x_k\|_{\ell_2^{n^m}} \leqslant \big(\sqrt{2}\big)^m \sup_{\lambda \in B_{\ell_\infty^N}} \Big\| \sum_{k=1}^N x_k \lambda_k \Big\|_{\ell_1^n \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_1^n}.$$

Indeed,

$$\begin{aligned}
\sum_{k=1}^{N} \|x_k\|_{\ell_2^{n^m}} &= \sum_{k=1}^{N} \Big(\sum_{i_1,\dots,i_m=1}^{n} |a_k(i_1,\dots,i_m)|^2 \Big)^{1/2} \qquad \text{multiple Khint. ineq. (13)} \\
&\leqslant (\sqrt{2})^m \sum_{k=1}^{N} \int \cdots \int \Big| \sum_{i_1,\dots,i_m=1}^{n} a_k(i_1,\dots,i_m)\varepsilon_{i_1}(\omega_1)\cdots\varepsilon_{i_m}(\omega_m) \Big| d\omega_1\cdots d\omega_m \\
&= 2^{m/2} \int \cdots \int \sum_{k=1}^{N} \Big| \sum_{i_1,\dots,i_m=1}^{n} a_k(i_1,\dots,i_m)\varepsilon_{i_1}(\omega_1)\cdots\varepsilon_{i_m}(\omega_m) \Big| d\omega_1\cdots d\omega_m \\
&\leqslant 2^{m/2} \int \cdots \int \sum_{k=1}^{N} \sup_{\eta^{(j)}\in B_{\ell_\infty^n}} \Big| \sum_{i_1,\dots,i_m=1}^{n} a_k(i_1,\dots,i_m)\eta_{i_1}^{(1)}\cdots\eta_{i_m}^{(m)} \Big| d\omega_1\cdots d\omega_m \\
&= 2^{m/2} \sup_{\eta^{(j)}\in B_{\ell_\infty^n}} \sum_{k=1}^{N} \Big| \sum_{i_1,\dots,i_m=1}^{n} a_k(i_1,\dots,i_m)\eta_{i_1}^{(1)}\cdots\eta_{i_m}^{(m)} \Big| \qquad \text{this is an } \ell_1 \text{ norm} \\
&= 2^{m/2} \sup_{\eta^{(j)}\in B_{\ell_\infty^n}} \sup_{\lambda\in B_{\ell_\infty^N}} \Big| \sum_{k=1}^{N} \big(\sum_{i_1,\dots,i_m=1}^{n} a_k(i_1,\dots,i_m)\eta_{i_1}^{(1)}\cdots\eta_{i_m}^{(m)} \big)\lambda_k \Big| \\
&= 2^{m/2} \sup_{\lambda\in B_{\ell_\infty^N}} \sup_{\eta^{(j)}\in B_{\ell_\infty^n}} \Big| \sum_{i_1,\dots,i_m=1}^{n} \Big(\sum_{k=1}^{N} a_k(i_1,\dots,i_m)\lambda_k \Big) \eta^{(1)}(e_{i_1})\cdots\eta^{(m)}(e_{i_m}) \Big| \\
&= 2^{m/2} \sup_{\lambda\in B_{\ell_\infty^N}} \Big\| \sum_{k=1}^{N} x_k\lambda_k \Big\|_{\ell_1^n\otimes_\varepsilon\cdots\otimes_\varepsilon\ell_1^n}.
\end{aligned}$$

The last equality holds by the very definition of the ε tensor norm. This completes the proof. ∎

We are also going to interpolate ℓ_p spaces with the complex method. It is well known (see [6, Theorem 5.1.1]) that

$$[\ell_{p_1}, \ell_{p_2}]_\theta = \ell_r,$$

where $\frac{1}{r} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Also, [6, Theorem 5.1.2] gives

$$[\ell_{p_1}(E), \ell_{p_2}(F)]_\theta = [\ell_{p_1}, \ell_{p_2}]_\theta([E,F]_\theta).$$

These two together give finally

$$[\ell_{p_1}(\ell_{q_1}), \ell_{p_2}(\ell_{q_2})]_\theta = \ell_r(\ell_s) \tag{36}$$

with $\frac{1}{r} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $\frac{1}{s} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$.

We are now ready to give the proof of Bohnenblust-Hille inequality in its reformulation (32). We proceed by induction. The case $m = 1$ in Proposition 2.5 gives $\pi_1(\mathrm{id} : \ell_1^n \hookrightarrow \ell_2^n) \leqslant \sqrt{2}$ for every n (this is the inequality due to Littlewood and Orlicz that was used in Section 2.1.1). By Theorem 2.4, for every n,

$$\| \mathrm{id} : \ell_1^n \otimes_\varepsilon \ell_1^n \longrightarrow \ell_1^n \otimes_1 \ell_2^n = \ell_1^n(\ell_2^n) \| \leqslant \sqrt{2}. \tag{37}$$

On the other hand, we consider the transposition operator t given by $e_i \otimes e_j \rightsquigarrow e_j \otimes e_i$. By the metric mapping property $\|t : \ell_1^n \otimes_\varepsilon \ell_1^n \to \ell_1^n \otimes_\varepsilon \ell_1^n\| = 1$ and by the integral Minkowski inequality $\|t : \ell_1^n(\ell_2^n) \to \ell_2^n(\ell_1^n)\| \leqslant 1$. Then we compose to get

$$\begin{array}{ccccccc}
\ell_1^n \otimes_\varepsilon \ell_1^n & \xrightarrow[\| \ \|=1]{t} & \ell_1^n \otimes_\varepsilon \ell_1^n & \xrightarrow[\| \ \| \leqslant \sqrt{2}]{\mathrm{id}} & \ell_1^n(\ell_2^n) & \xrightarrow[\| \ \| \leqslant 1]{t} & \ell_2^n(\ell_1^n) \\
e_i \otimes e_j & \rightsquigarrow & e_j \otimes e_i & \rightsquigarrow & e_j \otimes e_i & \rightsquigarrow & e_i \otimes e_j
\end{array}$$

Hence

$$\| \mathrm{id} : \ell_1^n \otimes_\varepsilon \ell_1^n \longrightarrow \ell_2^n(\ell_1^n) \| \leqslant \sqrt{2} \tag{38}$$

From (37) and (38) we can interpolate with the complex method to get that for every $0 < \theta < 1$ and

$$\| \mathrm{id} : \ell_1^n \otimes_\varepsilon \ell_1^n \longrightarrow [\ell_1^n(\ell_2^n), \ell_2^n(\ell_1^n)]_\theta \| \leqslant \sqrt{2}$$

By (36), $[\ell_1^n(\ell_2^n), \ell_2^n(\ell_1^n)]_\theta = \ell_r^n(\ell_s^n)$. We want $r = s$, hence $\frac{\theta}{1} + \frac{1-\theta}{2} = \frac{\theta}{2} + \frac{1-\theta}{2}$. This gives $\theta = 1/2$ and $r = s = 4/3$; that is

$$\| \mathrm{id} : \ell_1^n \otimes_\varepsilon \ell_1^n \longrightarrow \ell_{4/3}^{n^2} \| \leqslant \sqrt{2} \qquad \text{for every } n.$$

This is (31) and therefore a proof of the Littlewood's inequality.

We assume now $\mathrm{id} : \|\ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n \to \ell_{2(m-1)/m}^{n^{m-1}}\| \leqslant 2^{\frac{m-2}{2}}$ and proceed by induction.

On the one hand, by Lemma 2.5, the identity mapping is absolutely 1-summing. Then by Theorem 2.4

$$\Big\| \mathrm{id} = \mathrm{id} \otimes \mathrm{id} : \ell_1^n \otimes_\varepsilon \big(\ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n\big) \longrightarrow \ell_1^n\big(\ell_2^{n^{m-1}}\big) \Big\| \leqslant 2^{\frac{m-1}{2}}. \tag{39}$$

On the other hand, we know that $\pi_p(\mathrm{id} : \ell_1 \hookrightarrow \ell_2) \leqslant \sqrt{2}$ for every p; in particular for $p = \frac{2(m-1)}{m}$. We now block $\ell_1^n \otimes_\varepsilon \big(\ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n\big)$ and transpose blockwise in the following way

$$\begin{tikzcd}
\ell_1^n \otimes_\varepsilon \big(\ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n\big) \arrow[d, swap, "\text{transposition}"] & e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m} \arrow[d, "\|\ \| \leqslant 1"] \\
\big(\ell_1^n \otimes_\varepsilon \overset{m-1}{\cdots} \otimes_\varepsilon \ell_1^n\big) \otimes_\varepsilon \ell_1^n \arrow[d, swap, "\text{induction}"] & e_{i_2} \otimes \cdots \otimes e_{i_m} \otimes e_{i_1} \arrow[d, "\|\ \| \leqslant 2^{\frac{m-2}{2}}"] \\
\ell^{n^{m-1}}_{\frac{2(m-1)}{m}} \otimes_\varepsilon \ell_1^n \arrow[d, swap, "\frac{2(m-1)}{m}\text{-summing}"] & e_{i_2} \otimes \cdots \otimes e_{i_m} \otimes e_{i_1} \arrow[d, "\|\ \| \leqslant \sqrt{2}"] \\
\ell^{n^{m-1}}_{\frac{2(m-1)}{m}}\big(\ell_2^n\big) \arrow[d, swap, "\text{transposition}"] & e_{i_2} \otimes \cdots \otimes e_{i_m} \otimes e_{i_1} \arrow[d, "\|\ \| \leqslant 1 \text{ (Minkowski)}"] \\
\ell_2^n\big(\ell^{n^{m-1}}_{\frac{2(m-1)}{m}}\big) & e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m}
\end{tikzcd}$$

This means

$$\Big\| \mathrm{id} : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \longrightarrow \ell_2^n\big(\ell^{n^{m-1}}_{\frac{2(m-1)}{m}}\big) \Big\| \leqslant 2^{\frac{m-1}{2}}. \tag{40}$$

We again interpolate with the complex method from (39) and (40) and get

$$\Big\| \mathrm{id} : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \longrightarrow \big[\ell_1^n\big(\ell_2^{n^{m-1}}\big), \ell_2^n\big(\ell^{n^{m-1}}_{\frac{2(m-1)}{m}}\big)\big]_\theta = \ell_r^n(\ell_s^{n^{m-1}}) \Big\| \leqslant 2^{\frac{m-1}{2}}.$$

Again we want $r = s$, then

$$\begin{aligned}
\frac{1}{r} &= \frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{2} + \frac{\theta}{2} \\
\frac{1}{s} &= \frac{\theta}{2} + \frac{1-\theta}{\frac{2(m-1)}{m}} = \frac{\theta}{2} + \frac{(1-\theta)m}{2(m-1)} \\
&= \frac{1}{2}\Big(\frac{\theta m - \theta + m - \theta m}{m-1}\Big) = \frac{m}{2(m-1)} - \frac{\theta}{2(m-1)}.
\end{aligned}$$

Then $1 + \theta = \frac{m-\theta}{m-1}$ and $\theta = \frac{1}{m}$. This gives $r = s = \frac{2m}{m+1}$ and finally

$$\Bigg\| \mathrm{id} : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \longrightarrow \ell^{n^m}_{\frac{2m}{m+1}} \Bigg\| \leqslant 2^{\frac{m-1}{2}}.$$

This is (32) and finishes the proof of the Bohnenblust-Hille inequality. Let us note that with this proof we get the constant $C_m = 2^{\frac{m-1}{2}}$.

2.1.3. A variant of the original proof

An improvement of Lemma 1.1 allows to give a slightly different proof of Theorem I, improving the original constant of Bohnenblust and Hille and giving again $C_m = 2^{\frac{m-1}{2}}$. In [7, Lemma 5.3] we find it. Several variants of this inequality can be found in [34, 32, 78, 72].

Lemma 2.6 (Blei's inequality). *For $m,n \in \mathbb{N}$ fixed and every matrix $(a_{i_1,\dots,i_m})_{1\leqslant 1_1,\dots,i_m\leqslant n}$ we have*

$$\sum_{i_1,\dots,i_m=1}^{n} |a_{i_1,\dots i_m}|^{\frac{2m}{m+1}} \leqslant \prod_{k=1}^{m} \Big(\sum_{i_k=1}^{n} \Big(\sum_{\sim i_k} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{1}{2}} \Big)^{\frac{2}{m+1}}.$$

In the notation of Bohnenblust and Hille, $S^\rho \leqslant \prod_{k=1}^m T^{(k)\frac{2}{m+1}}$.

