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VARIATIONS OF THE RAMANUJAN POLYNOMIALS AND REMARKS ON $\zeta(2j+1)/\pi^{2j+1}$

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Abstract: We observe that five polynomial families have all of their roots on the unit circle. We prove the statements explicitly for four of the polynomial families. The polynomials have coefficients which involve Bernoulli numbers, Euler numbers, and the odd values of the Riemann zeta function. These polynomials are closely related to the Ramanujan polynomials, which were recently introduced by Murty, Smyth and Wang [MSW]. Our proofs rely upon theorems of Schinzel [S], and Lakatos and Losonczi [LL] and some generalizations.

Keywords: Ramanujan polynomials, Riemann zeta function values, reciprocal polynomials, roots on the unit circle, Bernoulli numbers, Euler numbers.

1. Introduction

In a recent paper, Murty, Smyth and Wang considered the *Ramanujan polynomials* [MSW]. They were defined by Gun, Murty and Rath [GMR] using

$$R_{2k+1}(z) := \sum_{j=0}^{k+1} \frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!} z^{2j}, \tag{1.1}$$

where B_j is the jth Bernoulli number. Among other fascinating results, Murty, Smyth and Wang showed that $R_{2k+1}(z)$ has all of its non-real roots on the unit circle. The purpose of this paper is to study some variants of $R_{2k+1}(z)$, which also have many roots on the unit circle.

Conjecture 1.1. *Let B_j denote the Bernoulli numbers, and let E_j denote the Euler numbers. Suppose that $k \geqslant 2$. The following polynomials have all of their*

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non-zero roots on the unit circle:

$$\begin{aligned}
P_k(z) &:= \frac{(2\pi)^{2k-1}}{(2k)!}\sum_{j=0}^{k}(-1)^j B_{2j}B_{2k-2j}\binom{2k}{2j}z^{2j} \\
&\quad + \zeta(2k-1)\left(z^{2k-1}+(-1)^k z\right), \\
Q_k(z) &:= \left(2^{2k}+1\right)P_k(z) - 2^{2k}P_k\left(z/2\right) - P_k(2z), \\
Y_k(z) &:= \frac{\pi}{2^{2k}}\left(Q_k(i\sqrt{z})+Q_k(-i\sqrt{z})\right) \\
W_k(z) &:= \left(2^{2k-1}+2\right)P_k(z) - 2^{2k}P_k\left(z/2\right) - P_k(2z), \\
S_k(z) &:= \sum_{j=0}^{k} E_{2j}E_{2k-2j}\binom{2k}{2j}z^j. \qquad (1.2)
\end{aligned}$$

We will offer a general proof of Conjecture 1.1 for $Q_k(z)$, $Y_k(z)$, $W_k(z)$, and $S_k(z)$. It seems that $P_k(z)$ is more difficult to handle. In Section 4 we offer several partial results concerning $P_k(z)$.

An important secondary goal of this work, is to highlight a connection with the odd values of the Riemann zeta function $\zeta(s)$. While it is a classical fact that $\zeta(2j)/\pi^{2j}$ is rational when $j \geqslant 1$, very little is known about the arithmetic nature of $\zeta(2j+1)$. The only theorems in this direction are celebrated irrationality results. For instance, Apéry showed that $\zeta(3)$ is irrational [A, P], Ball and Rivoal proved that infinitely many odd zeta values are irrational [BR, R] (generalized by Hessami Pilerud and Hessami Pilerud [HH]), and Zudilin established that at least one element of the set $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$ is also irrational [Z].

Therefore, let us briefly consider the identity which gave birth to the Ramanujan polynomials [B, p. 276]. The formula can be written as

$$\frac{1}{2}P_k(z) = -\sum_{n=1}^{\infty}\frac{1}{n^{2k-1}}\frac{z^{2k-1}}{e^{2\pi n/z}-1} + (-1)^{k+1}\sum_{n=1}^{\infty}\frac{1}{n^{2k-1}}\frac{z}{e^{2\pi n z}-1}. \qquad (1.3)$$

This identity holds whenever $z \notin i\mathbb{Q}$ [1]. The restriction is necessary to ensure that both infinite series converge. Not surprinsigly, this formula is also mentioned in [P] while dealing with the irrationality of $\zeta(3)$. Notice that Gun, Murty and Rath used (1.3) to express odd zeta values in terms of Eichler integrals [GMR]. Now consider the case when $k = 2$. A brief numerical calculation shows that the polynomial obtained from the left-hand side

$$z^4 + 5z^2 + 1 - \frac{90\zeta(3)}{\pi^3}\left(z^3+z\right) = 0, \qquad (1.4)$$

has all of its roots on the unit circle. Notice that if we truncate the right-hand

[1] This identity is often presented in the literature as valid for $z \in \mathbb{R}$ positive, but it is not hard to see that it extends by analytic continuation to the largest domain where the infinite series converge, i.e., $z \notin i\mathbb{Q}$

side of (1.3), then we can obtain an approximation for $\zeta(3)$:

$$\zeta(3) \approx \left(\frac{z^4+5z^2+1}{z^3+z}\right)\frac{\pi^3}{90}, \tag{1.5}$$

where

$$0 = \frac{z}{e^{2\pi/z}-1} + \frac{z^{-1}}{e^{2\pi z}-1}.$$

This approximation gives six decimal places of numerical accuracy. The accuracy can be increased by including higher order terms in the truncation. It would be extremely interesting if this idea could be used to say something about the irrationality of $\zeta(3)/\pi^3$. In this paper, we will settle for the more modest goal of studying the polynomial families listed in Conjecture 1.1.

Before proceeding, it is important to note that all of our polynomials are naturally occurring objects. The following identities are corollaries to (1.3):

$$2^{-2k}\, Q_k(z) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^{2k-1}}\frac{z^{2k-1}}{e^{\pi n/z}+1} + (-1)^k \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^{2k-1}}\frac{z}{e^{\pi n z}+1},$$

$$-\frac{1}{\pi}Y_k(-z^2) = z^{2k-1}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{\tanh\left(\frac{\pi n}{2z}\right)}{n^{2k-1}} + (-1)^k\, z \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{\tanh\left(\frac{\pi n z}{2}\right)}{n^{2k-1}},$$

$$-2^{1-2k}\, W_k(z) = z^{2k-1}\sum_{n=1}^{\infty} \frac{(-1)^n \operatorname{csch}\left(\frac{\pi n}{z}\right)}{n^{2k-1}} + (-1)^k\, z \sum_{n=1}^{\infty} \frac{(-1)^n \operatorname{csch}(\pi n z)}{n^{2k-1}}.$$

Proofs follow from the fact that $Q_k(z)$, $W_k(z)$, and $Y_k(z)$ are all linear combinations of $P_k(z)$'s. Ramanujan also discovered a similar formula for $S_k(z)$. We have stated that identity in (2.7).

