Functiones et Approximatio
49.2 (2013), 221–227
doi: 10.7169/facm/2013.49.2.2

THE CRITICAL VALUES OF *L*-FUNCTIONS OF CM-BASE CHANGE FOR HILBERT MODULAR FORMS

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Abstract: In this paper we generalize some results, obtained by Shimura, on the critical values of L-functions of l-adic representations attached to quadratic CM-base change of Hilbert modular forms twisted by finite order characters, to the case of the critical values of L-functions of arbitrary base change to CM-number fields of l-adic representations attached to Hilbert modular forms twisted by some finite-dimensional representations.

Keywords: critical values, CM-fields, Hilbert modular forms.

1. Introduction

For F a totally real number field, let J_F be the set of infinite places of F, and let $\Gamma_F := \mathrm{Gal}(\bar{\mathbb{Q}}/F)$. Let f be a normalized Hecke eigenform of $\mathrm{GL}(2)/F$ of weight $k = (k(\tau))_{\tau \in J_F}$, where all $k(\tau)$ have the same parity and $k(\tau) \geqslant 2$. We denote by Π the cuspidal automorphic representation of $\mathrm{GL}(2)/F$ generated by f. In this paper we assume that Π is non-CM. We denote by ρ_Π the l-adic representation attached to Π, for some prime number l (by fixing an isomorphism $\iota : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ one can regard ρ_Π as a complex-valued representation). Define $k_0 = \max\{k(\tau) | \tau \in J_F\}$ and $k^0 = \min\{k(\tau) | \tau \in J_F\}$. Any integer $m \in \mathbb{Z}$ such that $(k_0 - k^0)/2 < m < (k_0 + k^0)/2$ is called a critical point for f or Π. Throughout this paper we write $a \sim b$ for $a, b \in \mathbb{C}$ if $b \neq 0$ and $a/b \in \bar{\mathbb{Q}}$.

In this article we prove the following result.

Theorem 1.1. *Assume $k(\tau) \geqslant 3$ for all $\tau \in J_F$, and $k(\tau)$ mod 2 is independent of τ. Let M be a finite CM-extension of F, and let $\Gamma_M := Gal(\bar{\mathbb{Q}}/M)$. Assume that χ is a continuous complex-valued abelian representation of Γ_M, and that ϕ is a continuous complex-valued representation of Γ_M satisfyng the following property: $K := \bar{\mathbb{Q}}^{ker\phi}$ is a $(\mathbb{Z}/2\mathbb{Z})^r$-extension of a CM-number field for some non-negative*

2010 Mathematics Subject Classification: primary: 11F41; secondary: 11F80, 11R42, 11R80 *integer r. Let $\psi = \phi \otimes \chi$. Then*

$$L\Big(m, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi\Big) \sim \pi^{(m+1-k_0)[M:\mathbb{Q}]\dim\psi} \langle f, f\rangle^{\frac{[M:F]}{2}\dim\psi},$$

for any integer m satisfying

$$(k_0+1)/2 \leqslant m < (k_0+k^0)/2.$$

Theorem 1.1 is a generalization of Theorem 5.7 of [S1] (i.e. Proposition 2.1 below; the inner product $\langle f, f\rangle$ is normalized as in §2 below).

2. Known results

Consider F a totally real number field and let J_F be the set of infinite places of F. If Π is a cuspidal automorphic representation (discrete series at infinity) of weight $k = (k(\tau))_{\tau\in J_F}$ of $\mathrm{GL}(2)/F$, where all $k(\tau)$ have the same parity and all $k(\tau) \geqslant 2$, let $k_0 = \max\{k(\tau)|\tau \in J_F\}$ and $k^0 = \min\{k(\tau)|\tau \in J_F\}$. Let O be the coefficient ring of Π (i.e. O is the ring of integers of the field generated over $\mathbb{Q}$ by the eigenvalues $a_\wp$ defined by $T_\wp f = a_\wp f$, where $T_\wp$ is the Hecke operator at $\wp$, and $\wp$ runs over the prime ideals of F), and let λ be a prime ideal of O above some rational prime l. Then there exists ([T]) a λ-adic representation