Proof. In order to keep the notation as clear as possible we write $|a_{i_1,\dots i_m}| = f_k(i_1,\dots,i_m)$ and apply the m-fold Hölder inequality in the i_m sum with exponents $p_1 = \frac{m+1}{2}$ and $p_2 = \dots p_m = m+1$.

$$\begin{aligned}
\sum_{i_1,\dots,i_m=1}^{n} |a_{i_1,\dots i_m}|^{\frac{2m}{m+1}} &= \sum_{i_1,\dots,i_m=1}^{n} |a_{i_1,\dots i_m}|^{\frac{2}{m+1}} \cdots |a_{i_1,\dots i_m}|^{\frac{2}{m+1}} \\
&= \sum_{i_1,\dots,i_m=1}^{n} f_1(i_1,\dots,i_m)^{\frac{2}{m+1}} \cdots f_m(i_1,\dots,i_m)^{\frac{2}{m+1}} \\
&= \sum_{i_1,\dots,i_{m-1}=1}^{n} \Big[\Big(\sum_{i_m=1}^{n} (f_1^{\frac{2}{m+1}})^{\frac{m+1}{2}} \Big)^{\frac{2}{m+1}} \prod_{k=2}^{m} \Big(\sum_{i_m=1}^{n} (f_k^{\frac{2}{m+1}})^{m+1} \Big)^{\frac{1}{m+1}} \Big] \\
&= \sum_{i_1,\dots,i_{m-1}=1}^{n} \Big[\Big(\sum_{i_m=1}^{n} f_1 \Big)^{\frac{2}{m+1}} \prod_{k=2}^{m} \Big(\sum_{i_m=1}^{n} f_k^2 \Big)^{\frac{1}{m+1}} \Big]
\end{aligned}$$

We use now the m-fold Hölder inequality in the i_{m-1} sum with $p_2 = \frac{m+1}{2}$ and $p_1 = p_3 = \dots = p_m = m+1$. We write $\alpha_1 = \big(\sum_{i_m=1}^n f_1 \big)^{\frac{2}{m+1}}$ and $\alpha_k = \big(\sum_{i_m=1}^n f_k^2 \big)^{\frac{1}{m+1}}$ for $k = 2,\dots,m$. Then we have

$$\begin{aligned}
\sum_{i_1,\dots i_{m-2}=1}^{n} \sum_{i_{m-1}=1}^{n} \alpha_1\alpha_2\cdots\alpha_m \leqslant & \sum_{i_1,\dots i_{m-2}=1}^{n} \Big(\sum_{i_{m-1}=1}^{n} \alpha_1^{m+1} \Big)^{\frac{1}{m+1}} \\
& \times \Big(\sum_{i_{m-1}=1}^{n} \alpha_2^{\frac{m+1}{2}} \Big)^{\frac{2}{m+1}} \cdots \Big(\sum_{i_{m-1}=1}^{n} \alpha_m^{m+1} \Big)^{\frac{1}{m+1}}
\end{aligned}$$

But $\alpha_1^{m+1} = \big(\sum_{i_m=1}^n f_1\big)^2$, $\alpha_2^{\frac{m+1}{2}} = \big(\sum_{i_m=1}^n f_2^2\big)^{\frac{1}{2}}$ and $\alpha_k^{m+1} = \big(\sum_{i_m=1}^n f_k^2\big)$ for $k = 3, \dots, m$; then

$$\begin{aligned}
&\sum_{i_1,\dots,i_m=1}^n |a_{i_1,\dots i_m}|^{\frac{2m}{m+1}} \\
&\qquad \leqslant \sum_{i_1,\dots,i_{m-1}=1}^n \left[\Big(\sum_{i_m=1}^n f_1 \Big)^{\frac{2}{m+1}} \Big(\sum_{i_m=1}^n f_2^2 \Big)^{\frac{1}{m+1}} \prod_{k=3}^m \Big(\sum_{i_m=1}^n f_k^2 \Big)^{\frac{1}{m+1}} \right] \\
&\qquad \leqslant \sum_{i_1,\dots,i_{m-2}=1}^n \left[\Big(\sum_{i_{m-1}=1}^n \Big(\sum_{i_m=1}^n f_1 \Big)^2 \Big)^{\frac{1}{m+1}} \Big(\sum_{i_{m-1}=1}^n \Big(\sum_{i_m=1}^n f_2^2 \Big)^{\frac{1}{2}} \Big)^{\frac{2}{m+1}} \right. \\
&\qquad\qquad\qquad \left. \times \prod_{k=3}^m \Big(\sum_{i_{m-1}=1}^n \sum_{i_m=1}^n f_k^2 \Big)^{\frac{1}{m+1}} \right]
\end{aligned}$$

We repeat this procedure m times, using in the k-th step the m-fold Hölder inequality for the sum over i_{m-k+1} with $p_k = \frac{m+1}{2}$ and $p_j = m+1$ for $j \neq k$. We finally obtain

$$\begin{aligned}
\sum_{i_1,\dots,i_m=1}^n |a_{i_1,\dots i_m}|^{\frac{2m}{m+1}} &\leqslant \Big(\sum_{i_1,\dots,i_{m-1}=1}^n \Big(\sum_{i_m=1}^n f_1 \Big)^2 \Big)^{\frac{1}{m+1}} \\
&\quad \times \dots \times \left[\sum_{i_1,\dots,i_{m-k}=1}^n \Big(\sum_{i_{m-k+1}=1}^n \Big(\sum_{i_{m-k+2},\dots,i_m=1}^n f_k^2 \Big)^{\frac{1}{2}} \Big)^2 \right]^{\frac{1}{m+1}} \\
&\quad \times \dots \times \Big(\sum_{i_1=1}^n \Big(\sum_{i_2,\dots,i_m=1}^n f_m^2 \Big)^{\frac{1}{2}} \Big)^{\frac{2}{m+1}}
\end{aligned}$$

We apply now the integral Minkowski inequality in each term. For the first one we take $\int_Y = \sum_{i_1,\dots,i_{m-1}}$, $\int_X = \sum_{i_m}$, $f = f_1$ and $r = 2$ to get

$$\begin{aligned}
\Big(\sum_{i_1,\dots,i_{m-1}=1}^n \Big(\sum_{i_m=1}^n f_1 \Big)^2 \Big)^{\frac{2}{m+1}} &= \left[\Big(\sum_{i_1,\dots,i_{m-1}=1}^n \Big(\sum_{i_m=1}^n f_1 \Big)^2 \Big)^{\frac{1}{2}} \right]^{\frac{1}{m+1}} \\
&\leqslant \Big(\sum_{i_m=1}^n \Big(\sum_{i_1,\dots,i_{m-1}=1}^n f_1^2 \Big)^{\frac{1}{2}} \Big)^{\frac{2}{m+1}}.
\end{aligned}$$

For the intermediate factors ($k = 2, \dots, m-1$) we take $\int_Y = \sum_{i_1,\dots,i_{m-k}}$, $\int_X = \sum_{i_{m-k+1}}$, $f = \big(\sum_{i_{m-k+2},\dots,i_m} f_k^2\big)^{\frac{1}{2}}$ and $r = 2$; this gives

$$\Big[\Big(\sum_{i_1,\dots,i_{m-k}=1}^{n}\Big(\sum_{i_{m-k+1}=1}^{n}\Big(\sum_{i_{m-k+2},\dots,i_m=1}^{n} f_k^2\Big)^{\frac12}\Big)^2\Big)^{\frac12}\Big]^{\frac{2}{m+1}}$$

$$\leqslant\Big[\sum_{i_{m-k+1}=1}^{n}\Big(\sum_{i_1,\dots,i_{m-k}=1}^{n}\Big[\Big(\sum_{i_{m-k+2},\dots,i_m} f_k^2\Big)^{\frac12}\Big]^2\Big)^{\frac12}\Big]^{\frac{2}{m+1}}$$

$$=\Big[\sum_{i_{m-k+1}=1}^{n}\Big(\sum_{i_1,\dots,i_{m-k},i_{m-k+2},\dots,i_m} f_k^2\Big)^{\frac12}\Big]^{\frac{2}{m+1}}.$$

The m-th factor is already of this form. This altogether gives

$$\sum_{i_1,\dots,i_m=1}^{n}|a_{i_1,\dots i_m}|^{\frac{2m}{m+1}}\leqslant\prod_{k=1}^{m}\Big(\sum_{i_k=1}^{n}\Big(\sum_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_m}|a_{i_1,\dots i_m}|^2\Big)^{\frac12}\Big)^{\frac{2}{m+1}}$$

and completes the proof. ■

Proof of Theorem I. Using Lemma 2.6 and (14) we have

$$\big(S^\rho\big)^{1/\rho}\leqslant\Big(\prod_{k=1}^{m}T^{(k)\frac{2}{m+1}}\Big)^{1/\rho}\leqslant\Big(\prod_{k=1}^{m}\big(2^{\frac{m-1}{2}}\|L\|\big)^{\frac{2}{m+1}}\Big)^{\frac{m+1}{2m}}$$

$$=\Big(\big(2^{\frac{m-1}{2}}\|L\|\big)^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}}=2^{\frac{m-1}{2}}\|L\|.$$

This yields Theorem I with $C_m=2^{\frac{m-1}{2}}$. ■

2.1.4. A proof by induction

We present now a more or less direct proof of Theorem I by induction on m. Let us remark that the case $m=1$ follows easily from the Khintchine inequality and the case $m=2$ is Littlewood's 4/3-inequality. We suppose that the result holds for $m-1$.

We apply Hölder inequality with $p=\frac{m+1}{m-1}$ and $q=\frac{m+1}{2}$.

$$\sum_{i_1,\dots,i_m=1}^{n}|a_{i_1,\dots i_m}|^{\frac{2m}{m+1}}=\sum_{i_1,\dots,i_{m-1}=1}^{n}\sum_{i_m=1}^{n}|a_{i_1,\dots i_m}|^{\frac{2(m-1)}{m+1}}|a_{i_1,\dots i_m}|^{\frac{2}{m+1}}$$

$$\leqslant\sum_{i_1,\dots,i_{m-1}=1}^{n}\Big(\sum_{i_m=1}^{n}|a_{i_1,\dots i_m}|^{\frac{2(m-1)}{m+1}\frac{m+1}{m-1}}\Big)^{\frac{m-1}{m+1}}$$

$$\times\Big(\sum_{i_m=1}^{n}|a_{i_1,\dots i_m}|^{\frac{2}{m+1}\frac{m+1}{2}}\Big)^{\frac{2}{m+1}}$$

$$=\sum_{i_1,\dots,i_{m-1}=1}^{n}\Big(\sum_{i_m=1}^{n}|a_{i_1,\dots i_m}|^2\Big)^{\frac{m-1}{m+1}}\Big(\sum_{i_m=1}^{n}|a_{i_1,\dots i_m}|\Big)^{\frac{2}{m+1}}.$$

We apply again Hölder inequality, now with $p = \frac{m+1}{m}$ and $q = m+1$ to get

$$\begin{aligned}
\sum_{i_1,\dots,i_m=1}^{n} |a_{i_1,\dots i_m}|^{\frac{2m}{m+1}} &\leqslant \sum_{i_1,\dots,i_{m-1}=1}^{n} \Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{m-1}{m+1}} \Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}| \Big)^{\frac{2}{m+1}} \\
&\leqslant \Bigg(\sum_{i_1,\dots,i_{m-1}=1}^{n} \Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{m-1}{m+1}\frac{m+1}{m}} \Bigg)^{\frac{m}{m+1}} \\
&\quad \times \Bigg(\sum_{i_1,\dots,i_{m-1}=1}^{n} \Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}| \Big)^{\frac{2}{m+1}(m+1)} \Bigg)^{\frac{1}{m+1}} \\
&= \Bigg(\sum_{i_1,\dots,i_{m-1}=1}^{n} \Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{m-1}{m}} \Bigg)^{\frac{m}{m+1}} \\
&\quad \times \Bigg(\sum_{i_1,\dots,i_{m-1}=1}^{n} \Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}| \Big)^{2} \Bigg)^{\frac{1}{m+1}}.
\end{aligned}$$

