

A MAXIMALLY SEPARATED SEQUENCE

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Abstract: The paper builds on earlier published work by the author in which a measure for the slowness of clustering of a bounded real sequence, called *separation*, was introduced. Here a conjecture of the earlier paper is proved: that a particular sequence of rational numbers – the f sequence – defined in that paper is of maximal separation.

Keywords: bounded real sequence, separation, optimization.

1. Introduction

Following the notation of [1], a sequence of natural numbers a_i ($i \in \mathbb{N}$) is defined by the recurrence relation $a_{i+1} - 3a_i + a_{i-1} = 0$ ($i = 1, 2, \dots$) with $a_0 = 0$ and $a_1 = 1$. The same paper introduced, and showed to be well defined, a rational sequence $\mathbf{f} = (f_i : i \in \mathbb{N})$ specified by

$$f_n = \sum_{i=1}^r \frac{c_i}{a_i} \quad \text{iff} \quad n = \sum_{i=1}^r c_i a_i,$$

where the coefficients $c_i \in \{0, 1, 2\}$ ($i = 1, \dots, r$) are chosen to be as large as possible from the highest-indexed term downwards.

Definition 1. *Given any bounded real sequence $\mathbf{x} = (x_0, x_1, \dots)$, its spacing and span are defined by*

$$\begin{aligned} \text{spac } \mathbf{x} &= \inf\{|i - j||x_i - x_j| : i \neq j; i, j \in \mathbb{N}\}, \\ \text{span } \mathbf{x} &= \sup\{|x_i - x_j| : i, j \in \mathbb{N}\}. \end{aligned}$$

The separation of a nonconstant sequence \mathbf{x} is the ratio

$$\text{sep } \mathbf{x} = \frac{\text{spac } \mathbf{x}}{\text{span } \mathbf{x}}.$$

For completeness, the separation of a constant sequence is defined to be 0. The spacing, span, and separation of a finite sequence may be defined similarly in the obvious way. A sequence is unit-spaced if its spacing is 1, and it is maximally separated (ms) if no sequence has greater separation.

In [1], we proved that \mathbf{f} is unit-spaced and conjectured that it is also a maximally separated sequence. The purpose of the present paper is to show that \mathbf{f} is indeed an ms sequence.

The spacing, span, and separation of a sequence are clearly invariant under uniform shifting, say by $x_i \leftarrow x_i + c$ ($i \in \mathbb{N}$) with c a constant. The uniform scaling of a sequence, say by $x_i \leftarrow cx_i$ ($i \in \mathbb{N}$), with c a nonzero constant, scales the spacing and span equally, leaving the separation unchanged. Truncating a sequence, say by $x_i \leftarrow x_{n+i}$ ($i \in \mathbb{N}$) with $n \in \mathbb{N}$ a constant, cannot reduce the spacing or increase the span, and so cannot reduce the separation. Thus a truncated ms sequence is still an ms sequence. Starting with an arbitrary bounded sequence of nonzero spacing, by subtracting its infimum from each of its terms, we can shift the sequence to one with infimum 0; next, scaling the sequence by the reciprocal of its spacing yields a unit-spaced sequence; and, observing that such a sequence can have at most one null term since all the terms are different, we may truncate the sequence to remove its null term (if any) to produce a positive unit-spaced sequence whose separation is at least as great as that of the original sequence. Therefore, when considering maximality of separation, it will be sufficient to focus on unit-spaced positive sequences.

2. The proof of maximality

We will use the notation

$$b_n = 1 + \sum_{i=1}^n \frac{1}{a_i} \quad (n \in \mathbb{N}).$$

The following definition is the key idea of the present paper, although it will prove to be ephemeral.

Definition 2. For $n \in \mathbb{N}$, a finite sequence will be called left (resp. right) limboic of order n if it is a segment of a unit-spaced positive sequence and comprises $a_{n+1} + 1$ terms, none exceeding b_n , of which the initial (resp. final) term is the greatest.