2. $S_k(z)$, $Y_k(z)$ and the theorems of Schinzel, Lakatos and Losonczi

In this section we prove that $S_k(z)$ and $Y_k(z)$ have all of their non-zero roots on the unit circle. Our proofs follow from applying the theorems of Schinzel [S], Lakatos and Losonczi [LL], and Lakatos [L]. Lakatos proved that any reciprocal polynomial $\sum_{j=0}^{k} A_j z^j$, with real-valued coefficients, which satisfies

$$|A_k| \geqslant \sum_{j=0}^{k} |A_j - A_k|, \tag{2.1}$$

must have all of its roots on the unit circle. If the inequality is strict then the polynomial has only simple roots. Equation (2.1) is a very strong restriction. There have been a number of recent improvements to (2.1) with a similar flavor (see [S] and [LL]). Schinzel proved that any self-inversive polynomial which satisfies

$$|A_k| \geqslant \inf_{\substack{c,d\in\mathbb{C}\\ |d|=1}} \sum_{j=0}^{k} \left|cA_j - d^{k-j}A_k\right|, \tag{2.2}$$

must have all of its roots on the unit circle [S]. Self-inversive polynomials have complex-valued coefficients which satisfy $A_j = \epsilon\overline{A_{k-j}}$, for some fixed $|\epsilon| = 1$. Notice that the class of self-inversive polynomials includes both reciprocal ($\epsilon = 1$) and anti-reciprocal ($\epsilon = -1$) polynomials. In Theorems 2.1 and 2.2 we apply Schinzel's theorem with $d = 1$.

Theorem 2.1. *Suppose that $k \geqslant 1$. Then all of the roots of the polynomial given by* (1.2) *lie on the unit circle. Furthermore, they are all simple.*

Proof. With (2.2) in mind, let us begin by setting

$$A_j := E_{2j}E_{2k-2j}\binom{2k}{2j}.$$

The sign of E_{2n} is $(-1)^n$. This implies that all of the coefficients of $S_k(z)$ have sign $(-1)^k$. As $E_0 = 1$, we have $A_k = E_{2k}$.

Our proof consists of three main steps. First we remove the absolute values from the sum in (2.2). This is easily accomplished by showing that $(-1)^k(cA_j - A_{k-2}) > 0$ for $c = \frac{\pi}{4(1+3^{-1-2k})}$. Next we evaluate $\sum_{j=0}^{k} A_j$ explicitly, and finally we deduce the desired upper bound.

In order to remove the absolute value signs from (2.2), we need to demonstrate that $(-1)^k(cA_j - A_{k-2}) > 0$. Using the following bound for Euler numbers [AS, p. 805]:

$$\frac{4^{k+1}(2k)!}{\pi^{2k+1}} > |E_{2k}| > \frac{4^{k+1}(2k)!}{\pi^{2k+1}(1+3^{-1-2k})}, \tag{2.3}$$

leads to

$$\begin{aligned}\frac{4(1+3^{-1-2k})}{\pi}|A_k| &= \frac{4(1+3^{-1-2k})}{\pi}|E_{2k}E_0| \\ &> \frac{4^{k+2}(2k)!}{\pi^{2k+2}} \\ &> |E_{2j}E_{2k-2j}|\binom{2k}{2j} = |A_j|.\end{aligned}$$

The absolute values can be removed because both A_k and A_j have sign $(-1)^k$:

$$\frac{(-1)^k 4(1+3^{-1-2k})}{\pi}A_k > (-1)^k A_j > 0.$$

If we take

$$c = \frac{\pi}{4(1+3^{-1-2k})},$$

then the previous inequality implies $(-1)^k A_k - c(-1)^k A_j > 0$.

Let $E_n(z)$ denote the classical Euler polynomials, and recall a standard convolution identity [Di]:

$$\sum_{j=0}^{n}\binom{n}{j}E_j(v)E_{n-j}(w) = 2(1-w-v)E_n(v+w) + 2E_{n+1}(v+w).$$

Set $v = w = \frac{1}{2}$, and then use $E_n = 2^n E_n\left(\frac{1}{2}\right)$, to obtain an expression for $S_k(1)$. We have

$$|S_k(1)| = \left|2^{2k+1}E_{2k+1}(1)\right| = \frac{2^{2k+1}(2^{2k+2}-1)|B_{2k+2}|}{k+1}.$$

The evaluation of $E_{2k+1}(1)$ follows from [AS, p. 805]. Thus

$$\begin{aligned}\sum_{j=0}^{k}|cA_j - A_k| &= (-1)^k\sum_{j=0}^{k}(A_k - cA_j)\\ &= (k+1)(-1)^kA_k - c(-1)^kS_k(1)\\ &= (k+1)|A_k| - c|S_k(1)|\\ &= (k+1)|A_k| - \frac{\pi 2^{2k-1}(2^{2k+2}-1)|B_{2k+2}|}{(k+1)(1+3^{-1-2k})}.\end{aligned}$$

To finish the verification of (2.2), we need to show that the last expression is bounded above by $|A_k|$. This is equivalent to showing that

$$\frac{\pi 2^{2k-1}(2^{2k+2}-1)|B_{2k+2}|}{(k+1)(1+3^{-1-2k})} \geqslant k|A_k| = k|E_{2k}|. \tag{2.4}$$

We will resort to an inequality for Bernoulli numbers [AS, p. 805]:

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})}. \tag{2.5}$$

Thus we find

$$\begin{aligned}\frac{\pi 2^{2k-1}(2^{2k+2}-1)|B_{2k+2}|}{(k+1)(1+3^{-1-2k})} &> \frac{\pi 2^{2k-1}(2^{2k+2}-1)}{(k+1)(1+3^{-1-2k})}\frac{2(2k+2)!}{(2\pi)^{2k+2}}\\ &= \frac{2^{2k+1}(1-2^{-2-2k})}{(1+3^{-1-2k})}\frac{(2k+1)!}{\pi^{2k+1}}.\end{aligned} \tag{2.6}$$

On the other hand, we have already used the fact that Euler numbers are bounded by (2.3). Substituting (2.6) and (2.3) into (2.4) reduces the inequality to

$$\frac{2k+1}{2k} > \frac{1+3^{-1-2k}}{1-2^{-2-2k}}.$$

This final inequality is easily verified with elementary calculus for $k \geqslant 1$. Since the inequality is strict, we conclude immediately that $S_k(z)$ has only simple roots which all lie on the unit circle. ■

We have proved that $S_k(z)$ has all of its roots on the unit circle. Perhaps it is interesting to note that $S_k(z)$ satisfies

$$\begin{aligned}\frac{(\pi/2)^{2k+1}}{2(2k)!}S_k(-z^2) = z^{2k}&\sum_{n=1}^{\infty}\frac{\chi_{-4}(n)\operatorname{sech}(\pi n/2z)}{n^{2k+1}}\\ &+ (-1)^k\sum_{n=1}^{\infty}\frac{\chi_{-4}(n)\operatorname{sech}(\pi nz/2)}{n^{2k+1}},\end{aligned} \tag{2.7}$$

where $\chi_{-4}(n)$ is the non-principal character mod 4. This formula appears in Ramanujan's notebooks [B, p. 276]. As a result it is easy to approximate the roots of $S_k(z)$ by the roots of exponential polynomials. It remains to be seen whether or not there are any interesting applications for this observation.