We use the integral Minkowski inequality to bound the second term:

$$\Bigg(\sum_{i_1,\dots,i_{m-1}=1}^{n} \Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}| \Big)^{2} \Bigg)^{\frac{1}{2}\frac{2}{m+1}} \leqslant \Bigg(\sum_{i_m=1}^{n} \Big(\sum_{i_1,\dots,i_{m-1}=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{1}{2}} \Bigg)^{\frac{2}{m+1}}.$$

Then we have

$$\begin{aligned}
&\Bigg(\sum_{i_1,\dots,i_m=1}^{n} |a_{i_1,\dots i_m}|^{\frac{2m}{m+1}} \Bigg)^{\frac{m+1}{2m}} \\
&\quad \leqslant \Bigg(\sum_{i_1,\dots,i_{m-1}=1}^{n} \Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{m-1}{m}} \Bigg)^{\frac{1}{2}} \Bigg(\sum_{i_m=1}^{n} \Big(\sum_{i_1,\dots,i_{m-1}=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{1}{2}} \Bigg)^{\frac{1}{m}} \\
&\quad \leqslant \Bigg[\Bigg(\sum_{i_1,\dots,i_{m-1}=1}^{n} \bigg[\Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{1}{2}} \bigg]^{\frac{2(m-1)}{m}} \Bigg)^{\frac{m}{2(m-1)}} \Bigg]^{\frac{m-1}{m}} \\
&\quad \quad \times \Bigg(\sum_{i_m=1}^{n} \Big(\sum_{i_1,\dots,i_{m-1}=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{1}{2}} \Bigg)^{\frac{1}{m}}
\end{aligned} \tag{41}$$

We bound each term separately. For the first term we use Khintchine inequality

$$\begin{aligned}
&\Big[\Big(\sum_{i_m} |a_{i_1,\dots i_m}|^2 \Big)^{1/2} \Big]^{\frac{2(m-1)}{m}} \\
&\qquad \leqslant A^{-1}_{\frac{2(m-1)}{m}} \Bigg(\int \Big| \sum_{i_m} a_{i_1,\dots i_m} \varepsilon_{i_m}(\omega) \Big|^{\frac{2(m-1)}{m}} d\omega \Bigg)^{\frac{m}{2(m-1)} \frac{2(m-1)}{m}}
\end{aligned}$$

and the induction hypothesis

$$\begin{aligned}
&\Big(\sum_{i_1,\dots,i_{m-1}=1}^{n} \Big[\Big(\sum_{i_m=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{1}{2}} \Big]^{\frac{2(m-1)}{m}} \Big)^{\frac{m}{2(m-1)}} \\
&\quad \leqslant A_{\frac{2(m-1)}{m}}^{-1} \Big(\sum_{i_1,\dots,i_{m-1}=1}^{n} \int \Big| \sum_{i_m=1}^{n} a_{i_1,\dots i_m} \varepsilon_{i_m}(\omega) \Big|^{\frac{2(m-1)}{m}} d\omega \Big)^{\frac{m}{2(m-1)}} \\
&\quad = A_{\frac{2(m-1)}{m}}^{-1} \Big(\int \sum_{i_1,\dots,i_{m-1}=1}^{n} \Big| \sum_{i_m=1}^{n} a_{i_1,\dots i_m} \varepsilon_{i_m}(\omega) \Big|^{\frac{2(m-1)}{m}} d\omega \Big)^{\frac{m}{2(m-1)}} \\
&\quad \leqslant A_{\frac{2(m-1)}{m}}^{-1} C_{m-1} \Big(\int \Big(\sup_{\lambda^{(j)} \in B_{\ell_\infty^n}} \Big| \sum_{i_1,\dots,i_{m-1}=1}^{n} \sum_{i_m=1}^{n} a_{i_1,\dots i_m} \varepsilon_{i_m}(\omega) \\
&\qquad\qquad\qquad\qquad \times \lambda_{i_1}^{(1)} \cdots \lambda_{i_{m-1}}^{(m-1)} \Big| \Big)^{\frac{2(m-1)}{m}} d\omega \Big)^{\frac{m}{2(m-1)}} \\
&\quad \leqslant A_{\frac{2(m-1)}{m}}^{-1} C_{m-1} \Big(\int \Big(\sup_{\mu^{(m)} \in B_{\ell_\infty^n}} \sup_{\lambda^{(j)} \in B_{\ell_\infty^n}} \Big| \sum_{i_1,\dots,i_m=1}^{n} a_{i_1,\dots i_m} \mu_{i_m} \\
&\qquad\qquad\qquad\qquad \times \lambda_{i_1}^{(1)} \cdots \lambda_{i_{m-1}}^{(m-1)} \Big| \Big)^{\frac{2(m-1)}{m}} d\omega \Big)^{\frac{m}{2(m-1)}} \\
&\quad = A_{\frac{2(m-1)}{m}}^{-1} C_{m-1} \sup_{\mu^{(m)} \in B_{\ell_\infty^n}} \sup_{\lambda^{(j)} \in B_{\ell_\infty^n}} \Big| \sum_{i_1,\dots,i_m=1}^{n} a_{i_1,\dots i_m} \mu_{i_m} \lambda_{i_1}^{(1)} \cdots \lambda_{i_{m-1}}^{(m-1)} \Big| \\
&\quad = A_{\frac{2(m-1)}{m}}^{-1} C_{m-1} \|L\|.
\end{aligned}$$

To bound the second term in (41) we basically repeat the same calculation as in Lemma 2.5, using the multiple Khintchine inequality, to get

$$\begin{aligned}
\sum_{i_m=1}^{n} \Big(\sum_{i_1,\dots,i_{m-1}=1}^{n} |a_{i_1,\dots i_m}|^2 \Big)^{\frac{1}{2}} &\leqslant (\sqrt{2})^{m-1} \sup_{\lambda^{(j)} \in B_{\ell_\infty^n}} \Big| \sum_{i_1,\dots,i_m=1}^{n} a_{i_1,\dots i_m} \lambda_{i_1}^{(1)} \cdots \lambda_{i_m}^{(m)} \Big| \\
&= (\sqrt{2})^{m-1} \|L\|.
\end{aligned}$$

We use these two estimates in (41) and finally get

$$\begin{aligned}
\Big(\sum_{i_1,\dots,i_m=1}^{n} |a_{i_1,\dots i_m}|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} &\leqslant \big(A_{\frac{2(m-1)}{m}}^{-1} C_{m-1} \|L\| \big)^{\frac{m-1}{m}} \big((\sqrt{2})^{m-1} \|L\| \big)^{\frac{1}{m}} \\
&= C_m \|L\|.
\end{aligned}$$

This completes (once again) the proof of Theorem I. ■

2.1.5. Multiple summing operators

There are several generalisations of summing operators to the setting of multilinear mappings. One of them is the concept of multiple summing multilinear mappings, independently introduced by Bombal–Pérez García–Villanueva [16, Definition 2.1] and Matos [61, Definition 2.2]. Given Banach spaces $X_1, \dots, X_m, Y$ an m-linear mapping $A : X_1 \times \cdots \times X_m \to Y$ is said to be multiple $(r,1)$-summing if there exist a constant $\kappa > 0$ such that for every finite choice of vectors $\big(x_{i_j}^j\big)_{i_j=1}^{N_j}$ in X_j for $j = 1, \dots, m$ we have

$$\Big(\sum_{i_1,\dots,i_m=1}^{N_1,\dots,N_m} \|A(x_{i_1}^1, \dots, x_{i_m}^m)\|^r \Big)^{\frac{1}{r}} \leqslant \kappa \prod_{j=1}^{m} \sup_{x^* \in X_j^*} \sum_{i_j=1}^{N_j} \big|x^*(x_{i_j}^j)\big|.$$

The best constant in this inequality is denoted by $\pi_{(r,1)}^{mult}(A)$ and is a norm that makes the space of all multiple $(r,1)$-summing m-linear from $X_1 \times \cdots \times X_m$ to Y (denoted $\Pi_{(r,1)}^{mult}(X_1, \dots, X_m; Y)$) a Banach space.

Given finitely many $x_1, \dots, x_N \in X$, the operator $T : \ell_\infty^N \to X$ given by $T(e_i) = x_i$ satisfies that $\|T\| = \sup_{x^* \in X^*} \sum_{i=1}^N \big|x^*(x_i)\big|$. Keeping this in mind it is easy to see that the Bohnenblust–Hille inequality (Theorem I) can be re-stated as

Theorem 2.7. *Every m-linear form $A : X_1 \times \cdots \times X_m \to \mathbb{C}$ is multiple $\big(\frac{2m}{m+1}, 1\big)$-summing.*

This path was further explored by Defant, Popa and Schwarting in [34] (see also [78, 72]) in the following way: take $C \subseteq \{1, \dots, m\}$ and let $\complement C = \{1, \dots, m\} \setminus C$. Now, given Banach spaces $X_1, \dots, X_m$ and Y, for $x \in \prod_{j \in C} X_j$ we define $\tilde{x} \in X_1 \times \cdots \times X_m$ by

$$\tilde{x}_j = \begin{cases} x_j & \text{if } j \in C \\ 0 & \text{if } j \in \complement C \end{cases}$$

Then an m-linear mapping $A : X_1 \times \cdots \times X_m \to Y$ is said to be multiple $(r,1)$-summing in the coordinates of C if the mapping

$$A^C : \prod_{j \in \complement C} X_j \to \Pi_{(r,1)}^{mult}\Big(\prod_{j \in C} X_j ; Y \Big); \qquad x \rightsquigarrow [y \rightsquigarrow A(\tilde{x}, \tilde{y})]$$

is well defined (and then, by a closed graph argument, continuous).

Let us note that for a fixed $x \in \prod_{j \in \complement C} X_j$ the mapping $A^C x$ is just the multilinear mapping obtained by restricting A to the coordinates of C by fixing those of $\complement C$ through x.

For $q \geqslant 2$, two functions $\omega, f : [1, q[\times [1, q[\to \mathbb{R}_{\geqslant 0}$ are introduced in [34]:

$$\omega(x,y) = \frac{q^2(x+y) - 2qxy}{q^2 - xy}, \qquad f(x,y) = \frac{q^2 x - qxy}{q^2(x+y) - 2qxy}.$$

With this notation we have we have [34, Theorem 4.1] (see also [78, Corollary 2.6]).

Theorem 2.8. *Let Y be a cotype q space*[9]. *If $A : X_1 \times \cdots \times X_m \to Y$ is multiple $(r_1,1)$-summing on the coordinates of C_1 and multiple $(r_2,1)$-summing on the coordinates of C_2, then A is multiple $(\omega(r_1,r_2),1)$-summing and*

$$\pi^{mult}_{(\omega(r_1,r_2),1)}(A) \leqslant \sigma(r_1,r_2) \Big\| A^{C_2} : \prod_{j \in C_1} X_j \to \Pi^{mult}_{(r_2,1)}\Big(\prod\nolimits_{j\in C_2} X_j ; Y\Big)\Big\|^{f(r_2,r_1)}$$
$$\times \Big\| A^{C_1} : \prod_{j \in C_2} X_j \to \Pi^{mult}_{(r_1,1)}\Big(\prod\nolimits_{j\in C_1} X_j ; Y\Big)\Big\|^{f(r_1,r_2)}$$

where $\sigma(r_1,r_2) = \big(C_q(Y)^{|C_1|} K_{q,r_1}^{|C_1|}\big)^{f(r_1,r_2)} \big(C_q(Y)^{|C_2|} K_{q,r_2}^{|C_2|}\big)^{f(r_2,r_1)}$ and $K_{u,v}$ is the best constant in Kahane's inequality [41, 11.1].

We can now proceed by induction to define $\omega_k : [1,q[^k \to \mathbb{R}_{\geqslant 0}$ and $f_k = (f_k^1, \dots, f_k^k) : [1,q[^k \to \mathbb{R}^k_{\geqslant 0}$ by doing $\omega_1(r_1) = r_1$, $\omega_2(r_1,r_2) = \omega(r_1,r_2)$ and $\omega_k(r_1,\dots,r_k) = \omega(r_k, \omega_{k-1}(r_1,\dots,r_{k-1}))$ for $k \geqslant 3$. Also $f_2(r_1,r_2) = (f(r_1,r_2), f(r_2,r_1))$ and for $k \geqslant 3$ we consider

$$f_k^j(r_1,\dots,r_k) = f_{k-1}^j(r_1,\dots,r_{k-1}) \cdot f(\omega_{k-1}(r_1,\dots,r_{k-1}), r_r)$$

for $j = 1,\dots,k-1$ and $f_k^k(r_1,\dots,r_k) = f(r_r, \omega_{k-1}(r_1,\dots,r_{k-1}))$. This is how these numbers are defined in [34]. It can be shown by induction that, taking $R = \sum_{j=1}^k \frac{r_j}{q-r_j}$, we have

$$\omega_k(r_1,\dots,r_k) = \frac{qR}{1+R} \qquad \text{and} \qquad f_k^j(r_1,\dots,r_k) = \frac{r_j}{R(q-r_j)}.$$

This reformulation was introduced in [72]. With this we can state what is probably the most general result in this field [34, Theorem 5.1].