$$\rho_\Pi := \rho_{\Pi,\lambda} : \Gamma_F \to \mathrm{GL}_2(O_\lambda) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_l),$$

which satisfies $L(s, \iota\rho_{\Pi,\lambda}) = L\Big(s - \frac{(k_0-1)}{2}, \Pi\Big) = L\Big(s - \frac{(k_0-1)}{2}, f\Big)$, where $\iota : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ is a specific isomorphism (the above equality of L-functions is up to finitely many Euler factors), and the representation ρ_Π is unramified outside the primes dividing $\mathbf{n}l$. Because the line of convergence of $L(s,\Pi)$ is $\mathrm{Re}(s)=1$, we get that the line of convergence of $L(s, \rho_{\Pi,\lambda})$ is $\mathrm{Re}(s) = (k_0+1)/2$. Here f is the normalized Hecke eigenform of $\mathrm{GL}(2)/F$ of weight k corresponding to Π, $\mathbf{n}$ is the level of Π. We define

$$\langle f, f\rangle = \pi^{\sum_{\tau\in J_F} k(\tau)} \int_{Z_{\infty+}\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} f(x)\overline{f(x)}dx$$

where $Z_{\infty+} \simeq \mathbb{R}_+^\times$ is the connected component of the center of $\mathrm{GL}_2(\mathbb{R})$, and the measure is normalized such that $\mathrm{vol}(Z_{\infty+}\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)) = 1$.

We know (by Proposition 5.2 and Theorem 5.7 of [S1]; we actually use the fact that $L(s, \iota\rho_\Pi|_{\Gamma_M}\otimes\psi) = L(s, \iota\rho_\Pi \otimes \mathrm{Ind}_{\Gamma_M}^{\Gamma_F}\psi))$ in order to reduce Proposition 2.1 below to a particular case of Theorem 5.7 of [S1] where a convolution of two cuspidal automorphic representations (one non-CM, and the other CM) of $\mathrm{GL}(2)/F$ was considered; we remark that $\mathrm{Ind}_{\Gamma_M}^{\Gamma_F}\psi$ corresponds to a CM cuspidal automorphic representation of $\mathrm{GL}(2)/F$ of weight 1).

Proposition 2.1. *Assume $k(\tau) \geqslant 2$ for all $\tau \in J_F$ and $k(\tau)$ mod 2 is independent of τ. Let M be a quadratic CM-extension of F, and let ψ be a continuous one-dimensional representation of Γ_M. Then*

$$L\Big(m, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi\Big) \sim \pi^{(m+1-k_0)[M:\mathbb{Q}]}\langle f, f\rangle$$

for any integer m satisfying

$$(k_0+1)/2 \leqslant m < (k_0+k^0)/2.$$

3. The proof of Theorem 1.1 for ψ a character

We fix a non-CM cuspidal automorphic representation Π of $\mathrm{GL}(2)/F$ as in Theorem 1.1, and let M/F be a finite CM-extension. In this section we assume that ψ is an arbitrary one-dimensional continuous representation of Γ_M and prove Theorem 1.1 in this case.

We know the following result (Theorem 1.1 of [V1], or Theorem 2.1 of [V2], or Theorem A of [BGGT]):

Theorem 3.1. *Let Π be a cuspidal automorphic representation of weight $k = (k(\tau))_{\tau\in J_F}$ of $GL(2)/F$, where all $k(\tau)$ have the same parity and all $k(\tau) \geqslant 2$. Let F' be a totally real extension of F. Then there exists a totally real Galois extension F'' of F', such that $\rho_\Pi|_{\Gamma_{F''}}$ is cuspidal automorphic i.e. there exists a cuspidal automorphic representation Π'' of weight k'' of $GL(2)/F''$ such that $\rho_\Pi|_{\Gamma_{F''}} \cong \rho_{\Pi''}$.*

We denote by F' the maximal totally real subfield of M; hence M is a quadratic CM-extension of F'. Then from Theorem 3.1 we know that we can find a totally real Galois extension F'' of F', and a cuspidal automorphic representation Π'' of $\mathrm{GL}(2)/F''$ such that $\rho_\Pi|_{\Gamma_{F''}} \cong \rho_{\Pi''}$. Because Π is non-CM, we get that Π'' is non-CM.