To illustrate our method a second time, we prove that the polynomial $Y_k(z)$ has all of its non-zero roots on the unit circle. Notice that $Y_k(z)$ is a close analogue of $S_k(z)$, except that it involves Bernoulli numbers rather than Euler numbers.

Theorem 2.2. *Suppose that $k \geqslant 2$. The polynomial*

$$\begin{aligned} Y_k(z) &= \frac{\pi}{2^{2k}}\left(Q_k(i\sqrt{z}) + Q_k(-iz)\right) \\ &= \frac{\pi^{2k}}{(2k)!}\sum_{j=0}^{k} B_{2j}B_{2k-2j}(2^{2j}-1)(2^{2k-2j}-1)\binom{2k}{2j}z^j, \end{aligned}$$

has all of its non-zero roots on the unit circle. Furthermore, all of the roots are simple.

Proof. Observe that $Y_k(z)$ has degree $k-1$, since the coefficients of z^k and z^0 are identically zero. We prove that $Y_k(z)/z$ satisfies the hypothesis of Schinzel's theorem [S]. If we eliminate the trivial factor of z, then we obtain a polynomial of the form

$$\frac{Y_k(z)}{z} = \sum_{j=0}^{k-2} A_j z^j,$$

where

$$A_j = \frac{(2\pi)^{2k}}{(2k)!}\binom{2k}{2j+2}(1-2^{-2-2j})(1-2^{2-2k+2j})B_{2j+2}B_{2k-2j-2}.$$

Notice that $Y_k(z)/z$ is reciprocal, since $A_{k-2-j} = A_j$. By elementary properties of Bernoulli numbers, the sign of A_j is $(-1)^k$ for all j.

Schinzel's theorem can be applied if the following inequality holds:

$$|A_{k-2}| \geqslant \sum_{j=0}^{k-2} |cA_j - A_{k-2}|, \tag{2.8}$$

for some $c \in \mathbb{C}$. We prove that (2.8) holds when $c = \frac{\pi^2(1-2^{2-2k})}{8(1-2^{3-2k})}$. Our proof follows the same three steps as in the case of $S_k(z)$.

In order to remove the absolute value signs from (2.8), we need to demonstrate that $(-1)^k(cA_j - A_{k-2}) > 0$. We demonstrate this by comparing an upper bound on $(-1)^k A_{k-2}$, with a lower bound on $(-1)^k A_j$. The lower bound on $|A_j|$ is a consequence of an inequality from [Da]:

$$|B_{2n}| > \frac{2(2n)!}{(2\pi)^{2n}(1-2^{-2n})}.$$

In particular we find

$$(-1)^k A_j = |A_j| > 4. \tag{2.9}$$

By (2.5), we find an upper bound for $|A_{k-2}|$:

$$(-1)^k A_{k-2} = |A_{k-2}| < \frac{\pi^2}{2} \frac{1-2^{2-2k}}{1-2^{3-2k}}. \tag{2.10}$$

Comparing (2.10) and (2.9), allows us to easily conclude

$$(-1)^k (cA_j - A_{k-2}) > 0, \tag{2.11}$$

whenever $k > 2$.

Since we have proved (2.11), Schinzel's sum immediately reduces to

$$\sum_{j=0}^{k-2} |cA_j - A_{k-2}| = -(k-1)(-1)^k A_{k-2} + (-1)^k c \sum_{j=0}^{k-2} A_j. \tag{2.12}$$

Now we simplify the remaining sum. Let $B_j(z)$ denote the usual Bernoulli polynomials. By standard evaluations of Bernoulli polynomials [AS, p. 805], we have

$$A_j = \frac{(2\pi)^{2k}}{4(2k)!} \binom{2k}{2j+2} \left(B_{2j+2}\left(\frac{1}{2}\right) - B_{2j+2}(0) \right) \times \left(B_{2k-2j-2}\left(\frac{1}{2}\right) - B_{2k-2j-2}(0) \right).$$

Next we use a well known convolution identity for Bernoulli polynomials [Di]:

$$\sum_{j=0}^{n} \binom{n}{j} B_j(v) B_{n-j}(w) = n(w+v-1)B_{n-1}(v+w) - (n-1)B_n(v+w). \tag{2.13}$$

Considering all of the cases where $(v,w) \in \left\{ \left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), (0,0) \right\}$, leads to

$$\sum_{j=0}^{k-2} A_j = \frac{(2\pi)^{2k}}{(2k)!} \frac{(2k-1)}{2} \left(B_{2k}\left(\frac{1}{2}\right) - B_{2k}(0) \right)$$
$$= -\frac{(2\pi)^{2k}}{(2k)!} (2k-1)(1-2^{-2k}) B_{2k}. \tag{2.14}$$

Substituting (2.14) into (2.8), leads to a closed form expression for the sum we are interested in:

$$\sum_{j=0}^{k-2} |cA_j - A_{k-2}| = -(k-1)(-1)^k A_{k-2} - (-1)^k c \frac{(2\pi)^{2k}}{(2k)!} (2k-1)(1-2^{-2k}) B_{2k}.$$

The proof can be completed by showing that this last expression is bounded from above by $|A_{k-2}|$ or

$$\frac{\pi^2 (1-2^{2-2k})}{8(1-2^{3-2k})} \frac{(2\pi)^{2k}}{(2k)!} (2k-1)(1-2^{-2k}) |B_{2k}| < k |A_{k-2}|.$$

It is elementary to show that this inequality holds for $k > 1$ by applying (2.5). ■

3. Generalizing the criteria to other families

Conditions such as (2.2) appear to be too restrictive to apply to polynomial families such as $P_k(z)$, $W_k(z)$ and $Q_k(z)$. In this section we prove that $Q_k(z)$ and $W_k(z)$ have all their roots on the unit circle, by extending the theorems used in the previous section. Let us briefly recall how to derive results such as (2.1) and (2.2). For a real-valued reciprocal polynomial $V_k(z) = \sum_{j=0}^{k} A_j z^j$, the condition

$$|A_k| > \sum_{j=0}^{k} |cA_j - A_k|, \tag{3.1}$$

immediately implies that

$$1 > \left| \frac{c}{A_k} V_k(z) - v_k(z) \right|, \tag{3.2}$$

where $v_k(z) = \frac{z^{k+1}-1}{z-1}$. Notice that if $v_k(z)$ is expanded in a geometric series, then (3.2) can be derived from (3.1) as a simple consequence of the triangle inequality. Despite the fact that (3.2) does not imply (3.1), it turns out that (3.2) is *easily* strong enough to conclude that $V_k(z)$ has all of its roots on the unit circle. To demonstrate this, first restrict z to the unit circle, and write $z = e^{i\theta}$ with $\theta \in (0, 2\pi)$, and

$$\tilde{v}_k(\theta) = z^{-(k+1)/2} v_k(z) = \frac{\sin\left(\frac{(k+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

If $j < 2k+2$ is a positive odd integer, then it is easy to show that $\tilde{v}_k\left(\frac{j\pi}{k+1}\right)$ has sign $(-1)^{(j-1)/2}$, and $\left|\tilde{v}_k\left(\frac{j\pi}{k+1}\right)\right| \geqslant 1$. This implies that $\tilde{v}_k(\theta)$ has at least $k+1$ interlacing positive and negative values in the interval $(0, 2\pi)$, and it has absolute value $\geqslant 1$ at each of those points. By (3.2) we can write $\frac{c}{A_k} z^{-(k+1)/2} V_k(z) = \tilde{v}_k(\theta) + ET$, where the error term ET has absolute value less than 1. It follows that $\frac{c}{A_k} z^{-(k+1)/2} V_k(z)$ changes sign at least k times for $\theta \in (0, 2\pi)$. By the intermediate value theorem we conclude that $V_k(z)$ has at least k roots on the unit circle. Since the polynomial has at most k roots, all of its roots must lie on the unit circle.