Theorem 2.9. *Let $\{1,\dots,m\}$ be a disjoint union of k non-void sets C_j, Y a cotype q space and $1 \leqslant r_1,\dots,r_k < q$. If $A : X_1 \times \cdots \times X_m \to Y$ is multiple $(r_j,1)$-summing in each set of coordinates C_j then A is multiple $(\omega_k(r_1,\dots,r_k),1)$-summing and*

$$\pi^{mult}_{(\omega_k(r_1,\dots,r_k),1)}(A) \leqslant \sigma_k(r_1,\dots,r_k) \prod_{j=1}^k$$
$$\times \Big\| A^{C_j} : \prod_{i \in \complement C_j} X_j \to \Pi^{mult}_{(r_j,1)}\Big(\prod\nolimits_{i\in C_j} X_i ; Y\Big)\Big\|^{f_k^j(r_1,\dots,r_k)}$$

where $\sigma_k(r_1,\dots,r_k)$ only depends on k, $|C_1|$, $\dots$, $|C_k|$, $r_1,\dots,r_k$, q and $C_q(Y)$.

[9] A Banach space X has cotype q (with $2 \leqslant q < \infty$) if there exists a constant $\kappa \geqslant 1$ such that for every finite choice of vectors $x_1,\dots,x_n \in X$, $\Big(\sum_{j=1} \|x_j\|^q\Big)^{1/q} \leqslant \kappa \Big(\int \Big\| \sum_{j=1} \varepsilon_j(\omega) x_j \Big\|^2 d\omega\Big)^{1/2}$, where the ε's are independent indentically distributed Rademacher random variables, see [41, Chapter 11]. The best constant in this inequality is denoted by $C_q(X)$.

Let us see now how Theorem 2.7 follows from Theorem 2.9. Take $C_j = \{j\}$ for $j = 1, \dots, m$ and $Y = \mathbb{C}$ (then $q = 2$ and $r_1 = \ldots = r_m = 1$ since every functional is absolutely summing). We have $R = m$, $\omega_m(r_1, \dots, r_m) = \frac{2m}{m+1}$ and $f_k^j(r_1, \dots, r_k) = \frac{1}{k}$; then Theorem 2.9 gives that for every m-linear form $A : X_1 \times \cdots \times X_m \to \mathbb{C}$,

$$\pi^{mult}_{(\omega_k(r_1,\dots,r_k),1)}(A) \leqslant \sigma_m \|A\|.$$

Similar variants in this direction have been recently obtained in [72].

2.1.6. The constants

It is also important to have a good control of the constant appearing in the Bohnenblust-Hille inequality in Theorem I. If we denote by $\mathrm{B}_m^{\mathrm{mult}}$ the best constant in the inequality we have that the original proof of Bohnenblust and Hille shows that $\mathrm{B}_m^{\mathrm{mult}} \leqslant m^{\frac{m+1}{2}} 2^{\frac{m-1}{2}}$ and the proof of Kaijser (Sections 2.1.1 and 2.1.2) gives $\mathrm{B}_m^{\mathrm{mult}} \leqslant 2^{\frac{m-1}{2}}$. The $\sqrt{2}$ factor comes from the constant in the Khintchine's inequality (7) that is used in (14) and in Lemma 2.6; hence a smaller factor can come from considering a similar inequality with a smaller constant. Such an inequality was obtained by Sawa in [77]. Instead of using Rademacher random varibles, Sawa uses Steinhaus random varibles. These are complex valued random variables whose values are uniformly distributed on the unit circle of $\mathbb{C}$ (i.e. they are random variables with distribution function $t \rightsquigarrow e^{it}$). Then Sawa shows that for each $1 \leqslant p < \infty$ there is a constant $\mathfrak{S}_p$ such that if $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $s_1, \dots, s_n$ are independent equally distributed Steinhaus random variables then

$$\Big(\sum_{i=1}^n |\alpha_i|^2 \Big)^{1/2} \leqslant \mathfrak{S}_p \Big(\int \big| \sum_{i=1}^n \alpha_i s_i(\omega) \big|^p d\omega \Big)^{\frac{1}{p}}. \tag{42}$$

The best constants in this inequality are known to be

$$\mathfrak{S}_p = \begin{cases} \frac{2}{\sqrt{\pi}} & \text{if } p = 1 \quad [77, \text{Theorem A}] \\ \Gamma\big(\frac{p+2}{2}\big)^{-\frac{1}{p}} & \text{if } 1 < p < 2 \quad [56, 1] \\ 1 & \text{if } 2 \leqslant p < \infty \quad \text{(obvious)} \end{cases} \tag{43}$$

A multilinear version of (42) like (13) follows immediately and it can be used in (14) or Lemma 2.5 to get that $\pi_1(\mathrm{id} : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \longrightarrow \ell_2^{n^m}) \leqslant (\frac{2}{\sqrt{\pi}})^m$. Then, proceeding as in Section 2.1.2 or Section 2.1.3 get $\mathrm{B}_m^{\mathrm{mult}} \leqslant \big(\frac{2}{\sqrt{\pi}}\big)^{m-1}$ (this is essentially done by Queffélec in [74]).

The introduction of separately multiple summing mappings has allowed a substantial improvement of the estimation of $\mathrm{B}_m^{\mathrm{mult}}$. It is easy to see that for every $A : X_1 \times \cdots \times X_m \to \mathbb{C}$ we have

$$\pi^{mult}_{(\frac{2m}{m+1},1)}(A) \leqslant \mathrm{B}_m^{\mathrm{mult}} \|A\|. \tag{44}$$

Pellegrino and Seoane-Sepúlveda in a series of papers with different coauthors [65, 69, 43, 67, 66] show, through a careful analysis and a deep understanding of [34], that $\mathrm{B}_m^{\mathrm{mult}}$ in many respects grows much slower in m as expected. For example, they show that $\lim_m \frac{\mathrm{B}_m^{\mathrm{mult}}}{\mathrm{B}_{m-1}^{\mathrm{mult}}} = 1$, that $\liminf \mathrm{B}_m^{\mathrm{mult}} - \mathrm{B}_{m-1}^{\mathrm{mult}} \leqslant 0$ or that

$$\mathrm{B}_m^{\mathrm{mult}} \leqslant 1.41(m-1)^{0.34975} - 0.04.$$

We present now a proof of the polynomial growth entirely based in Theorem 2.8 that can be found in [78, Theorem 2.7]. First of all, Theorem 2.8 is a vector valued result, in which the Kahane inequalities are used. Since we are in the scalar case, this role is played by (42) and the we get $\mathfrak{S}_r$ instead of the constant $K_{r,2}$ that appears in Theorem 2.8. A result of Bayart [3, Theorem 9] gives that $\mathfrak{S}_r \leqslant \sqrt{\frac{2}{r}}$ (this also follows from (43)). With this we can show

$$\mathrm{B}_m^{\mathrm{mult}} \leqslant \begin{cases} \mathrm{B}_{\frac{m}{2}}^{\mathrm{mult}} \sqrt{\frac{m+2}{m}}^{\frac{m}{2}} & \text{for } m \text{ even} \\ \Big(\mathrm{B}_{\frac{m-1}{2}}^{\mathrm{mult}} \sqrt{\frac{m+1}{m-1}}^{\frac{m+1}{2}} \Big)^{\frac{m-1}{2m}} \Big(\mathrm{B}_{\frac{m+1}{2}}^{\mathrm{mult}} \sqrt{\frac{m+3}{m+1}}^{\frac{m-1}{2}} \Big)^{\frac{m+1}{2m}} & \text{for } m \text{ odd} \end{cases} \tag{45}$$

Indeed, we take first an even m and split $\{1, \ldots, m\}$ into C_1 and C_2, with $|C_1| = |C_2| = \frac{m}{2}$. We know that any scalar valued, m-linear A is multiple $(\frac{2m}{m+2}, 1)$-summing in the coordinates of both C_1 and C_2 (because $\frac{2\frac{m}{2}}{\frac{m}{2}+1} = \frac{2m}{m+2}$). We have

$$\omega\big(\tfrac{2m}{m+2}, \tfrac{2m}{m+2}\big) = \tfrac{2m}{m+1} \qquad \text{and} \qquad f\big(\tfrac{2m}{m+2}, \tfrac{2m}{m+2}\big) = \tfrac{1}{2}.$$

Since $\mathbb{C}$ has cotype 2 (and $C_2(\mathbb{C}) = 1$), Theorem 2.8 and (44) give

$$\begin{aligned} \pi^{mult}_{(\frac{2m}{m+2},1)}(A) &\leqslant \Big[\Big(\sqrt{\frac{2(m+2)}{2m}} \Big)^{\frac{m}{2}} \Big]^{\frac{1}{2}} \Big[\Big(\sqrt{\frac{2(m+2)}{2m}} \Big)^{\frac{m}{2}} \Big]^{\frac{1}{2}} \|A^{C_1}\| \cdot \|A^{C_2}\| \\ &\leqslant \Big(\sqrt{\frac{m+2}{m}} \Big)^{\frac{m}{2}} \mathrm{B}_{\frac{m}{2}}^{\mathrm{mult}} \|A\|. \end{aligned}$$

Using again (44) we get the conclusion.

For m odd we take sets with $|C_1| = \frac{m-1}{2}$ and $|C_2| = \frac{m+1}{2}$. Then $\frac{2\frac{m-1}{2}}{\frac{m-1}{2}+1} = \frac{2m-2}{m+1}$ and $\frac{2\frac{m+1}{2}}{\frac{m+1}{2}+1} = \frac{2m+2}{m+3}$. With this

$$\omega\big(\tfrac{2m-2}{m+1}, \tfrac{2m+2}{m+3}\big) = \tfrac{2m}{m+1}, \quad f\big(\tfrac{2m-2}{m+1}, \tfrac{2m+2}{m+3}\big) = \tfrac{m-1}{2m} \quad \text{and} \quad f\big(\tfrac{2m+2}{m+3}, \tfrac{2m-2}{m+1}\big) = \tfrac{m+1}{2m}.$$

Proceeding as before we finally get the proof of (45).

Theorem 2.10. *There is a universal constant $D > 0$ such that*

$$\mathrm{B}_m^{\mathrm{mult}} \leqslant m^D.$$

Proof. Let us assume first that $m = 2^k$ for some k. Clearly $\left(\frac{m+2}{m}\right)^{\frac{m}{4}}$ is increasing and converges to $\sqrt{e}$; then (45) gives

$$\mathrm{B}_m^{\mathrm{mult}} \leqslant \sqrt{e}\mathrm{B}_{\frac{m}{2}}^{\mathrm{mult}} \leqslant \left(\sqrt{e}\right)^k = \left(\sqrt{e}\right)^{\log_2 m} = m^{\frac{1}{2\log 2}}.$$

Now, for an arbitrary m let $[\log_2 m]$ be the smallest integer bigger that or equal $\log_2 m$; then $\mathrm{B}_m^{\mathrm{mult}} \leqslant \mathrm{B}_{2^{[\log_2 m]}}^{\mathrm{mult}}$ and a straightforward argument gives $\mathrm{B}_m^{\mathrm{mult}} \leqslant m^{\frac{1}{2\log 2}}$. ■

The fact that the constant in the Bohnenblust-Hille inequality grows at most like a polynomial on m and not exponentially has recently found some applications in the field of quantum information and XOR-games [64].