From Theorem 15.10 of [CR] we know that there exist some subfields $M_i \subseteq MF''$ such that $M \subseteq M_i$ and $\mathrm{Gal}(MF''/M_i)$ are solvable, and some integers n_i, such that the trivial representation

$$1_M : \mathrm{Gal}(MF''/M) \to \mathbb{C}^\times,$$

can be written as

$$1_M = \sum_{i=1}^{u} n_i \mathrm{Ind}_{\mathrm{Gal}(MF''/M_i)}^{\mathrm{Gal}(MF''/M)} 1_{M_i},$$

(an equality in the character ring of $\mathrm{Gal}(MF''/M)$), where $1_{M_i} : \mathrm{Gal}(MF''/M_i) \to \mathbb{C}^\times$ is the trivial representation. In particular we have $1 = \sum_{i=1}^{u} n_i[M_i : M]$. Then

$$\begin{aligned} L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi) &= \prod_{i=1}^{u} L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \mathrm{Ind}_{\Gamma_{M_i}}^{\Gamma_M} 1_{M_i} \otimes \psi)^{n_i} \\ &= \prod_{i=1}^{u} L(s, \mathrm{Ind}_{\Gamma_{M_i}}^{\Gamma_M} (\iota\rho_\Pi|_{\Gamma_{M_i}}) \otimes \psi)^{n_i} \\ &= \prod_{i=1}^{u} L(s, \iota\rho_\Pi|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}})^{n_i}. \end{aligned}$$

Since $\rho_{\Pi}|_{\Gamma_{F''}}$ is cuspidal automorphic and MF'' is a quadratic extension of F'' we get that $\rho_{\Pi}|_{\Gamma_{MF''}}$ is cuspidal automorphic, and because $\mathrm{Gal}(MF''/M_i)$ is solvable, one gets easily (see §4 of [V4]) that $\rho_{\Pi}|_{\Gamma_{M_i}}$ is cuspidal automorphic.

Hence the function $L(s, \iota\rho_{\Pi}|_{\Gamma_M} \otimes \psi)$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation because each function $L(s, \iota\rho_{\Pi}|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}})$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation. Moreover, since each function $L(s, \iota\rho_{\Pi}|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}})$ has no poles or zeros for $\mathrm{Re}(s) \geqslant (k_0+1)/2$ (see Proposition 5.2 of [S1] and Proposition 4.16 of [S2]), we get that the function $L(s, \iota\rho_{\Pi}|_{\Gamma_M} \otimes \psi)$ has no poles or zeros for $\mathrm{Re}(s) \geqslant (k_0+1)/2$. Thus for any integer m satisfying

$$k_0 + 1 \leqslant m,$$

we get the identity

$$L\Big(m, \iota\rho_{\Pi}|_{\Gamma_M} \otimes \psi\Big) = \prod_{i=1}^{u} L\Big(m, \iota\rho_{\Pi}|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}}\Big)^{n_i}.$$

Let F_i be the maximal totally real subfield of M_i. Since $\rho_{\Pi}|_{\Gamma_{M_i}}$ is cuspidal automorphic and M_i/F_i is quadratic, one can prove easily that $\rho_{\Pi}|_{\Gamma_{F_i}}$ is cuspidal automorphic (see Lemma 1.3 of [BGHT]), so $\rho_{\Pi}|_{\Gamma_{F_i}} \cong \rho_{\Pi_i}$ for some cuspidal automorphic representation Π_i of $\mathrm{GL}(2)/F_i$. We denote by f_i the normalized Hecke eigenform of $\mathrm{GL}(2)/F_i$ associated to Π_i. Then f_i has weight $k_i = (k_i(\tau))_{\tau \in J_{F_i}}$, where J_{F_i} is the set of infinite places of F_i, and $k_i(\tau) = k(\tau|F)$ for any $\tau \in J_{F_i}$.