We can easily extend this idea by selecting a different $v_k(z)$.[2] This typically entails constructing $v_k(z)$ to approximate specific polynomial families.

Definition 3.1. *Let $f : (a, b) \to \mathbb{R}$ be a continuous function. We call $f(\theta)$ a* kth order alternating function on (a, b), *if it assumes alternating positive and negative (or negative and positive) values at points p_i, where $a < p_1 < \cdots < p_{k+1} < b$. We say that $f(\theta)$ has oscillation distance d, if $|f(p_i)| > d$ for each $i \in \{1, \ldots, k+1\}$.*

[2] This principle was inspired by a careful study of the proof in [MSW]

Lemma 3.2. *Suppose that $f(\theta)$ is a kth order alternating function on (a,b), with oscillation distance d. Let $F:(a,b)\to\mathbb{R}$ be a continuous function such that $|F(\theta)-f(\theta)|<d$ for all θ. Then $F(\theta)$ has at least k roots.*

Proof. This lemma is essentially a restatement of the intermediate value theorem. The proof follows immediately from the method described in the previous discussion. ■

3.1. The roots of $W_k(z)$ lie on the unit circle

The main result of this subsection is the following theorem:

Theorem 3.3. *Suppose that $k\geqslant 2$. The polynomial*

$$\begin{aligned}W_k(z) &= (2^{2k-1}+2)P_k(z)-2^{2k}P_k(z/2)-P_k(2z)\\ &= \frac{(2\pi)^{2k-1}2^{2k}}{(2k)!}\sum_{j=0}^{k}(-1)^j B_{2j}B_{2k-2j}(1-2^{1-2j})(1-2^{1-2k+2j})\binom{2k}{2j}z^{2j}\end{aligned}$$

has all of its roots on the unit circle. Furthermore, they are all simple.

In order to prove Theorem 3.3 we first need to establish that a certain trigonometric polynomial possesses the alternating property with oscillation distance 0.3.

Lemma 3.4. *Suppose that $k>10$. The function*

$$w_k(\theta):=2\cos(k\theta)+\frac{\pi^2}{3}\cos((k-2)\theta)+\frac{2}{(1-2^{1-2k})}\frac{\sin((k-3)\theta)}{\sin\theta}$$

is an alternating function of order $2k$ on $(-\pi,\pi)$, with oscillation distance 0.3.

Proof. We need to demonstrate that $|w_k(\theta)|>0.3$ for $2k+1$ values of $\theta\in(-\pi,\pi)$. We must also show that the sign of $w_k(\theta)$ alternates over successive points in this set. Since $w_k(\theta)$ is even with respect to θ, and since $w_k(0)>3$, we only need to demonstrate that there are an additional k such points in $(0,\pi)$. Suppose that $k>10$, let α be defined by

$$\alpha=\frac{1}{\pi}\arccos\left(\frac{0.3}{\frac{\pi^2}{3}-2}\right)=0.42\ldots, \tag{3.3}$$

and let

$$j_0=[(k-1)\alpha]+1. \tag{3.4}$$

We claim that $w_k(\theta)$ satisfies the necessary conditions on the following set of points:

$$\begin{aligned}S=&\left\{\frac{\pi}{k-1},\ldots,\frac{(j_0-1)\pi}{k-1}\right\}\cup\left\{\frac{(j_0-1/2)\pi}{k-1},\ldots,\frac{(k-j_0-1/2)\pi}{k-1}\right\}\\ &\cup\left\{\frac{(k-j_0)\pi}{k-1},\ldots,\frac{(k-2)\pi}{k-1}\right\}\cup\left\{\frac{(k-(1-\epsilon))\pi}{k-1}\right\},\end{aligned}$$

where $\epsilon > 0$ is sufficiently small. First expand $w_k(\theta)$ using trigonometric identities

$$\begin{aligned} w_k(\theta) = \cos((k-1)\theta)\cos(\theta)\left(\frac{\pi^2}{3}+2-\frac{4}{1-2^{1-2k}}\right) \\ + \sin((k-1)\theta)\left(\left(\frac{\pi^2}{3}-2-\frac{4}{1-2^{1-2k}}\right)\sin\theta + \frac{2}{1-2^{1-2k}}\csc\theta\right). \end{aligned}$$

Notice that S (essentially) arises from cases where either $\cos((k-1)\theta) = 0$, or $\sin((k-1)\theta) = 0$.

Begin by considering the cases where $\theta = \frac{j\pi}{k-1}$ and $j \in \{1,\dots,k-2\}$. Then

$$w_k\left(\frac{j\pi}{k-1}\right) = (-1)^j \cos\left(\frac{j\pi}{k-1}\right)\left(\frac{\pi^2}{3}+2-\frac{4}{1-2^{1-2k}}\right). \tag{3.5}$$

In order to have $|w_k\left(\frac{j\pi}{k-1}\right)| > 0.3$, we need to restrict j so that

$$\left|\cos\left(\frac{j\pi}{k-1}\right)\right| > \frac{0.3}{\frac{\pi^2}{3}+2-\frac{4}{1-2^{1-2k}}}.$$

In other words we must have

$$\frac{j}{k-1} \notin (\alpha_k, 1-\alpha_k),$$

where

$$\alpha_k = \frac{1}{\pi}\arccos\left(\frac{0.3}{\frac{\pi^2}{3}+2-\frac{4}{1-2^{1-2k}}}\right).$$

Since $k > 10$ we have $(k-1)(\alpha_k - \alpha) \ll 1$. Therefore it is sufficient that

$$\frac{j}{k-1} \notin (\alpha, 1-\alpha),$$

where α is defined in (3.3). This implies that $j \in \{1,2,\dots,j_0-1\} \cup \{k-j_0,\dots, k-2\}$, with j_0 defined in (3.4). If $j \in \{1,2,\dots,j_0-1\}$, then by (3.5) $w_k\left(\frac{j\pi}{k-1}\right)$ has sign $(-1)^j$. If $j \in \{k-j_0,\dots,k-2\}$ then the cosine in (3.5) contributes an extra minus sign, and $w_k\left(\frac{j\pi}{k-1}\right)$ has sign $(-1)^{j+1}$.