2.1.7. Vector valued variants

Some attention has been recently paid to vector valued Bohnenblust–Hille type inequalities. Within the setting of multiple summing multilinear mappings Bombal, Pérez-García and Villanueva showed [16, Theorem 3.2] (see also [83]) that if X is a cotype q space, then for every m-linear mapping $L : c_0 \times \cdots \times c_0 \to X$ the following holds

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \|L(e_{i_1},\dots,e_{i_m})\|_X^q \right)^{\frac{1}{q}} \leqslant \mathrm{C}_q(X)^m \|L\|. \tag{46}$$

Moreover it can be shown that the exponent in this inequality cannot be strictly smaller than $\cot(X) = \inf\{q \colon X \text{ has cotype } q\}$. We see that, while in the scalar valued case the exponent in the Bohnenblust–Hille inequality heavily depends on the degree of the multilinear mapping, in the result of Bombal, Pérez-García and Villanueva this dependency totally vanishes. This situation was studied in [38] and related with some summability properties. There the concept of $(\rho, 1)$–summing operator of order m was introduced as those operators $v : X \to Y$ between Banach spaces such that for each m there exists a constant C_m such that for every m-linear $L : c_0 \times \cdots \times c_0 \to X$ the following holds

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \|vL(e_{i_1},\dots,e_{i_m})\|_Y^\rho \right)^{\frac{1}{\rho}} \leqslant \mathrm{C}_m \|L\|. \tag{47}$$

Let us note that $m = 1$ gives the classical concept of $(r, 1)$–summing operator that can be found in for example [41, Chapter 10]. With this notation, Theorem I says that $\mathrm{id}_{\mathbb{C}}$ is $(\frac{2m}{m+1}, 1)$–summing of order m and (46) says that the identity of a cotype q space is $(q, 1)$–summing of order m.

A good understanding of the tensor product proof presented in Section 2.1.2 allows to generalize it to the vector valued setting. Then, [38, Lemma 3] shows that if Y has cotype 2, then every $(r, 1)$–summing operator with $1 \leqslant r \leqslant 2$ is $(\frac{2m}{m+2(1/r-1/2)}, 1)$–summing of order m. This was applied to the inclusion id :

$\ell_p \hookrightarrow \ell_q$ with $1 \leqslant p \leqslant q \leqslant \infty$ to show [38, Theorem 1] that there exists a constant $C = C_{m,p,q} > 0$ such that for every m-linear mapping $L : c_0 \times \cdots \times c_0 \to \ell_p$ the following holds

$$\Big(\sum_{i_1,\dots,i_m=1}^{\infty} \|L(e_{i_1},\dots,e_{i_m})\|_q^{\rho} \Big)^{\frac{1}{\rho}} \leqslant \|L : c_0 \times \cdots \times c_0 \to \ell_p\| \,, \tag{48}$$

where

$$\rho = \begin{cases} \frac{2m}{m+2(\frac{1}{p}-\frac{1}{q})} & \text{if } 1 \leqslant p \leqslant q \leqslant 2 \\ \frac{2m}{m+2(\frac{1}{p}-\frac{1}{2})} & \text{if } 1 \leqslant p \leqslant 2 \leqslant q \\ p & \text{if } 2 \leqslant p \leqslant q \end{cases} \tag{49}$$

This result is obviously still contains the classical Bohnenblust–Hille inequality from Theorem I as a special case (doing $p = 1$, $q = 2$ and considering only mappings L that have their range in the first coordinate of ℓ_1). But also (and this is in principle not so obvious), doing $m = 1$, it contains the celebrated Bennett–Carl inequalities, independently obtained in [5, 19], that characterize those r's for which the inclusion $\mathrm{id} : \ell_p \hookrightarrow \ell_q$ is $(r, 1)$-summing.

This study was carried on and extended in [34]. A careful analysis of the original proof of Bohnenblust and Hille allowed them to prove the following result, [34, Theorem 5.1 and Corollary 5.2], that generalizes all the results in this section.

Theorem 2.11. *Let Y be a Banach space with cotype q and $v : X \to Y$ an $(r,1)$-summing operator (with $1 \leqslant r \leqslant q$). Define*

$$\rho = \frac{qrm}{q + (m-1)r}.$$

Then there is a constant $C > 0$ such that for every m-linear mapping $L : c_0 \times \cdots \times c_0 \to X$ the following holds

$$\Big(\sum_{i_1,\dots,i_m=1}^{\infty} \|v\big(L(e_{i_1},\dots,e_{i_m})\big)\|_Y^{\rho} \Big)^{1/\rho} \leqslant C_m \|L\|.$$

In fact, the proof presented in Section 2.1.4 is just an adaptation of the proofs of Theorem 4.1, Theorem 5.1 and Corollary 5.2 in [34] to the particular case $X = Y = \mathbb{C}$ and $v = \mathrm{id}_{\mathbb{C}}$. This very particular case allows to simplify a lot the general proof.

2.2. Optimality of the exponent

The fact that the exponent in Theorem I is optimal is proved in Section 1.3 by producing an extremelly clever example of a m-linear mapping. Using probabilistic tools we can show the optimality of the exponent; that is, if r is such that (15) holds (i.e. $\big(\sum_{i_1,\dots,i_m} |a_{i_1,\dots,i_m}|^r\big)^{1/r} \leqslant C_m \|L\|$) for every $L \in \mathcal{L}(^m c_0)$, then $r \geqslant \frac{2m}{m+1}$. There are two slightly different approaches.

2.2.1. First approach

This is a slight modification of an argument given by Boas [8] and begins with the following result [54, Chapter 6, Theorem 3].

Theorem 2.12 (Kahane-Salem-Zygmund inequality). *Let us consider a finite family of trigonometric polynomials $(f_n)_n$ of degree less than or equal m in n variables and a family $(\xi_n)_n$ of independent indentically distributed subnormal random variables. We consider*

$$P(t_1,\dots,t_n)=\sum \xi_n f_n(t_1,\dots,t_n).$$

Then, denoting by $\|\cdot\|_\infty$ the norm given by the supremum taken on $t_1,\dots,t_n\in[0,2\pi]$ we have

$$\mathbb{P}\Big(\|P\|_\infty \geqslant C\sqrt{n\sum\|f_n\|_\infty^2\log m}\Big)\leqslant \frac{1}{m^2e^n},$$

where C is a universal constant and $\mathbb{P}$ stands for the probabilty.

An example of subnormal random variables are the Rademacher random variables ε, that take values $+1$ and -1 with probability $1/2$. Then, a family ε_n of Rademacher random variables can be seen as a random choice of signs and what Theorem 2.12 says is that, given a choice of signs, the probability that the supremum of the polynomial P exceeds a certain value is small. This implies that there is at least one choice of signs for wich the value of the supremum is actually smallest that the value in Theorem 2.12. This is [54, Chapter 6, Theorem 4]:

Theorem 2.13. *Given complex numbers c_α with $\alpha\in\mathbb{N}_0^n$ and $|\alpha|\leqslant m$ there exists a choice of signs $\varepsilon_\alpha=\pm1$ such that*

$$\sup_{t_1,\dots,t_n\in[0,2\pi]}\Big|\sum_\alpha \varepsilon_\alpha c_\alpha e^{i(\alpha_1t_1+\cdots+\alpha_nt_n)}\Big|\leqslant C\sqrt{n\sum_\alpha|c_\alpha|^2\log m}.$$

By the Maximum Modulus Principle we have

$$\begin{aligned}\sup_{|z_1|\leqslant1,\dots,|z_n|\leqslant1}\Big|\sum_\alpha w_\alpha z_1^{\alpha_1}\cdots z_n^{\alpha_n}\Big| &= \sup_{|z_1|=1,\dots,|z_n|=1}\Big|\sum_\alpha w_\alpha z_1^{\alpha_1}\cdots z_n^{\alpha_n}\Big|\\ &= \sup_{t_1,\dots,t_n\in[0,2\pi]}\Big|\sum_\alpha w_\alpha e^{i(\alpha_1t_1+\cdots+\alpha_nt_n)}\Big|.\end{aligned}$$

for every finite choice $w_\alpha\in\mathbb{C}$. Following the proof of Proposition 1.3 we easily have that if (15) holds for m-linear mappings for some exponent r then (20) holds for m-homogeneous polynomials with the same exponent. Then, if r is optimal in (20), then so also is it in (15). Let us now produce an m-homogeneous polynomial Q on ℓ_∞^n with unimodular coefficients and small norm. Theorem 2.13 implies (taking $c_\alpha=1$) that there exists a choice of signs $\varepsilon_\alpha=\pm1$ such that the polynomial $Q:\ell_\infty^n\to\mathbb{C}$ defined by $Q(z)=\sum_\alpha\varepsilon_\alpha z^\alpha$ satisfies

$$\|Q\|\leqslant C\sqrt{n\cdot n^m\log m}.$$

On the other hand

$$\sum_{\alpha} |\varepsilon_\alpha|^r = \binom{m+n-1}{m} \geqslant \frac{1}{m!} n^m.$$

Then, if (15) holds we have

$$n^{\frac{m}{r}} \leqslant C_m n^{\frac{m+1}{2}}.$$

Since this holds for every n, it implies $\frac{m}{r} \leqslant \frac{m+1}{2}$ and then $r \geqslant \frac{2m}{m+1}$, proving again that the exponent in Proposition 1.3 (and hence in Theorem I) is optimal.

2.2.2. Second approach

The second approach uses the tensor version of the inequality as presented in (32). Let us assume that $r \geqslant 1$ is such that

$$\sup_n \| \operatorname{id} : \ell_1^n \otimes_\varepsilon \overset{m}{\cdots} \otimes_\varepsilon \ell_1^n \to \ell_r^{n^m} \| = C < \infty. \tag{50}$$

We want to show that $r \geqslant \frac{2m}{m+1}$. We take as starting point a different probabilistic tool, namely the following multilinear version of Chevét's inequality (see [81, (43.2)] for the bilinear version) presented in [23, Lemma 6].

Proposition 2.14. *Let X be a Banach space and $x_1, \dots, x_n \in X$. Then for independent identically distributed Gaussian random variables $(g_{i_1,\dots,i_m})_{i_1,\dots,i_m=1,\dots n}$ and $(g_i)_{i=1,\dots,n}$ we have*

$$\int \Big\| \sum_{i_1,\dots,i_m=1}^{n} g_{i_1,\dots,i_m}(\omega) x_{i_1} \otimes \cdots \otimes x_{i_m} \Big\|_{\otimes_\varepsilon^m X} d\omega \\ \leqslant d_m \int \Big\| \sum_{i=1}^n g_i(\omega) x_i \Big\|_X d\omega \sup_{\|x^*\|_{X^*} \leqslant 1} \Big(\sum_{i=1}^n |x^*(x_i)|^2 \Big)^{\frac{m-1}{2}},$$

where d_m is a universal constant depending only on m.