Now from Proposition 2.1 we get that

$$L\Big(m, \iota\rho_{\Pi}|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}}\Big) \sim \pi^{(m+1-k_0)[M_i:\mathbb{Q}]}\langle f_i, f_i\rangle,$$

for any integer m satisfying

$$(k_0+1)/2 \leqslant m < (k_0 + k^0)/2.$$

But we know that (see the paragraph just before Remark 5.1 of [V3])

$$\langle f_i, f_i\rangle \sim \langle f, f\rangle^{[F_i:F]},$$

and using the fact that $1 = \sum_{i=1}^{u} n_i[M_i : M]$, we obtain

$$\begin{aligned} L\Big(m, \iota\rho_{\Pi}|_{\Gamma_M} \otimes \psi\Big) &\sim \pi^{\sum_{i=1}^{u}(m+1-k_0)[M_i:\mathbb{Q}]n_i} \prod_{i=1}^{u} \langle f_i, f_i\rangle^{n_i} \\ &\sim \pi^{\sum_{i=1}^{u}(m+1-k_0)[M_i:\mathbb{Q}]n_i}\langle f, f\rangle^{\sum_{i=1}^{u}[F_i:F]n_i} \\ &\sim \pi^{(m+1-k_0)[M:\mathbb{Q}]}\langle f, f\rangle^{\frac{[M:F]}{2}}, \end{aligned}$$

for any integer m satisfying

$$(k_0+1)/2 \leqslant m < (k_0 + k^0)/2,$$

which proves Theorem 1.1 for ψ a one-dimensional representation. ■

4. The proof of Theorem 1.1 for general ψ

Let $\psi = \phi \otimes \chi$ be a finite-dimensional representation of Γ_M as in Theorem 1.1. We denote by M' the maximal CM-subfield of $K := \bar{\mathbb{Q}}^{\ker\phi}$ which contains M. Applying the conditions in Theorem 1.1, we see that K is a $(\mathbb{Z}/2\mathbb{Z})^r$-extension of M' for some r, and χ is a direct sum of one-dimensional representations.

From the beginning of §15 of [CR] we know that there exist some subfields $E_j \subseteq K$ such that $M \subseteq E_j$ and $\mathrm{Gal}(K/E_j)$ are solvable (actually we don't use this solvability), and some integers m_j, such that the representation

$$\phi : \mathrm{Gal}(K/M) \to \mathrm{GL}_N(\mathbb{C}),$$

can be written as

$$[K:M]\phi = \sum_{j=1}^{v} m_j \mathrm{Ind}_{\mathrm{Gal}(K/E_j)}^{\mathrm{Gal}(K/M)} 1_{E_j},$$

where $1_{E_j} : \mathrm{Gal}(K/E_j) \to \mathbb{C}^\times$ is the trivial representation. In particular we have $[K:M]\dim\phi = \sum_{j=1}^{v} m_j [E_j : M]$. Then

$$\begin{aligned} L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \phi)^{[K:M]} &= \prod_{j=1}^{v} L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \mathrm{Ind}_{\Gamma_{E_j}}^{\Gamma_M} 1_{E_j})^{m_j} \\ &= \prod_{j=1}^{v} L(s, \mathrm{Ind}_{\Gamma_{E_j}}^{\Gamma_M}(\iota\rho_\Pi|_{\Gamma_{E_j}}))^{m_j} \\ &= \prod_{j=1}^{v} L(s, \iota\rho_\Pi|_{\Gamma_{E_j}})^{m_j}. \end{aligned}$$

Let $M_j := E_j \cap M'$. Then E_j is a $(\mathbb{Z}/2\mathbb{Z})^{r_j}$-extension of M_j for some r_j (this is true because from the fact that K is a $(\mathbb{Z}/2\mathbb{Z})^r$-extension of M', we get that $E_j M'$ is a $(\mathbb{Z}/2\mathbb{Z})^{r_j}$-extension of M' for some r_j, and hence E_j is a $(\mathbb{Z}/2\mathbb{Z})^{r_j}$-extension of $M_j = E_j \cap M'$). Thus

$$\begin{aligned} L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \phi)^{[K:M]} &= \prod_{j=1}^{v} L(s, \iota\rho_\Pi|_{\Gamma_{E_j}})^{m_j} \\ &= \prod_{j=1}^{v} \prod_{\phi_j : \mathrm{Gal}(E_j/M_j) \to \mathbb{C}^\times} L(s, \iota\rho_\Pi|_{\Gamma_{M_j}} \otimes \phi_j)^{m_j}. \end{aligned}$$