Now consider the case where $\theta = \frac{(j-1/2)\pi}{(k-1)}$ and $j \in \{j_0,\dots,k-j_0\}$. By elementary properties of trigonometric functions, $w_k(\theta)$ reduces to

$$\begin{aligned} w_k\left(\frac{(j-1/2)\pi}{k-1}\right) = (-1)^{j+1}\left(\left(\frac{\pi^2}{3}-2-\frac{4}{1-2^{1-2k}}\right)\sin\left(\frac{(j-1/2)\pi}{k-1}\right)\right. \\ \left. + \frac{2}{1-2^{1-2k}}\csc\left(\frac{(j-1/2)\pi}{k-1}\right)\right). \end{aligned} \tag{3.6}$$

In order to place a lower bound on this expression, first choose an interval $(\beta, 1-\beta)$, which contains the set of rational numbers $\{\frac{j_0-1/2}{k-1}, \ldots, \frac{k-j_0-1/2}{k-1}\}$. This can be accomplished by selecting

$$\beta = \begin{cases} \alpha & \text{if } j_0 > \alpha(k-1) + \frac{1}{2}, \\ \alpha - \frac{1}{2(k-1)} & \text{if } j_0 < \alpha(k-1) + \frac{1}{2}. \end{cases}$$

Notice that one of these situations must occur, because (3.4) guarantees that $j_0 \in (\alpha(k-1), \alpha(k-1)+1)$. We obtain the following lower bound from (3.6):

$$\left| w_k\left(\frac{(j-1/2)\pi}{k-1}\right)\right| \geqslant \min_{\theta \in (\pi\beta, \pi(1-\beta))} \left| \left(\frac{\pi^2}{3} - 2 - \frac{4}{1-2^{1-2k}}\right)\sin\theta + \frac{2}{1-2^{1-2k}}\csc\theta \right|.$$

The right-hand side is minimized at the end points of the interval $(\pi\beta, \pi(1-\beta))$, so it follows that

$$\left| w_k\left(\frac{(j-1/2)\pi}{k-1}\right)\right| \geqslant \left| \left(\frac{\pi^2}{3} - 2 - \frac{4}{1-2^{1-2k}}\right)\sin\pi\beta + \frac{2}{1-2^{1-2k}}\csc\pi\beta \right|$$

Consider both choices of β, and recall the assumption that $k > 10$. A few easy calculations are sufficient to obtain

$$\left| w_k\left(\frac{(j-1/2)\pi}{k-1}\right)\right| > \begin{cases} 0.57 & \text{if } j_0 > \alpha(k-1) + \frac{1}{2}, \\ 0.34 & \text{if } j_0 < \alpha(k-1) + \frac{1}{2}, \end{cases}$$

for all values of $j \in \{j_0, \ldots, k-2j_0\}$. It is easy to deduce from (3.6) that the sign of $w_k\left(\frac{(j-1/2)\pi}{k-1}\right)$ is $(-1)^j$.

Finally consider the value of $w_k\left(\frac{(k-(1-\epsilon))\pi}{k-1}\right)$. Notice that

$$w_k(\pi) = (-1)^k \left(\frac{\pi^2}{3} + 2 + \frac{2k-6}{1-2^{1-2k}}\right).$$

Since $k > 10$ it follows easily that $|w_k(\pi)| > 19$, and $w_k(\pi)$ has sign $(-1)^k$. If ϵ is sufficiently small then $w_k\left(\frac{(k-(1-\epsilon))\pi}{k-1}\right)$ also has sign $(-1)^k$, and absolute value much larger than 0.3.

To briefly summarize the sign values of $w_k(\theta)$, we have the following table:

θ	$Sign\,(w_k(\theta))$
0	$(-1)^0$
$\frac{\pi}{k-1}$	$(-1)^1$
$\frac{2\pi}{k-1}$	$(-1)^2$
$\vdots$	$\vdots$
$\frac{(j_0-1)\pi}{k-1}$	$(-1)^{j_0-1}$
$\frac{(j_0-1/2)\pi}{k-1}$	$(-1)^{j_0}$
$\vdots$	$\vdots$
$\frac{(k-j_0-1/2)\pi}{k-1}$	$(-1)^{k-j_0}$
$\frac{(k-j_0)\pi}{k-1}$	$(-1)^{k-j_0+1}$
$\vdots$	$\vdots$
$\frac{(k-2)\pi}{k-1}$	$(-1)^{k-1}$
$\frac{(k-(1-\epsilon))\pi}{k-1}$	$(-1)^k$

This table shows that $w_k(\theta)$ changes sign at least k times over the interval $(0,\pi)$. ■

Now we use Lemma 3.4 to prove our main result.

Proof of Theorem 3.3. Let us define A_j using

$$\begin{aligned} W_k(iz) &= \frac{(2\pi)^{2k-1}2^{2k}}{(2k)!}\sum_{j=0}^{k} B_{2j}B_{2k-2j}(1-2^{1-2j})(1-2^{1-2k+2j})\binom{2k}{2j}z^{2j} \\ &= \sum_{j=0}^{k} A_j z^{2j}. \end{aligned}$$

By Lemma 3.2 it suffices to prove that $\left|\frac{z^{-k}W_k(iz)}{A_0} - w_k(z)\right| < 0.3$, where

$$w_k(z) = (z^k + z^{-k}) + \frac{\pi^2}{6}(z^{k-2}+z^{2-k}) + \frac{2}{(1-2^{1-2k})}\frac{z^{k-3}-z^{3-k}}{z-z^{-1}}, \tag{3.7}$$

and $z = e^{i\theta}$. Thus we write

$$\begin{aligned} \left|\frac{z^{-k}W_k(iz)}{A_0} - w_k(z)\right| &= \left|\sum_{j=0}^{k}\frac{A_j}{A_0}z^{2j-k} - (z^k+z^{-k}) - \frac{\pi^2}{6}(z^{k-2}+z^{2-k})\right. \\ &\qquad \left. - \frac{2}{(1-2^{1-2k})}\frac{z^{k-3}-z^{3-k}}{z-z^{-1}}\right| \\ &\leqslant 2\left|\frac{A_1}{A_0} - \frac{\pi^2}{6}\right| + \sum_{j=2}^{k-2}\left|\frac{A_j}{A_0} - \frac{2}{(1-2^{1-2k})}\right|, \end{aligned} \tag{3.8}$$

where the second step makes use of a geometric series and the triangle inequality.

If we recall that $B_0 = -1/2$, and use both inequalities from (2.5), then we find

$$\frac{A_j}{A_0} = \frac{B_{2j}B_{2k-2j}(1-2^{1-2j})(1-2^{1-2k+2j})\binom{2k}{2j}}{-B_{2k}(1-2^{1-2k})} < \frac{2}{(1-2^{1-2k})}.$$

Additionally we have, by $\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}$,

$$\left|\frac{A_1}{A_0} - \frac{\pi^2}{6}\right| = \left|\frac{\pi^2}{6}\frac{\zeta(2k-2)}{\zeta(2k)} - \frac{\pi^2}{6}\right|.$$

This second expression goes to zero as $k \to \infty$. For example, it is not hard to see that the absolute value is less than 0.01 for $k > 4$.