Let us remark first that

$$\sup_{\|x^*\|_{X^*} \leqslant 1} \Big(\sum_{i=1}^n |x^*(x_i)|^2 \Big)^{\frac{1}{2}} = \| \operatorname{id} : \ell_2^n \to X \, , \, e_i \rightsquigarrow x_i \|.$$

On the other hand, it is a well known fact [41, Proposition 12.11] that $\int \| \sum \varepsilon_i z_i \|_Z \leqslant \int \| \sum g_i z_i \|_Z$ holds for every finite choice of vectors z_i in a Banach space Z, where the g_i and the ε_i are respectively Gaussian and Rademacher random variables. Then

$$\int \Big\| \sum_{i_1,\dots,i_m=1}^{n} \varepsilon_{i_1,\dots,i_m}(\omega) x_{i_1} \otimes \cdots \otimes x_{i_m} \Big\|_{\otimes_\varepsilon^m X} d\omega \\ \leqslant d_m \int \Big\| \sum_{i=1}^n g_i(\omega) x_i \Big\|_X d\omega \, \| \operatorname{id} : \ell_2^n \to X \|^{m-1}.$$

We take $X = \ell_1^n$ and $x_i = e_i$ for $i = 1, \dots, n$ to get

$$\int \Big\| \sum_{i_1,\dots,i_m=1}^{n} \varepsilon_{i_1,\dots,i_m}(\omega) e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\|_{\otimes_\varepsilon^m \ell_1^n} d\omega \\ \leqslant d_m \int \Big\| \sum_{i=1}^{n} g_i(\omega) e_i \Big\|_{\ell_1^n} d\omega \, \| \mathrm{id} : \ell_2^n \to \ell_1^n \|^{m-1}.$$

Now we have that $\int \big\| \sum_{i=1}^n g_i(\omega) e_i \big\|_{\ell_p^n} \leqslant k n^{\frac{1}{p}}$ [33, (4)] and $\| \mathrm{id} : \ell_2^n \to \ell_p^n \| = n^{\frac{1}{p} - \frac{1}{2}}$ for every $1 \leqslant p < \infty$; then

$$\int \Big\| \sum_{i_1,\dots,i_m=1}^{n} \varepsilon_{i_1,\dots,i_m}(\omega) e_{i_1} \otimes \cdots \otimes e_{i_m} \Big\|_{\otimes_\varepsilon^m \ell_1^n} d\omega \leqslant d_m k n^1 n^{\frac{1}{2}(m-1)} = d_m k n^{\frac{m+1}{2}}.$$

The left-hand side of the previous inequality is an average over all possible choices of signs; hence there must exist at least one choice of signs for which the norm is smaller than the integral. This implies that there are signs $\varepsilon_{i_1,\dots,i_m} = \pm 1$ such that $z = \sum_{i_1,\dots,i_m=1}^{n} \varepsilon_{i_1,\dots,i_m} e_{i_1} \otimes \cdots \otimes e_{i_m}$ satisfies

$$\|z\|_{\otimes_\varepsilon^m \ell_1^n} \leqslant d_m k n^{\frac{m+1}{2}}.$$

Now, if (50) holds then

$$\|z\|_{\ell_r^{n^m}} \leqslant C \|z\|_{\otimes_\varepsilon^m \ell_1^n}.$$

But $\|z\|_{\ell_r^{n^m}} = \big(\sum_{i_1,\dots,i_m=1}^{n} |\varepsilon_{i_1,\dots,i_m}|^r \big)^{\frac{1}{r}} = n^{\frac{m}{r}}$ and this gives

$$n^{\frac{m}{r}} \leqslant K n^{\frac{m+1}{2}},$$

and since this holds for every n this implies $r \geqslant \frac{2m}{m+1}$.

2.2.3. Optimality of the exponents in the vector valued case

As we have already mentioned in Section 2.1.7, Bohnenblust–Hille type inequalities have been lately obtained for operators between Banach spaces, like that in Theorem 2.11. That sort of results show that the inequality holds for a certain exponent, but not that the exponent is optimal. Getting the optimality of the exponent requires to find examples and that is also not at all easy. For the case of the inclusion $\mathrm{id} : \ell_p \hookrightarrow \ell_q$ [38, Theorem 1] shows that the exponents given in (49) for (48) are actually optimal. For the case $1 \leqslant p \leqslant q \leqslant 2$ the example uses the probabilistic tools as presented in Section 2.2.2. For the case $1 \leqslant p \leqslant 2 \leqslant q$ the example given in [38] is an adaptation of the classical example of Bohnenblust and Hillle given in Section 1.3. The optimality of the exponent when $2 \leqslant p \leqslant q$ follows from the fact that the Bennett-Carl inequalities are optimal.

2.3. Polynomials

What is used in many cases is actually the polynomial version of the Bohnenblust-Hille inequality (19). But not only the inequality is important, also the fact that the exponent is optimal is relevant. As we have seen in Section 1.4, in principle this does not follow from the optimality for multilinear mappings and a new (ingenious and sophisticated) example has to be given.

In Section 2.2.1 we have produced, using Theorem 2.13, a polynomial that shows that the exponent in Proposition 1.3 is optimal.

Alternatively, the following result [28, Corollary 3.2] (a symmetric version of Proposition 2.14) can be used to get the optimality of the exponent for polynomials.

Proposition 2.15. *Let $X = (\mathbb{K}^n, \|\cdot\|)$ be an n-dimensional Banach space, and consider families of independent standard Gaussian random variables $(g_\alpha)_{|\alpha|=m}$ and $(g_k)_{1\leqslant k\leqslant n}$. Then for each choice of scalars c_α, $|\alpha| = m$,*

$$\int \sup_{\|z\|\leqslant 1} \Big| \sum_{|\alpha|=m} c_\alpha g_\alpha(\omega) z^\alpha \Big| d\omega$$
$$\leqslant C \sup_{|\alpha|=m} \Big(|c_\alpha| \sqrt{\frac{\alpha!}{m!}} \Big) \sup_{\|z\|\leqslant 1} \Big(\sum_{k=1}^n |z_k|^2 \Big)^{\frac{1}{2}} \int \sup_{\|z\|\leqslant 1} \Big| \sum_{k=1}^n g_k(\omega) z_k \Big| d\omega \,, \quad (51)$$

where $C > 0$ is a universal constant depending only on m.

Taking $c_\alpha = 1$ and proceeding as in Section 2.2.2 we again can produce a polynomial that shows that the exponent in Proposition 1.3 is optimal. Since the Rademacher averages are dominated by the Gauss averages, and since we are talking about averages, then there exists a choice of sings $\varepsilon_\alpha = \pm 1$ such that $\sup_{\|z\|\leqslant 1} \big| \sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha \big|$ is bounded by the right-hand side of (51). Finally we use the bounds for the usual norm and the Gaussian averages given in Section 2.2.2 to reach the conclusion.

The tensor formulation given in Section 2.1.2 gives a new insight in this direction, using techniques from the metric theory symmetric tensor products. We use a technique that was first considered in [18, 30], later used in [23, 33, 46] and finally presented in its more general form in [35]. For each fixed $n \in \mathbb{N}$ and every $i = 1, \ldots, m$ we consider mappings

$$\begin{aligned} I_i : \mathbb{C}^n &\longrightarrow \mathbb{C}^{mn} \\ \textstyle\sum_{j=1}^n \lambda_j e_j &\rightsquigarrow \textstyle\sum_{j=1}^n \lambda_j e_{n(i-1)+j} \end{aligned} \qquad \begin{aligned} P_i : \mathbb{C}^{mn} &\longrightarrow \mathbb{C}^n \\ \textstyle\sum_{j=1}^{mn} \lambda_j e_j &\rightsquigarrow \textstyle\sum_{j=1}^n \lambda_{n(i-1)+j} e_j . \end{aligned}$$

On the other hand, there are the natural embedding and simmetrisation (see [45] for the definitions)

$$\iota_m : \bigotimes^{m,s} \mathbb{C}^{mn} \longrightarrow \bigotimes^m \mathbb{C}^{mn} \qquad \text{and} \qquad \sigma_m : \bigotimes^m \mathbb{C}^{mn} \longrightarrow \bigotimes^{m,s} \mathbb{C}^{mn}.$$

From all this it can be easily deduced that the following diagram is commutative (see [45]):

$$\begin{tikzcd}
\bigotimes^m \mathbb{C}^n \arrow[r, "\otimes^m \mathrm{id}_n"] \arrow[d, "I_1\otimes\cdots\otimes I_m"'] & \bigotimes^m \mathbb{C}^n \\
\bigotimes^m \mathbb{C}^{mn} \arrow[d, "\sigma_m"'] & \bigotimes^m \mathbb{C}^{mn} \arrow[u, "m!\,P_1\otimes\cdots\otimes P_m"'] \\
\bigotimes^{m,s} \mathbb{C}^{mn} \arrow[r, "\otimes^{m,s} \mathrm{id}_{mn}"] & \bigotimes^{m,s} \mathbb{C}^{mn} \arrow[u, "\iota_m"']
\end{tikzcd} \tag{52}$$

Suppose now that a Bohnenblust-Hille inequality holds for polynomials for a certain exponent $r > 1$; that is, there exists a constant K_m such that for every $P \in \mathcal{P}({}^m c_0)$

$$\Big(\sum_{\alpha} |c_\alpha|^r \Big)^{\frac{1}{r}} \leqslant K_m \|P\|.$$

This, in terms of tensors is equivalent to

$$\sup_n \| \mathrm{id} : \bigotimes\nolimits_{\varepsilon_s}^{m,s} \ell_1^n \longrightarrow \ell_r^{d(m,n)} \| \leqslant K_m \,, \tag{53}$$

here $d(m,n) = \dim \bigotimes_r^{m,s} \ell_r^n = \binom{m+n-1}{n-1}$. We consider then the injective and the r norms in (52) and we get (let us recall that $\ell_r^{d(m,n)} = \bigotimes_r^{m,s} \ell_r^n$)

$$\begin{tikzcd}
\bigotimes_\varepsilon^m \ell_1^n \arrow[r, "\otimes^m \mathrm{id}_n"] \arrow[d, "I_1\otimes\cdots\otimes I_m"'] & \bigotimes_r^m \ell_r^n \\
\bigotimes_\varepsilon^m \ell_1^{mn} \arrow[d, "\sigma_m"'] & \bigotimes_r^m \ell_r^{mn} \arrow[u, "m!\,P_1\otimes\cdots\otimes P_m"'] \\
\bigotimes_{\varepsilon_s}^{m,s} \ell_1^{mn} \arrow[r, "\otimes^{m,s} \mathrm{id}_{mn}"] & \bigotimes_r^{m,s} \ell_r^{mn} \arrow[u, "\iota_m"']
\end{tikzcd}$$

We now conclude from the metric mapping property of the injective norm and our assumption from (53),

$$\begin{aligned}\|I_1 \otimes \cdots \otimes I_m : \textstyle\bigotimes_\varepsilon^m \ell_1^n \longrightarrow \bigotimes_\varepsilon^m \ell_1^{mn}\| &\leqslant 1\\ \|\sigma_m : \textstyle\bigotimes_\varepsilon^m \ell_1^{mn} \longrightarrow \bigotimes_{\varepsilon_s}^{m,s} \ell_1^{mn}\| &\leqslant 1\\ \|\otimes^{m,s} \mathrm{id}_{mn} : \textstyle\bigotimes_{\varepsilon_s}^{m,s} \ell_1^{mn} \longrightarrow \ell_r^{d(m,mn)}\| &\leqslant K_m\\ \|\iota_m : \ell_r^{d(m,mn)} \longrightarrow \ell_r^{(mn)^m}\| &\leqslant 1\\ \|m!\, P_1 \otimes \cdots P_m : \ell_r^{(mn)^m} \longrightarrow \ell_r^{n^m}\| &\leqslant m!.\end{aligned}$$

This altogehter gives

$$\sup_n \| \mathrm{id} : \textstyle\bigotimes_\varepsilon^m \ell_1^n \longrightarrow \ell_r^{n^m} \| \leqslant m!\, K_m \,,$$

which, in view of (32) means that a the Bohnenblust-Hille inequality for m-linear mappings holds with exponent r.

It was already shown in Proposition 1.3 that if the Bohnenblust-Hille inequality holds for m-linear mappings with a certain exponent r, then it also holds for m-homogeneous polynomials with the same exponent. Hence we have just proved that the inequality holds in the multilinear case if and only if it holds (with the same exponent) for polynomials. This implies that if an exponent is optimal for multilinear mappings then it is also optimal for homogeneous polynomials. This in principle would only be a new way to show that the exponent $\frac{2m}{m+1}$ is also optimal for polynomials. This technique, however, allows to prove a more general fact: essentially that whenever we have a Bohnenblust-Hille type inequality for multilinear mappings, we also have it for polynomials and vice-versa [38, Lemma 5].

Proposition 2.16. *Let E be a Banach sequence space, $v : X \longrightarrow Y$ an operator, $1 \leqslant r < \infty$ and $m \in \mathbb{N}$. Consider the following two statements:*

(a) *There is $C_m > 0$ such that for every m-linear mapping $L : E \times \cdots \times E \longrightarrow X$*

$$\Big(\sum_{i_1,\dots,i_m} \|vL(e_{i_1},\dots,e_{i_m})\|_Y^r \Big)^{1/r} \leqslant C_m \|L\|.$$

(b) *There is $K_m > 0$ such that for every m-homogeneous polynomial $P : E \longrightarrow X$*

$$\Big(\sum_{|\alpha|=m} \|vc_\alpha(P)\|_Y^r \Big)^{1/r} \leqslant K_m \|P\|.$$

Then (a) always implies (b) with $K_m \leqslant (m!)^{1-1/r} c(m,E) C_m$. Conversely, if E is symmetric, then (b) implies (a) with $C_m \leqslant m! K_m$.