As in [1], we denote by α and $\bar{\alpha}$ respectively the larger and the smaller root of the quadratic equation $x^2 - 3x + 1 = 0$. The following three results will be used in the proof of our key lemma (Lemma 3).

Lemma 1. $a_n < \bar{\alpha}a_{n+1} < a_n + 1$ ($n \in \mathbb{N}$).

Proof. Since $0 < \bar{\alpha} < 1$, the result holds for $n = 0$. Suppose now that it has been proved for $n = k - 1$:

$$a_{k-1} < \bar{\alpha}a_k < a_{k-1} + 1.$$

Then the recurrence relation $a_{k+1} - 3a_k + a_{k-1} = 0$ gives

$$3a_k - a_{k+1} < \bar{\alpha}a_k < 3a_k - a_{k+1} + 1.$$

Adding $a_{k+1} - \bar{\alpha}a_k$ throughout yields

$$(3 - \bar{\alpha})a_k < a_{k+1} < (3 - \bar{\alpha})a_k + 1.$$

After multiplying through by $\bar{\alpha}$ and using the identity $\bar{\alpha}(3 - \bar{\alpha}) = 1$, we get

$$a_k < \bar{\alpha}a_{k+1} < a_k + \bar{\alpha} < a_k + 1,$$

which is the result for $n = k$. The general result follows by induction. ■

Corollary 1. $a_{n+1} > 2a_n$ ($n \in \mathbb{N}$).

Lemma 2. $a_n^2 - 3a_n a_{n+1} + a_{n+1}^2 = 1$ ($n \in \mathbb{N}$).

Proof. The result obviously holds for $n = 0$. Suppose that it holds for $n = k - 1$. Then

$$\begin{aligned} 1 &= a_{k-1}^2 - 3a_{k-1}a_k + a_k^2 = (3a_k - a_{k+1})^2 - 3(3a_k - a_{k+1})a_k + a_k^2 \\ &= a_k^2 - 3a_k a_{k+1} + a_{k+1}^2, \end{aligned}$$

which is the result for $n = k$, as required for induction. ■

Lemma 3. *There are no limboic segments, left or right, of any order.*

Proof. A limboic segment of order 0, if it exists, comprises two terms, both positive and at least 1 apart. So the greater exceeds 1, contrary to the condition of being bounded above by 1 ($= b_0$). So there is no limboic segment of order 0. Now suppose, contrary to the lemma, that there is some $n \in \{1, 2, \dots\}$ such that there is a limboic segment of order n but none of any lower order. We consider just the case when this is left limboic; the argument in the right-hand case is essentially the same by looking at the terms of the considered sequence segment in reverse order. Thus we have a finite sequence of terms, which for convenience we will label x_0, \dots, x_p , where

$$p = a_{n+1}$$

and $0 < x_1, \dots, x_p < x_0 \leq b_n$, such that

$$x_0 - x_j \geq \frac{1}{j} \quad \text{for } j = 1, \dots, p.$$

Let x_m be the largest of x_1, \dots, x_p . Then

$$x_m - x_j \geq \frac{1}{m - j} \quad \text{for } j = 1, \dots, m - 1.$$

For convenience, we will now write

$$q = a_n.$$

Consider the case when $m \leq q$. By Corollary 1, we have $2q < p$, and so $m + q < p$. Hence

$$x_{m+1}, \dots, x_{m+q} < x_m \leq x_0 - \frac{1}{m} \leq b_n - \frac{1}{q} = b_{n-1}.$$

That is, (x_m, \dots, x_{m+q}) is left limboic of order $n-1$, contrary to our supposition.