Therefore we can remove the absolute value signs from (3.8). We find that

$$\begin{aligned}\left|\frac{z^{-k}W_k(iz)}{A_0} - w_k(z)\right| &\leqslant 2\frac{A_1}{A_0} - \frac{\pi^2}{3} + \frac{2(k-3)}{1-2^{1-2k}} - \sum_{j=2}^{k-2}\frac{A_j}{A_0} \\ &= 2 + 4\frac{A_1}{A_0} - \frac{\pi^2}{3} + \frac{2(k-3)}{1-2^{1-2k}} - \sum_{j=0}^{k}\frac{A_j}{A_0} \\ &= 2 + \frac{2\pi^2}{3}\frac{\zeta(2k-2)}{\zeta(2k)} - \frac{\pi^2}{3} + \frac{2(k-3)}{1-2^{1-2k}} - \frac{2k-1}{1-2^{1-2k}} \\ &= 2 + \frac{2\pi^2}{3}\frac{\zeta(2k-2)}{\zeta(2k)} - \frac{\pi^2}{3} - \frac{5}{1-2^{1-2k}} \\ &\leqslant -3 + \left(2\frac{1-2^{-2k}}{1-2^{3-2k}} - 1\right)\frac{\pi^2}{3}.\end{aligned}$$

Notice that we evaluated $\sum_j A_j$ using the same Bernoulli convolution identity (2.13). In addition, we have used the inequality

$$\frac{1}{1-2^{-n}} < \zeta(n) < \frac{1}{1-2^{1-n}},$$

which are easy to deduce from the Euler product formula and the Dirichlet eta function. As $k \to \infty$ this final upper bound approaches a limit of $\frac{\pi^2}{3} - 3 \approx .289$. It is easy to see that it becomes < 0.3 for $k > 6$.

In summary, we have proved the theorem for $k > 10$. The cases for $k \leqslant 10$ are easily checked with the numerical method outlined in section 4. ■

3.2. The roots of $Q_k(z)$ lie on the unit circle

Notice that the coefficients of $Q_k(z)$ involve the odd values of the Riemann zeta function. The primary goal of this subsection is to prove the following theorem:

Theorem 3.5. *Suppose that $k \geqslant 2$. The polynomial*

$$\begin{aligned} Q_k(z) &= (2^{2k}+1)P_k(z) - 2^{2k}P_k(z/2) - P_k(2z) \\ &= \frac{(2\pi)^{2k-1}}{(2k)!}\sum_{j=0}^{k}(-1)^j B_{2j}B_{2k-2j}(2^{2j}-1)(2^{2k-2j}-1)\binom{2k}{2j}z^{2j} \\ &\quad + \zeta(2k-1)(2^{2k-1}-1)((-1)^k z + z^{2k-1}), \end{aligned}$$

has all of its non-zero roots on the unit circle. Furthermore, all of the roots are simple.

As in the proof of Theorem 3.3, the first step is to construct a trigonometric polynomial which approximates $Q_k(z)$. Notice that $Q_k(z)$ has degree $2k-1$, and that it has a trivial root at $z = 0$. Therefore we need to prove that it has $2k-2$ roots on the unit circle.

Lemma 3.6. *Suppose that $k > 5$. Then*

$$q_k(\theta) := 2\cos((k-2)\theta) + \frac{4}{\pi}\sin((k-1)\theta) + \frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\frac{\sin((k-3)\theta)}{\sin\theta}$$

is an alternating function of order $2k-2$ on $(-\pi,\pi)$, with oscillation distance 0.03.

Proof. We need to demonstrate that $|q_k(\theta)| > 0.03$ for $2k-1$ values of $\theta \in (-\pi,\pi)$. We must also show that the sign of $q_k(\theta)$ alternates over successive points in this subset. The proof is similar to the proof of Lemma 2, so we will be brief. Let α be defined by

$$\alpha = \frac{1}{\pi}\arccos\left(\frac{0.03}{2-\frac{16}{\pi^2}}\right) = 0.47\ldots, \tag{3.9}$$

and let

$$j_0 = [(k-1)\alpha] + 1. \tag{3.10}$$

We claim that $|q_k(\theta)| > 0.03$ on the following set of $2k+1$ points:

$$\begin{aligned} S = \{0\} &\cup \left\{\pm\frac{\pi}{k-1},\ldots,\pm\frac{(j_0-1)\pi}{k-1}\right\} \cup \left\{\pm\frac{(j_0-1/2)\pi}{k-1},\ldots,\pm\frac{(k-j_0-1/2)\pi}{k-1}\right\} \\ &\cup \left\{\pm\frac{(k-j_0)\pi}{k-1},\ldots,\pm\frac{(k-2)\pi}{k-1}\right\} \cup \left\{\pm\frac{(k-(1-\epsilon))\pi}{k-1}\right\}. \end{aligned}$$

If we consider $S \setminus \{\frac{(j_0-1/2)\pi}{k-1}, \frac{(k-j_0-1/2)\pi}{k-1}\}$, then we obtain a subset of $2k-1$ points where the sign of $q_k(\theta)$ alternates over successive points.

In order to prove this claim, first expand $q_k(\theta)$ using trigonometric identities

$$\begin{aligned} q_k(\theta) &= 2\cos((k-1)\theta)\cos(\theta)\left(1-\frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\right) \\ &\quad + \sin((k-1)\theta)\left(2\sin(\theta)\left(1-\frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\right)+\frac{4}{\pi}+\frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\csc\theta\right). \end{aligned}$$

Now consider the cases where $\theta = \frac{j\pi}{k-1}$, with $-(k-2) \leqslant j \leqslant (k-2)$ and $j \neq 0$. We have

$$q_k\left(\frac{j\pi}{k-1}\right) = 2(-1)^j \cos\left(\frac{j\pi}{k-1}\right)\left(1 - \frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\right). \tag{3.11}$$

To ensure that $|q_k\left(\frac{j\pi}{k-1}\right)| > 0.03$, we need to restrict j so that

$$\left|\cos\left(\frac{j\pi}{k-1}\right)\right| > \frac{0.03}{2\left(1 - \frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\right)}.$$

By similar reasoning to that in the proof of Lemma 3.4, it is sufficient that

$$\frac{j}{k-1} \notin (-(1-\alpha), -\alpha) \cup (\alpha, 1-\alpha),$$

where α is defined in (3.9). This immediately implies that $j \in \{\pm 1, \ldots, \pm(j_0-1)\} \cup \{\pm(k-j_0), \ldots, \pm(k-2)\}$, where j_0 is defined in (3.10). A careful inspection of (3.11) reveals that the function has sign $(-1)^j$ for $j \in \{\pm 1, \ldots, \pm(j_0-1)\}$, and sign $(-1)^{j+1}$ for $j \in \{\pm(k-j_0), \ldots, \pm(k-2)\}$.