2.3.1. The constants

Again, having a good control over the constants in the polynomial version fo the Bohnenblust-Hille inequality is a major problem for several applications. Let us denote by $\mathrm{B}_m^{\mathrm{pol}}$ the optimal constant in (19). Let us give a short historical account of the improvements on the control of the growth of $\mathrm{B}_m^{\mathrm{pol}}$. First of all, the proof of the inequality that we have presented here starts from the multilinear inequality and applies polarisation. Then the constant $\mathrm{B}_m^{\mathrm{pol}}$ depends both on $\mathrm{B}_m^{\mathrm{mult}}$ and on the polarisation constant (that relates the norm of a symmetric m-linear form and of its associated m-homogeneous polynomial as in (18)). Bohnenblust and Hille's proof of Theorem I gives $\mathrm{B}_m^{\mathrm{mult}} \leqslant m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$ then their constant is

$$\mathrm{B}_m^{\mathrm{pol}} \leqslant (m!)^{\frac{m-1}{2m}} m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}} \frac{m^m}{m!} = 2^{\frac{m-1}{2}} \frac{m^{m+\frac{m+1}{2m}}}{(m!)^{\frac{m+1}{2m}}}.$$

As we have seen, with the proofs presented in Sections 2.1.2 and 2.1.3 we get a better estimate for $\mathrm{B}_m^{\mathrm{mult}}$. This is what we have used in Proposition 1.3, giving

$$\mathrm{B}_m^{\mathrm{pol}} \leqslant (m!)^{\frac{m-1}{2m}} 2^{\frac{m-1}{2}} \frac{m^m}{m!} = 2^{\frac{m-1}{2}} \frac{m^m}{(m!)^{\frac{m+1}{2m}}}.$$

This improvement comes from having a better estimates for $\mathrm{B}_m^{\mathrm{mult}}$; another way to improve the constants is by finding a better constant relating the norms of the linear forms and the polynomial. It is well known (see [50]; see also [63, 82]) that, if we have m-homogeneous polynomials and m-linear forms on c_0 then the constant in (18) can be improved to $\frac{m^{m/2}(m+1)^{(m+1)/2}}{2^m m!}$. With this we have a new improvement

$$\mathrm{B}_m^{\mathrm{pol}} \leqslant (m!)^{\frac{m-1}{2m}} 2^{\frac{m-1}{2}} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m m!} = \big(\sqrt{2}\big)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}. \tag{54}$$

Konyagin and Queffélec proved in [57, Theorem 4.3] a sort of refinement of the classical result that $T = \frac{1}{2}$. More precisely, they show that there are constants α, $\beta > 0$ so that for every Dirichlet series

$$\sum_{n=1}^{N} |a_n| \leqslant \alpha N^{\frac{1}{2}} e^{-\beta\sqrt{\log N \log\log N}} \sup_{t\in\mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{it} \Big|. \tag{55}$$

We will not get now into the question of why is this a refinement of the Bohr-Bohnenblust-Hille Theorem (see Section 2.5.2). We will simply say that the Theorem can be deduced from (55) and that the exponent $\frac{1}{2}$ on N is what gives $T = \frac{1}{2}$. To prove this Konyagin and Queffélec need that $\mathrm{B}_m^{\mathrm{pol}} \leqslant m^{\frac{m}{2}}$. This does not follow from (54), basically because the $\sqrt{2}$ factor is too big as to make all the rest small enough. As we have already mentioned the $\sqrt{2}$ factor comes from using Khintchine's inequality and it can be improved by using instead Sawa's inequality [77] for Steinhaus random variables. This was done by Queffelec in [74], where he shows

$$\mathrm{B}_m^{\mathrm{pol}} \leqslant \Big(\frac{2}{\sqrt{\pi}}\Big)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}.$$

This constant is better than that in (54) since $\frac{2}{\sqrt{\pi}} < \sqrt{2}$ and indeed gives [57, Lemma 4.1] that $\mathrm{B}_m^{\mathrm{pol}} \leqslant m^{\frac{m}{2}}$.

Proving the polynomial inequality using the multilinear one forces us to use polarisation, and this will always ruin the constant. Avoiding this requires a direct proof of the polynomial inequality. This was done by Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip in [27]. Two are the main steps in the proof of the inequality:

- Split the original sum into 'mixed' sums of the kind $\sum_{i_k} \Big(\sum_{\tilde{i}_k} |a_{i_1 \dots i_m}|^2 \Big)^{\frac{1}{2}}$ (like in Lemma 2.6).
- Bound each one of these mixed sums... this needs a sort of Khintchine inequality (either the original one or the Sawa one).

If we start with a sum of coefficients of a polynomial $\big(\sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}} \big)^{\frac{m+1}{2m}}$, the first step can be carried on exactly in the same way. The problem comes in the second step. There a sort of polynomial version of Khintchine is needed. It was proved by Bayart in [3, Theorem 9] (see also [25, Section 3.2]) using results on the contractivity of the Poisson kernel [17, 84]: For every m-homogeneous polynomial in N variables we have

$$\Big(\sum_{|\alpha|=m} |c_\alpha|^2 \Big)^{\frac{1}{2}} \leqslant 2^{\frac{m}{2}} \int_{\mathbb{T}^N} \Big| \sum_{|\alpha|=m} c_\alpha z^N \Big| d\mu^N(z),$$

where μ^N is the normalized Lebesgue measure on the N-dimensional Torus. With this Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip prove in [27, Theorem 1] the following result.

Theorem 2.17. *Given an m-homogeneous polynomial $P : \mathbb{C}^N \longrightarrow \mathbb{C}$, $P = \sum_{|\alpha|=m} c_\alpha z^\alpha$, then*

$$\Big(\sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leqslant (1 + \frac{1}{m-1})^{m-1} \sqrt{m} (\sqrt{2})^{m-1} \sup_{z \in \mathbb{D}^N} \Big| \sum_{|\alpha|=m} c_\alpha z^\alpha \Big|.$$

This implies that there exists a universal constant constant $K > 0$ such that

$$\mathrm{B}_m^{\mathrm{pol}} \leqslant K^m.$$

This allows to solve in [27] several important open problems on Sidon constants, unconditionality constants, optimality in inequalities like (55) or to determine the precise asymptotic behaviour of the Bohr radius of the unit ball of ℓ_∞^n, a problem that had been open for more that 10 years [9].

Estimates for the constants in the vector valued case are given in e.g. [38, Lemma 3] or, in a far more general setting, [34, Theorem 5.1, Corollary 5.2]; these constants take into account the geometry of the Banach space, by the appearance of the cotype constant. In [32, Theorem 4.2] it is shown that in a fairly general setting the vector valued polynomial Bohnenblust-Hille inequality is also hypercontractive.

Theorem 2.18. *Let Y be a q-concave Banach lattice, with $2 \leqslant q < \infty$, and $v : X \to Y$ an $(r,1)$-summing bounded operator with $1 \leqslant r \leqslant q$. Define*

$$\rho = \frac{qrm}{q+(m-1)r}.$$

Then there is a constant $K > 0$ such that for every m-homogeneous polynomial $P : \ell_\infty^n \to X$ the following holds

$$\left(\sum_{\alpha \in \Lambda(m,n)} \|v(c_\alpha(P))\|_Y^\rho \right)^{1/\rho} \leqslant K^m \sup_{x \in B_{\ell_\infty^n}} \|P(x)\|_X \ ,$$

where $\Lambda(m,n) = \{\alpha \in \mathbb{N}_0^n \ : |\alpha| = m\}$.

2.4. Sets of monomial convergence

The fact that $T = \frac{1}{S}$ (see (5)) links the strips of convergence of Dirichlet series with the sets of monomial convergence. Bohr proved [12, page 462] that $\ell_2 \cap B_{c_0} \subseteq \operatorname{mon} H_\infty(B_{c_0})$; giving $T \leqslant \frac{1}{2}$. As we have seen, the final step was given in Section 1.5, where Bohnenblust and Hille show that $\ell_{\frac{2m}{m-1}} \subseteq \mathcal{P}({}^m c_0)$ (Theorem III) and that $\ell_{\frac{2m}{m-1}+\varepsilon} \not\subseteq \mathcal{P}({}^m c_0)$ for every $\varepsilon > 0$ (Theorem IV). Using the notation in (10) this means $S_m = \frac{2m}{m-1}$. From this they conclude that $S = 2$, which also implies that $\ell_{2+\varepsilon} \cap B_{c_0} \not\subseteq H_\infty(B_{c_0})$ for every $\varepsilon > 0$.

This shows that a bounded, holomorphic function on B_{c_0} does not admit a power series expansion at every point. Then the question remains open: for which points in B_{c_0} does every function in $H_\infty(B_{c_0})$ admit a monomial series expansion? (or, in other words: detemine $\operatorname{mon} H_\infty(B_{c_0})$). It is more or less clear by now that a first step in this way is to find out what $\operatorname{mon} \mathcal{P}({}^m c_0)$ is. These two problems were considered and carefully analyzed in [31], where the following two descriptions are given as particular cases of very general results:

$$\ell_{\frac{2m}{m-1}} \subseteq \mathcal{P}({}^m c_0) \subseteq \ell_{\frac{2m}{m-1}+\varepsilon} \qquad \text{for every } \varepsilon > 0 \qquad \text{[31, Example 4.6]}$$

$$\ell_2 \cap B_{c_0} \subseteq \operatorname{mon} H_\infty(B_{c_0}) \subseteq \ell_{2+\varepsilon} \cap B_{c_0} \qquad \text{for every } \varepsilon > 0 \qquad \text{[31, Example 4.9]}$$

That restricts a lot the possibilities for the sets of monomial convergence, that cannot be far away from $\ell_{\frac{2m}{m-1}}$ or ℓ_2, but still does not give a precise description of the sets. This has only very recently been possible for m-homogenous polynomials, using techniques and tools from Dirichlet series due to Queffélec and others [2, 62]. Then, in [26] we have that the set of monomial expansion of the polynomials is actually a Lorentz sequence space

$$\ell_{\frac{2m}{m-1},\infty} = \operatorname{mon} \mathcal{P}({}^m \ell_\infty).$$

For the space of bounded, holomorphic functions we have in [26] that (denoting

x^* for the decerasing rearrangement of the sequence x)

$$\Big\{x \in \ell_\infty \colon \limsup_n \tfrac{1}{\log n} \textstyle\sum_{j=1}^n x_n^{*\,2} < \tfrac{1}{2}\Big\} \subseteq \operatorname{mon} H_\infty(B_{c_0}) \subseteq \Big\{x \in \ell_\infty \colon \limsup_n \frac{1}{\log n} \sum_{j=1}^n x_n^{*\,2} \leqslant 1\Big\}.$$

The problem of describing holomorphic functions in infinitely many variables via monomial series expansions goes back to the very beginning of infinite dimensional holomorphy as a theory in the beginning of the 20th century. Hilbert himself suggested in [52] that as a the proper definition of a holomorphic function in infinitely many variables. It soon became clear (partly because of the results presented here) that this approach was not good enough and the approaches of Gâteaux and Fréchet (in terms of differentiability) and of Mazur and Orlicz (in terms of m-homogeneous polynomials, understood as diagonals of m-linear mappings) finally proved to be more convenient. But the idea of power series expansions is appealing and always kept the interest, now reformulated as: 'for wich holomorphic functions can a power series expansion be found?' and/or 'how are the sets on which every holomorphic function has such an expansion?' Boland and Dineen gave a first answer to these questions in [15, Corollary 2].

Theorem 2.19. *Let E be a fully nuclear space with a basis and U an open polydisk. Then $\sum_\alpha c_\alpha(f) x^\alpha$ converges absolutely to $f(x)$ for every holomorphic function f on U and every x.*

Nevertheless the question remained open for Banach spaces, since no infinitely dimensional Banach space is fully nuclear. A positive result for Banach spaces was first given by Ryan, who showed [76, Theorem 4.6] (in our language) that $\operatorname{mon} H(\ell_1) = \ell_1$ i.e., every holomorphic function on ℓ_1 admits a power series expansion at any point of ℓ_1. Some time later Lempert, when dealing with the $\bar{\partial}$-equation, proved [58, Theorem 4.4] that $\operatorname{mon} H(rB_{\ell_1}) = rB_{\ell_1}$ for every $r > 0$. A question remained open: 'is ℓ_1 the only space on which this happens?'

Power series expansions were again considered in [31], where a careful study of the subject was performed in a very general setting. There results and fairly accurate descriptions of the sets of monomial convergence of any family of holomorphic functions containing the polynomials defined on a Reinhardt domain on a Banach space are given. Also, for the spaces of m-homogeneous polynomials. For instance, a partial answer to the question that remained open after Lempert's result was given in [31, Theorem 7.1].