Also one has

$$\begin{aligned} L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi)^{[K:M]} &= L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \phi \otimes \chi) \\ &= \prod_{j=1}^{v} \prod_{\phi_j : \mathrm{Gal}(E_j/M_j) \to \mathbb{C}^\times} L(s, \iota\rho_\Pi|_{\Gamma_{M_j}} \otimes \phi_j \otimes \chi|_{\Gamma_{M_j}})^{m_j}. \end{aligned}$$

Hence the function $L(s,\iota\rho_\Pi|_{\Gamma_M}\otimes\psi)^{[K:M]}$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation because from §3 we know that each function $L(s,\iota\rho_\Pi|_{\Gamma_{M_j}}\otimes\phi_j\otimes\chi|_{\Gamma_{M_j}})$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation. Also, since each function $L(s,\iota\rho_\Pi|_{\Gamma_{M_j}}\otimes\phi_j\otimes\chi|_{\Gamma_{M_j}})$ has no poles or zeros for $\mathrm{Re}(s)\geqslant(k_0+1)/2$, we get that the function $L(s,\iota\rho_\Pi|_{\Gamma_M}\otimes\psi)$ has no poles or zeros for $\mathrm{Re}(s)\geqslant(k_0+1)/2$. Thus for any integer m satisfying

$$k_0+1\leqslant m,$$

we get the identity

$$L\Big(m,\iota\rho_\Pi|_{\Gamma_M}\otimes\psi\Big)^{[K:M]}=\prod_{j=1}^{v}\prod_{\phi_j:\mathrm{Gal}(E_j/M_j)\to\mathbb{C}^\times}L\Big(m,\iota\rho_\Pi|_{\Gamma_{M_j}}\otimes\phi_j\otimes\chi|_{\Gamma_{M_j}}\Big)^{m_j}.$$

From §3 we know that

$$L\Big(m,\iota\rho_\Pi|_{\Gamma_{M_j}}\otimes\phi_j\otimes\chi|_{\Gamma_{M_j}}\Big)\sim\pi^{(m+1-k_0)[M_j:\mathbb{Q}]\dim\chi}\langle f,f\rangle^{\frac{[M_j:F]}{2}\dim\chi},$$

for any integer m satisfying

$$(k_0+1)/2\leqslant m<(k_0+k^0)/2.$$

Hence from $[K:M]\dim\phi=\sum_{j=1}^{v}m_j[E_j:M]$ we get that

$$\begin{aligned}L\Big(m,\iota\rho_\Pi|_{\Gamma_M}\otimes\psi\Big)^{[K:M]}&=\prod_{j=1}^{v}\prod_{\phi_j:\mathrm{Gal}(E_j/M_j)\to\mathbb{C}^\times}L\Big(m,\iota\rho_\Pi|_{\Gamma_{M_j}}\otimes\phi_j\otimes\chi|_{\Gamma_{M_j}}\Big)^{m_j}\\&\sim\pi^{\sum_{j=1}^{v}(m+1-k_0)[E_j:\mathbb{Q}]m_j\dim\chi}\langle f,f\rangle^{\sum_{j=1}^{v}\frac{[E_j:F]}{2}m_j\dim\chi}\\&\sim\pi^{(m+1-k_0)[K:\mathbb{Q}]\dim\psi}\langle f,f\rangle^{\frac{[K:F]}{2}\dim\psi},\end{aligned}$$

and thus

$$L\Big(m,\iota\rho_\Pi|_{\Gamma_M}\otimes\psi\Big)\sim\pi^{(m+1-k_0)[M:\mathbb{Q}]\dim\psi}\langle f,f\rangle^{\frac{[M:F]}{2}\dim\psi},$$

for any integer m satisfying

$$(k_0+1)/2\leqslant m<(k_0+k^0)/2.$$

This concludes the proof of Theorem 1.1. ■

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Received: 17 August 2011; **revised:** 2 October 2012