Thus only the case $m \geq q+1$ remains to be considered. In this case, the terms x_1, \dots, x_{m-1} , and so x_{m-q}, \dots, x_{m-1} , are bounded according to

$$x_j \leq x_0 - \frac{1}{j} \quad \text{and} \quad x_j \leq x_m - \frac{1}{m-j} \quad \text{for } j = 1, \dots, m-1.$$

If we plot these (j, x_j) points in the cartesian (t, x) plane, we see that they are confined to the region

$$\left\{ (t, x) \in \mathbb{R}^2 : 0 < t < m; 0 < x \leq x_0 - \frac{1}{t}; 0 < x \leq x_m - \frac{1}{m-t} \right\}.$$

Note that $x_0 \leq b_n$ and $m \leq p$, so that $x_m \leq x_0 - 1/m \leq b_n - 1/p$. Then (j, x_j) ($j = 1, \dots, m-1$) must lie within the region

$$\left\{ (t, x) \in \mathbb{R}^2 : 0 < t < p; 0 < x < b_n - \frac{1}{t}; 0 < x < b_n - \frac{1}{p} - \frac{1}{p-t} \right\}$$

(see Fig. 7). Since the curve $x = b_n - 1/t$ ($t > 0$) is strictly increasing, while the curve $x = b_n - 1/p - 1/(p-t)$ ($t < p$) is strictly decreasing, the curves meet at a point whose ordinate (x) value bounds all the x_i ($i = 1, \dots, m-1$). This x value corresponds to the t value given by

$$\frac{1}{t} = \frac{1}{p} + \frac{1}{p-t}$$

or $t^2 - 3pt + p^2 = 0$. The solution of this equation in the range $0 < t < p$ is $t = \bar{\alpha}p$. Now $\bar{\alpha}p$ lies between q and $q+1$, by Lemma 1. Since $m \geq q+1 > \bar{\alpha}p$, the downward-sloping curve $x = b_n - 1/p - 1/(p-t)$ ($t < p$) to the right of $t = \bar{\alpha}p$ provides the effective bound for x_m . That is, x_m is bounded by the curve's

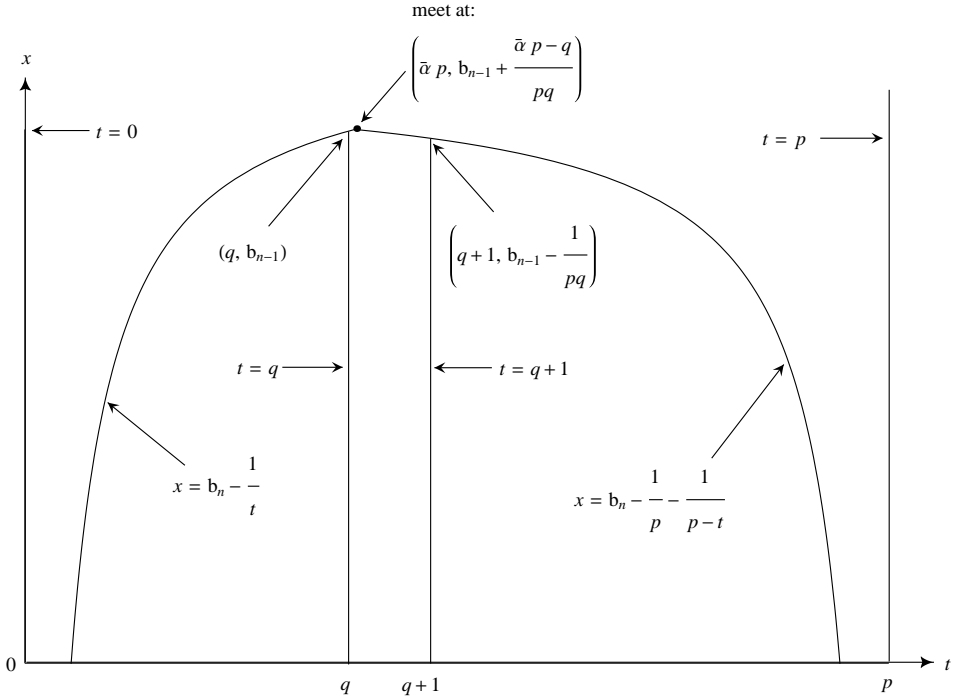


Figure 7: The bounding region for the points (j, x_j) ($j = 1, \dots, m - 1$).