Theorem 2.20. *Let R be a Reinhardt domain in a Banach sequence space X and $\mathcal{F}(R)$ a set of holomorphic functions on R which contains all polynomials. Assume that* $\operatorname{mon} \mathcal{F}(R)$ *is an absorbant subset of X . Then $\ell_1 \subseteq X \subseteq \ell_{1+\varepsilon}$ for all $\varepsilon > 0$.*

This, lousy speaking, is saying that if a space satisfy the Lempert's condition (i.e. $\operatorname{mon} H(R) = R$), then it is very close to ℓ_1 (see also [4, Corollary 5.4]).

Finally, we mention that some results for sets of monomial convergence of vector valued holomorphic functions are given in [40].

2.5. Strips

2.5.1. Proving again that $T \leqslant \frac{1}{2}$

As we have already mentioned, Bohr proved that $T \leqslant \frac{1}{2}$ by going to formal power series in infinitely many variables and proving that $S \geqslant 2$. Boas takes in [8] a different point of view and gives an elementary proof of this fact. We first need that for every finite family of complex numbers $a_1, \ldots, a_N$ the following Parseval–type inequality, due to Carlson [20], holds

$$\sum_{n=1}^{N} |a_n|^2 = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \Big| \sum_{n=1}^{N} a_n n^{it} \Big|^2 dt\,, \tag{56}$$

indeed, we have

$$\begin{aligned} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \Big| \sum_{n=1}^{N} a_n n^{it} \Big|^2 dt &= \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \Big(\sum_{n=1}^{N} a_n n^{it} \Big) \Big(\sum_{m=1}^{N} \overline{a}_m m^{-it} \Big) dt \\ &= \sum_{n,m=1}^{N} a_n \overline{a}_m \Big(\lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} n^{it} m^{-it} dt \Big) \\ &= \sum_{n,m=1}^{N} a_n \overline{a}_m \delta_{n,m} = \sum_{n=1}^{N} |a_n|^2 . \end{aligned}$$

Now, if $\sum_n a_n/n^s$ is a Dirichlet series that converges uniformly on the line $\operatorname{Re} s = \sigma$, let us show that it converges absolutely for $\operatorname{Re} s \geqslant \sigma + \varepsilon + 1/2$ for every ε (i.e. $\sum_n |a_n|/n^{\sigma+\varepsilon+1/2}$ is finite). By the Cauchy-Schwarz inequality we have

$$\sum_n a_n \frac{1}{n^{\sigma+\varepsilon+1/2}} \leqslant \Big(\sum_n \frac{|a_n|^2}{n^{2\sigma}} \Big)^{1/2} \Big(\sum_n \frac{1}{n^{2\varepsilon+1}} \Big)^{1/2} .$$

Since the second series clearly converges, it is enough to see that the first one is finite. But by hypothesis the Dirichlet series converges uniformly on the line $\operatorname{Re} s = \sigma$, hence its partial sums are uniformly bounded by say M on that line. Then we can apply (56) to get

$$\sum_{n=1}^{N} \Big| \frac{a_n}{n^\sigma} \Big|^2 = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \Big| \sum_{n=1}^{N} \frac{a_n}{n^\sigma} n^{it} \Big|^2 dt = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \Big| \sum_{n=1}^{N} a_n \frac{1}{n^{\sigma-it}} \Big|^2 dt \leqslant M^2 .$$

Since N is arbitrary the series $\sum_n |a_n|^2/n^{2\sigma}$ converges and this gives that $\sigma_a - \sigma_u \leqslant \frac{1}{2}$ for every Dirichlet series.

The proof can be made even more compact by using the following Hadamard–type formulas to compute the abscissas of convergence of a Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$ such that $\sum_{n=1}^{\infty} a_n$ diverges:

$$\sigma_u = \limsup_{N\to\infty} \frac{\log \sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{it} \right|}{\log N} \tag{57}$$

$$\sigma_a = \limsup_{N\to\infty} \frac{\log \sum_{n=1}^{N} |a_n|}{\log N} \tag{58}$$

Then by the Cauchy-Schwarz inequality and (56) we have

$$\begin{aligned}\sum_{n=1}^{N} |a_n| \leqslant N^{\frac{1}{2}} \Big(\sum_{n=1}^{N} |a_n|^2 \Big)^{\frac{1}{2}} &= N^{\frac{1}{2}} \Big(\lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \big| \sum_{n=1}^{N} a_n n^{it} \big|^2 dt \Big)^{\frac{1}{2}} \\ &\leqslant N^{\frac{1}{2}} \sup_{t\in\mathbb{R}} \big| \sum_{n=1}^{N} a_n n^{it} \big| .\end{aligned}$$

Taking now logarithm and lim sup we have that $\sigma_a - \sigma_u \leqslant \frac{1}{2}$ for every Dirichlet series such that $\sum_{n=1}^{\infty} a_n$ diverges. If the series does not diverge, then a translation argument gives that the upper bound holds for every Dirichlet series and hence $T \leqslant \frac{1}{2}$.

2.5.2. Refinements of the strips

We have seen that $T = \frac{1}{2}$ and $T_m = \frac{m-1}{2m}$; this implies that if a Dirichlet series (or an m-homogeneous Dirichlet series) $\sum_n a_n/n^s$ is in $\mathcal{H}_\infty$ then $\sum_n |a_n| \frac{1}{n^{\frac{1}{2}+\varepsilon}}$ (or $\sum_n |a_n| \frac{1}{n^{\frac{m-1}{2m}+\varepsilon}}$ in the m-homogeneous case) is finite for every $\varepsilon > 0$. A natural question then is to ask if one can even get that this holds for $\varepsilon = 0$ (i.e. if every Dirichlet series in $\mathcal{H}_\infty$ converges on the line [Re $= 1/2$]). Recently it has been shown that not only is this the case, but we can even have more. In [2, Theorem 1.2] it is proved that every Dirichlet series in $\mathcal{H}_\infty$ satisfies

$$\sum_n |a_n| \frac{e^{c\sqrt{\log n \log\log n}}}{n^{\frac{1}{2}}} < \infty \tag{59}$$

for $c > 0$ small enough. This clearly implies that $\sum_n |a_n| \frac{1}{n^{\frac{1}{2}}} < \infty$. The proof of this fact relies on the result of Konyagin and Quéffelec from (55). This kind of inequalities have been improved in [22] and [27], where Theorem 3 gives

$$\sum_{n=1}^{N} |a_n| \leqslant N^{\frac{1}{2}} e^{-\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log N \log\log N}} \sup_{t\in\mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{it} \Big| .$$

This estimate is optimal, as was shown in [22]. From this [27, Corollary 2] gives that the supremum of the set of positive c such that (59) holds for every Dirichlet series in $\mathcal{H}_\infty$ is $\frac{1}{\sqrt{2}}$.

For m-homogeneous Dirichlet series the corresponding result is [2, Theorem 1.4]: for every m-homogeneous Dirichlet series in $\mathcal{H}_\infty$ the following holds

$$\sum_n |a_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} < \infty.$$

As a consequence we have

$$\sum_{n=1}^N |a_n| \leqslant \alpha \frac{N^{\frac{m-1}{2m}}}{(\log N)^{\frac{m-1}{2}}} \sup_{t\in\mathbb{R}} \Big| \sum_{n=1}^N a_n n^{it} \Big|.$$

Moreover the exponent in the log-term is optimal; this is proved in two different ways in [62], first using a deterministic argument similar to that in Section 1.3 and then once again using probabilistic tools like in Section 2.2.1.

Results with this same spirit have been obtained in [3] for bigger classes of Dirichlet series, namely the Hardy spaces of Dirichlet series, $\mathcal{H}_p$. These can be seen as the image through the Bohr transform of the Hardy spaces of functions on the infinite dimensional torus. Bayart in [3] considers $T_p = \sup\{\sigma_a \colon$ Dirichlet series in $\mathcal{H}_p\}$ (it is not difficult to see that $T = T_\infty$) and shows that $T_p = \frac{1}{2}$ for every $1 \leqslant p \leqslant \infty$. In [2, Theorem 1.1] it was shown that, unlike in $\mathcal{H}_\infty$, the case '$\varepsilon = 0$' does not hold for $\mathcal{H}_p$ with $1 \leqslant p < \infty$.

Also, similar problems for vector valued Dirichlet series and operators between Banach spaces have been recently addressed in [37].

2.5.3. *T* and *S*

The key point in Bohr's approach to the absolute convergence problem for Dirichlet series was the relation between these and power series in infinitely many variables by means of the equality $T = \frac{1}{S}$. This equality can be taken a little bit further. Given a family of formal power series $\mathbb{P} \subset \mathfrak{P}$, then its set of monomial convergence can be defined as

$$\operatorname{mon} \mathbb{P} = \left\{ z \in \ell_\infty \colon \sum_\alpha |c_\alpha z^\alpha| < \infty \text{ for all series in } \mathbb{P} \right\}$$

A number $S(\mathbb{P})$ can be defined in an analogous way as in (4). Also, $\mathfrak{B}(\mathbb{P})$ is a family of Dirichlet series for which we can consider $T_p = \sup\{\sigma_a :$ Dirichlet series in $\mathfrak{B}(\mathbb{P})\}$. Following Bohr's original proof, it can be shown that if $\operatorname{mon}\mathbb{P}$ is stable under reordering and under changing finitely many coordinates, then

$$T(\mathfrak{B}(\mathbb{P})) = \frac{1}{S(\mathbb{P})}.$$

Recent results in [26] give $\operatorname{mon} H_p(\mathbb{T}^\infty) = \ell_2 \cap B_{c_0}$. In this way the results of Bayart about T_p are recovered.

In [29] this point of view has proved to be fruitful also for vector valued Dirichlet series, where $T(X)$ is defined for a Banach space X as the supremum of $\sigma_a - \sigma_u$ ranging over al Dirichlet series $\sum_n a_n n^{-s}$ with $a_n \in X$. Also the set of monomial convergence of $H_\infty(B_{c_0}, Y)$ (the bounded, holomorphic functions defined on B_{c_0} with values in Y) is defined in the same spirit of Section 2.4 and the corresponding number $S(H_\infty(B_{c_0}, Y))$ is considered. Then [29, Theorem 3] gives $T(X) = \frac{1}{S(H_\infty(B_{c_0},Y))}$. The main result [29, Theorem 1] states that $T(X) = 1 - \frac{1}{\cot X}$ for every Banach space ($\cot X$ is the infimum over all p such that X has cotype p). An analog result for vector valued $\mathcal{H}_p$ Dirichlet series has been recently obtained in [36].

A similar problem is studied in [39] for operators: given an operator between Banach spaces $v : X \to Y$ for each Dirichlet series $\sum_n a_n/n^s$ in X its abscissa of uniform convergence is denoted by σ_u^X; then $\sum_n v(a_n)/n^s$ is a Dirichlet series in Y whose abscissa of absolute convergence is denoted by σ_a^Y. The number $T(v)$ is defined as the supremum of the difference of these two abscissas ranging over all Dirichlet series in X. This number somehow gives an idea of how does v modify the convergence properties of Dirichlet series.

On the other hand, the space $H_v(B_{c_0}, Y))$ is defined as consisting of those f such that there exists $g \in H_\infty(B_{c_0}, X)$ satisfying $f = vg$ then we have

$$T(v) = \frac{1}{S(H_v(B_{c_0}, Y))}.$$

If we consider m-homogeneous Dirichlet series we can define analogous $T_m(X)$ and in [29] we have that if X is infinite dimensional then $T_m(X) = T(X) = 1 - \frac{1}{\cot X}$. We see that if the space is finite dimensional then the corresponding T_m depends on m but if the space is infinite dimensional then this dependency on m vanishes. This situation was analyzed in [39, 40], where the corresponding $T_m(v)$ is considered and is related with certain summability properties of the operator v (namely that v is $(r,1)$-summing for some r). It is well known (see e.g. [41, 24]) that id_X for infinite dimensional X is never $(r,1)$-summing for $r < 2$, this somehow is the hidden reason why $T_m(X) = T_m(\mathrm{id}_X)$ does not depend on m. If we consider operators with nicer properties like $\mathrm{id} : \ell_p \to \ell_q$ (this is done in [39]) the dependency on m appears again.

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