Next consider the cases where $\theta = \pm\frac{(j-1/2)\pi}{(k-1)}$ and $j \in \{j_0, \ldots, (k-j_0)\}$. We obtain

$$\begin{aligned} q_k\left(\pm\frac{(j-1/2)\pi}{2(k-1)}\right) = (-1)^{j-1}&\left(\pm 2\sin\left(\frac{(j-1/2)\pi}{(k-1)}\right)\left(1 - \frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\right)\right. \\ &\left.\pm\frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\csc\left(\frac{(j-1/2)\pi}{(k-1)}\right) + \frac{4}{\pi}\right). \end{aligned} \tag{3.12}$$

Now select β so that $\left\{\frac{j_0-1/2}{k-1}, \ldots, \frac{k-j_0-1/2}{k-1}\right\} \subset (\beta, 1-\beta)$. Following Lemma 3.4, this is accomplished by selecting

$$\beta = \begin{cases} \alpha & \text{if } j_0 > \alpha(k-1) + \frac{1}{2}, \\ \alpha - \frac{1}{2(k-1)} & \text{if } j_0 < \alpha(k-1) + \frac{1}{2}. \end{cases}$$

Therefore we obtain

$$\begin{aligned} &\left|q_k\left(\pm\frac{(j-1/2)\pi}{2(k-1)}\right)\right| \\ &\quad\geqslant \min_{\theta\in(\pi\beta,\pi(1-\beta))}\left|2\left(1 - \frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\right)\sin\theta + \frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\csc\theta \pm \frac{4}{\pi}\right| \\ &\quad\geqslant \left|2\left(1 - \frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\right)\sin\pi\beta + \frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})}\csc\pi\beta \pm \frac{4}{\pi}\right|. \end{aligned}$$

Checking both possible values of β, and both possible signs of $\pm$, leads to a lower bound which holds for $k > 5$:

$$\left| q_k \left(\pm \frac{(2j-1)\pi}{2(k-1)} \right) \right| > 0.08...$$

The final signs are summarized in the table below.

The only remaining cases are when $j \in \{0\} \cup \{\pm \frac{(k-(1-\epsilon))\pi}{k-1}\}$. These cases can be easily dispensed with by elementary properties of trigonometric functions.

To briefly summarize the sign values of $q_k(\theta)$, we have the following table:

θ	$Sign\,(q_k(\theta))$	θ	$Sign\,(q_k(\theta))$
$-\frac{(k-(1-\epsilon))\pi}{k-1}$	$(-1)^k$	0	$(-1)^0$
$-\frac{(k-2)\pi}{k-1}$	$(-1)^{k-1}$	$\frac{\pi}{k-1}$	$(-1)^1$
$\vdots$	$\vdots$	$\frac{2\pi}{k-1}$	$(-1)^2$
$-\frac{(k-j_0)\pi}{k-1}$	$(-1)^{k-j_0+1}$	$\vdots$	$\vdots$
$-\frac{(k-j_0-1/2)\pi}{k-1}$	$(-1)^{k-j_0}$	$\frac{(j_0-1)\pi}{k-1}$	$(-1)^{j_0-1}$
$\vdots$	$\vdots$	$\frac{(j_0-1/2)\pi}{k-1}$	$(-1)^{j_0-1}$
$-\frac{(j_0-1/2)\pi}{k-1}$	$(-1)^{j_0}$	$\vdots$	$\vdots$
$-\frac{(j_0-1)\pi}{k-1}$	$(-1)^{j_0-1}$	$\frac{(k-j_0-1/2)\pi}{k-1}$	$(-1)^{k-j_0-1}$
$\vdots$	$\vdots$	$\frac{(k-j_0)\pi}{k-1}$	$(-1)^{k-j_0+1}$
$-\frac{2\pi}{k-1}$	$(-1)^2$	$\vdots$	$\vdots$
$-\frac{\pi}{k-1}$	$(-1)^1$	$\frac{(k-2)\pi}{k-1}$	$(-1)^{k-1}$
		$\frac{(k-(1-\epsilon))\pi}{k-1}$	$(-1)^k$

Notice that there are precisely $2k+1$ values of θ in this table. If we exclude the cases where $\theta \in \{\frac{(j_0-1/2)\pi}{k-1}, \frac{(k-j_0-1/2)\pi}{k-1}\}$, then the sign of $q_k(\theta)$ alternates over the remaining $2k-1$ values of θ. ■

Next we use Lemma 3.6 to establish that $Q_k(z)$ has all of its non-zero roots on the unit circle for $k \geqslant 2$.

Proof of Theorem 3.5. Let us define A_j using

$$\begin{aligned} Q_k(iz) = {} & \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{j=0}^{k} B_{2j} B_{2k-2j} (2^{2j}-1)(2^{2k-2j}-1) \binom{2k}{2j} z^{2j} \\ & + i(-1)^k \zeta(2k-1)(2^{2k-1}-1)(z - z^{2k-1}) \\ = {} & \sum_{j=0}^{k} A_j z^{2j} + i(-1)^k \zeta(2k-1)(2^{2k-1}-1)(z - z^{2k-1}). \end{aligned}$$

In order to simplify the following analysis, we have intentionally defined A_j to only involve the even coefficients of $Q_k(iz)$. Notice that $A_0 = A_k = 0$, and that

$$\begin{aligned} A_1 &= \frac{(2\pi)^{2k-1}}{(2k)!} B_{2k-2}(2^{2k-2}-1)\frac{k(2k-1)}{2} \\ &= (-1)^k \zeta(2k-2)(2^{2k-2}-1). \end{aligned}$$

Suppose that $k > 5$. Then by Lemma 3.2 it suffices to prove that

$$\left| \frac{z^{-k}Q_k(iz)}{A_1} - q_k(z) \right| < 0.03$$

where

$$q_k(z) = (z^{k-2} + z^{2-k}) - \frac{2i}{\pi}(z^{k-1} - z^{1-k}) + \frac{8}{\pi^2}\frac{(1-2^{3-2k})}{(1-2^{2-2k})}\frac{z^{k-3}-z^{3-k}}{z-z^{-1}}, \quad (3.13)$$

and $z = e^{i\theta}$.

Therefore let us begin by writing

$$\begin{aligned} &\left| \frac{z^{-k}Q_z(iz)}{A_1} - q_k(z) \right| \\ &\quad = \left| \sum_{j=0}^{k} \frac{A_j}{A_1} z^{2j-k} + (-1)^k i \frac{\zeta(2k-1)(2^{2k-1}-1)}{A_1}(z^{1-k} - z^{k-1}) \right. \\ &\quad\quad \left. -(z^{k-2}+z^{2-k}) + \frac{2i}{\pi}(z^{k-1}-z^{1-k}) - \frac{8}{\pi^2}\frac{(1-2^{3-2k})}{(1-2^{2-2k})}\frac{z^{k-3}-z^{3-k}}{z-z^{-1}} \right| \\ &\quad \leqslant \sum_{j=2}^{k-2} \left| \frac{A_j}{A_1} - \frac{8}{\pi^2}\frac{(1-2^{3-2k})}{(1-2^{2-2k})} \right| + 2\left| (-1)^k \frac{\zeta(2k-1)(2^{2k-1}-1)}{A_1} - \frac{2}{\pi} \right|. \end{aligned}$$

The second step follows from substituting a geometric series, and then applying the triangle inequality. We know from equation (2.11) (after noting the change in definition of A_j), that

$$\frac{A_j}{A_1} > \frac{8}{\pi^2}\frac{(1-2^{3-2k})}{(1-2^{2-2k})}.$$

In addition

$$\left| (-1)^k \frac{\zeta(2k-1)(2^{2k-1}-1)}{A_1} - \frac{2}{\pi} \right| = \left| \frac{2}{\pi}\frac{\zeta(2k-1)(1-2^{1-2k})}{\zeta(2k-2)(1-2^{2-2k})} - \frac{2}{\pi} \right|.$$

This limit tends to zero. A simple calculation shows that the quantity is less than 0.001 for $k > 3$.