ordinate at the leftmost integral value $q + 1$ of the abscissa t above $\bar{\alpha}p$:

$$\begin{aligned}
 x_m &\leq b_n - \frac{1}{p} - \frac{1}{p - (q + 1)} \\
 &= b_n - \frac{1}{q} - \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{p - q - 1} \right) \\
 &= b_n - \frac{1}{q} - \frac{p - q - (p^2 - 3pq + q^2)}{pq(p - q - 1)} \\
 &= b_n - \frac{1}{q} - \frac{1}{pq} \quad (\text{by Lemma 2}) \\
 &< b_n - \frac{1}{q} = b_{n-1}.
 \end{aligned}$$

It follows that (x_{m-q}, \dots, x_m) is right limboic of order $n - 1$, and so does not exist, giving our sought contradiction. ■

From [1], each f_n is of the form $\sum_{i=1}^r c_i/a_i$, where $c_i \in \{0, 1, 2\}$ ($i = 1, \dots, r$) and all choices of c_i are possible provided that a digit 0 appears between any

two occurrences of the digit 2 in the string $c_1 \cdots c_r$. Thus, by Corollary 1, the successively highest terms of \mathbf{f} after the initial term are $\frac{1}{1}, \frac{2}{1}, \frac{2}{1} + \frac{1}{3}, \frac{2}{1} + \frac{1}{3} + \frac{1}{8}, \frac{2}{1} + \frac{1}{3} + \frac{1}{8} + \frac{1}{21}, \dots$ and generally $1 + \sum_{i=1}^r 1/a_i$ ($r = 0, 1, \dots$). The supremum of these is

$$\beta = \lim_{n \rightarrow \infty} b_n = 1 + \sum_{i=1}^{\infty} \frac{1}{a_i}.$$

Now the infimum of the terms of \mathbf{f} is 0 (the initial term, and also the infimum of its terms $1/a_n$), and so $\text{span } \mathbf{f} = \beta$. Also $\text{spac } \mathbf{f} = 1$ since, by Theorem 2 of [1], $\text{spac } \mathbf{f} \geq 1$ and this bound is attained: for example, $f_1 - f_0 = 1 - 0 = 1$. It follows that $\text{sep } \mathbf{f} = 1/\beta$.

Theorem 1. *The sequence \mathbf{f} is of maximal separation.*

Proof. Let $\mathbf{x} = (x_i : i \in \mathbb{N})$ be any bounded unit-spaced positive sequence of zero infimum. Then $\text{sep } \mathbf{x} = 1/x_s$, where

$$x_s = \sup\{x_i : i \in \mathbb{N}\}.$$

Let us suppose, for the purpose of contradiction, that $\text{sep } \mathbf{x} > 1/\beta$. Then $x_s < \beta = \sup\{b_j : j \in \mathbb{N}\}$, which implies that $x_s \leq b_n$ for some $n \in \mathbb{N}$, and so $x_i \leq b_n$ for all $i \in \mathbb{N}$. Now take any segment of $2a_{n+1} + 1$ terms of \mathbf{x} . Wherever the greatest term x_m of this segment lies, there will be a subsegment of a_{n+1} terms either immediately below or immediately above x_m which, together with x_m , constitutes a limboic segment of order n . By Lemma 3, this is impossible, and so our contradiction is reached. ■

3. A question of density

When we look at the terms of \mathbf{f} , we notice that there are none in the intervals $[\beta - \frac{5}{3}, 1)$ or $[\beta - \frac{2}{3}, 2)$ just below 1 and 2. Indeed, there is a gap below (though never above) every term. Thus $\{f_i : i \in \mathbb{N}\}$ is nowhere dense in $[0, \beta]$. The sequence $\boldsymbol{\alpha} = (\alpha_i : i \in \mathbb{N})$, specified in [1] by $\alpha_i \equiv i \pmod{\alpha}$ with $\alpha_i \in [0, \alpha)$ ($i \in \mathbb{N}$), has slightly lower-than-maximal separation ($\text{sep } \boldsymbol{\alpha} = \bar{\alpha} \approx 0.38197$; cf. $1/\beta \approx 0.39442$) and is dense in the interval that it spans. So a question arises: does any dense sequence have a separation exceeding $\bar{\alpha}$?

References

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