Therefore we can remove the absolute value signs from the inequality. We are left with

$$\begin{aligned}
&\left|\frac{z^{-k}Q_z(iz)}{A_1} - q_k(z)\right| \\
&\qquad \leqslant \sum_{j=2}^{k-2} \frac{A_j}{A_1} - (k-3)\frac{8}{\pi^2}\frac{(1-2^{3-2k})}{(1-2^{2-2k})} + \frac{4}{\pi} - 2(-1)^k \frac{\zeta(2k-1)(2^{2k-1}-1)}{A_1} \\
&\qquad = \sum_{j=1}^{k-1} \frac{A_j}{A_1} - 2 - (k-3)\frac{8}{\pi^2}\frac{(1-2^{3-2k})}{(1-2^{2-2k})} + \frac{4}{\pi} - \frac{4}{\pi}\frac{\zeta(2k-1)(1-2^{1-2k})}{\zeta(2k-2)(1-2^{2-2k})} \\
&\qquad = (2k-1)\frac{4}{\pi^2}\frac{\zeta(2k)(1-2^{-2k})}{\zeta(2k-2)(1-2^{2-2k})} - 2 - (k-3)\frac{8(1-2^{3-2k})}{\pi^2(1-2^{2-2k})} \\
&\qquad\quad + \frac{4}{\pi} - \frac{4}{\pi}\frac{\zeta(2k-1)(1-2^{1-2k})}{\zeta(2k-2)(1-2^{2-2k})}.
\end{aligned}$$

As usual, we have evaluated $\sum_j A_j$ using (2.13). The limit of the upper bound is $\frac{20}{\pi^2} - 2$. It is easy to see that it becomes < 0.03 for $k > 6$. The cases for $k < 6$ are easily proved with the numerical method described in Section 4. ■

We conclude this section by deriving a second approximation for $\zeta(3)/\pi^3$. If we truncate the exponential series for $Q_k(z)$, we can obtain

$$\zeta(3) \approx \frac{z}{1+z^2}\frac{\pi^3}{14}, \tag{3.14}$$

where

$$0 = \frac{z}{e^{\pi/z}+1} + \frac{z^{-1}}{e^{\pi z}+1}. \tag{3.15}$$

Selecting the root given by $z \approx 0.92 + 0.38i$, yields 4 decimal places of numerical accuracy. Notice that this approximation is slightly worse than (1.5).

4. Partial results on $P_k(z)$

We have made a number of unsuccessful attempts to apply the theorems of Schinzel, Lakatos and Losonczi, and their generalizations to the case of $P_k(z)$.[3] A piece of evidence indicating that these methods may not work is given by the result in [MSW] which shows that $R_{2k+1}(z)$ has four roots that do not lie in the unit circle (by comparison $Y_k(z)$ has all of its roots on the unit circle).

[3]The most general result that we can prove is that $P_k(z)$ has at least $k-1$ roots in half of the unit circle, using the construction from [MSW].

We will briefly describe one instance were Lakatos's condition (2.1) fails, because it leads to an interesting formula. Let us define the A_j's as follows:

$$\begin{aligned}\sum_{j=0}^{4k} A_j z^j &= |P_k(iz)|^2 \\ &= \left(\frac{2}{\pi}\sum_{j=0}^{k}\zeta(2j)\zeta(2k-2j)z^{2j}\right)^2 + \zeta^2(2k-1)\left(z^{2k-1}-z\right)^2 .\end{aligned}$$

Notice that $|P_k(iz)|^2$ has all of its roots on the unit circle, if and only if $P_k(z)$ also has all of its roots on the unit circle. Computational experiments helped us to make the following observation:

Observation 4.1. Suppose that $k \geqslant 2$, then

$$4k(k-1)|A_{4k}| = \sum_{j=0}^{4k} |A_{4k} - A_j| . \tag{4.1}$$

Formula (4.1) can be proved with the identities for Bernoulli numbers that we used in Theorem 2.2. This immediately rules out the possibility of applying (2.1) (condition (2.2) can also be ruled out by slightly different methods). It is curious to note that the right-hand side of (4.1) appears to involve odd zeta values, whereas the left-hand side does not. It turns out that when (4.1) is explicitly calculated, the odd zeta values drop out.

Theorem 4.2. *Suppose that $2 \leqslant k < 1000$. Then all of the roots of $P_k(z)$ lie on the unit circle. Furthermore, all of the roots are simple.*

While we have not been able to prove a general theorem concerning $P_k(z)$, we have been able to prove Theorem 4.2 for $k < 1000$. The proof uses a standard computational method based on the intermediate value theorem. Notice that the map

$$z + z^{-1} \mapsto 2u,$$

sends the unit circle to the real interval $[-1, 1]$. Under this transformation, we also have

$$z^k + z^{-k} \mapsto 2T_k(u) ,$$

where $T_k(u)$ is the usual Chebyshev Polynomial. If we write $(z^{2k} + (-1)^k)P_k(z) = z^{2k}\left(P_k(z) + P_k(1/z)\right) = 2z^{2k}P_k^*(u)$, then it follows that $P_k(z)$ has all of its roots on the unit circle, if and only if

$$\begin{aligned}P_k^*(u) := \frac{(2\pi)^{2k-1}}{(2k)!}&\sum_{j=0}^{k}(-1)^j B_{2j}B_{2k-2j}\binom{2k}{2j}T_{2j}(u) \\ &+ \zeta(2k-1)\left(T_{2k-1}(u) + (-1)^k T_1(u)\right) ,\end{aligned}$$

has all of its roots in the interval $[-1,1]$. It is easy to count real roots of real-valued polynomials. The intermediate value theorem allows one to find roots by detecting sign changes. Since $P_k^*(u)$ has degree $2k$, it is only necessary to detect $2k$ sign changes in $[-1,1]$ (fewer sign changes are required if roots lie at $u=\pm 1$). We have successfully carried out these calculations for $k<1000$.

5. Conclusion

In conclusion, we have shown that $S_k(z)$, $Y_k(z)$, $W_k(z)$, and $Q_k(z)$ have all of their non-zero roots on the unit circle. These polynomials have a strong connection to the Ramanujan polynomials. We were disappointed that we were unable to deduce a similar theorem concerning $P_k(z)$ with our methods, however we are hopeful that the approach outlined in Section 3 might eventually succeed in this case.

We remark that the theorem for $P_k(z)$ is true as it has recently proved in [LS] by different methods.

An additional avenue might involve studying the roots of a truncated version of the right-hand side of (1.3). Notice that the roots of $P_k(z)$ are very well approximated by the roots of

$$0=\frac{z^{k-1}}{e^{2\pi/z}-1}+(-1)^{k+1}\frac{z^{1-k}}{e^{2\pi z}-1}.$$

Thus, it should be a worthwhile endeavor to study the roots of these auxiliary functions.

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