Functiones et Approximatio
47.2 (2012), 241–266
doi: 10.7169/facm/2012.47.2.8

NUMBERS WITH INTEGER EXPANSION IN THE NUMERATION SYSTEM WITH NEGATIVE BASE

Petr Ambrož, Daniel Dombek, Zuzana Masáková, Edita Pelantová

Abstract: In this paper, we study representations of real numbers in the positional numeration system with negative base, as introduced by Ito and Sadahiro. We focus on the set $\mathbb{Z}_{-\beta}$ of numbers whose representation uses only non-negative powers of $-\beta$, the so-called $(-\beta)$-integers. We describe the distances between consecutive elements of $\mathbb{Z}_{-\beta}$. In case that this set is non-trivial we associate to β an infinite word $\boldsymbol{v}_{-\beta}$ over an (in general infinite) alphabet. The self-similarity of $\mathbb{Z}_{-\beta}$, i.e., the property $-\beta\mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta}$, allows us to find a morphism under which $\boldsymbol{v}_{-\beta}$ is invariant. On the example of two cubic irrational bases β we demonstrate the difference between Rauzy fractals generated by $(-\beta)$-integers and by β-integers.

Keywords: numeration system, negative base, Pisot numbers, morphism.

1. Introduction

In [7], Ito and Sadahiro have introduced a new numeration system, using negative base $-\beta < -1$ for expansion of real numbers. The method for obtaining the $(-\beta)$-expansion is analogous to the one for positive bases introduced by Rényi [11]. During the 50 years since the publication of this paper, the β-expansions of Rényi were extensively studied. On the other hand, analogous study of $(-\beta)$-expansions has yet to be performed.

The Rényi expansions of real numbers $x \in [0, 1)$ with positive base $\beta > 1$ are constructed using the transformation T_β of the unit interval. Simple adaptation leads to the greedy algorithm which allows one to uniquely expand any positive real number in the β-numeration system. Essential tool for determining which digit strings are admissible in β-expansions is the Rényi expansion of 1. The criterion for admissibility was given in [10] using the lexicographical ordering of

---

We acknowledge financial support by the Czech Science Foundation grant 201/09/0584 and by the grants MSM6840770039 and LC06002 of the Ministry of Education, Youth, and Sports of the Czech Republic. The work was also partially supported by the CTU in Prague grant SGS11/162/OHK4/3T/14.

2010 Mathematics Subject Classification: primary: 11K16; secondary: 68R15

strings. Frougny and Solomyak [6], and then many others have considered questions about arithmetics on β-expansions. Another point of view on β-numeration is to study combinatorial properties of the set of positive real numbers with integer β-expansion. The so-called β-integers were considered for example by Burdík et al. [3] as a natural counting system for coordinates of points in quasicrystal models. The description of distances between consecutive β-integers is due to Thurston [14]. Consequently, in case that the Rényi expansion of 1 is eventually periodic, the sequence of β-integers can be coded by an infinite word, denoted $\boldsymbol{u}_\beta$, over a finite alphabet. Fabre shows that this word is substitution invariant, see [4]. A generalization of the β-transformation T_β is studied from various points of view in [8].

Ito and Sadahiro in their paper [7] show some fundamental properties of their new numeration system with negative basis. Most of them are analogous to β-expansions, though often more complicated. For example, the characterization of admissible digit sequences is given using the alternate order on digit strings. Ito and Sadahiro also provided a criterion for the $(-\beta)$-shift to be sofic, and determined explicitly the absolutely continuous invariant measure of the $(-\beta)$-transformation $T_{-\beta}$ which is used in the expansion algorithm. Frougny and Lai [5] have studied arithmetical aspects of the $(-\beta)$-numeration. In particular, they have shown that for β Pisot, the $(-\beta)$-shift is a sofic system and addition is realizable by a finite transducer.

Our aim is to deepen the knowledge about $(-\beta)$-expansions. We focus, in particular, on the properties of the set $\mathbb{Z}_{-\beta}$ of $(-\beta)$-integers which are defined in natural analogy to the classical β-integers. We describe the bases for which this set is non-trivial, and we show that $\mathbb{Z}_{-\beta}$ does not have accumulation points. We try to give an insight to admissibility and alternate ordering of finite strings in order to describe the distances between consecutive $(-\beta)$-integers. This is achieved for a large class of bases $-\beta$. In case that $\mathbb{Z}_{-\beta}$ is non-trivial we associate to β an infinite word $\boldsymbol{v}_{-\beta}$ over an infinite alphabet. The self-similarity of $\mathbb{Z}_{-\beta}$, i.e., the property $-\beta\mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta}$, allows us to find a morphism under which $\boldsymbol{v}_{-\beta}$ is invariant. If, moreover, the corresponding $(-\beta)$-shift is sofic, then we associate an infinite word $\boldsymbol{u}_{-\beta}$ over a finite alphabet which is invariant under a primitive morphism. The infinite word $\boldsymbol{u}_{-\beta}$ is simply constructed from $\boldsymbol{v}_{-\beta}$ by a letter-to-letter projection. Similar question is studied in [12] using a different approach. For a Pisot number β, the set of β-integers corresponds naturally to the so-called Rauzy fractal, see [1, 2]. Analogous fractal can be constructed using $(-\beta)$-expansions. For two examples of cubic Pisot numbers β we compare the fractal tiles arising from $(-\beta)$-expansions with the Rauzy fractal given by the classical Rényi β-expansions.

2. β-expansions versus $(-\beta)$-expansions

2.1. Rényi β-expansions

Consider a real base $\beta > 1$ and the transformation $T_\beta : [0,1) \to [0,1)$ defined by the prescription $T_\beta(x) := \beta x - \lfloor \beta x \rfloor$. The representation of a number $x \in [0,1)$ of

the form
$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots ,$$
where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$, is called the *$\beta$-expansion* of x. The coefficients x_i are called *digits*. We write $\mathrm{d}_\beta(x) = x_1x_2x_3\cdots$ or $x = \bullet x_1x_2x_3\cdots$. If the string $x_1x_2x_3\cdots$ ends in suffix 0^ω, i.e., infinite repetition of 0, we omit it. Since $\beta T_\beta(x) \in [0,\beta)$, the β-expansion of x is an infinite word in the alphabet $\{0,1,\ldots,\lceil\beta\rceil-1\}$.

The β-expansion of an arbitrary real number $x \geqslant 1$ can be naturally defined in the following way: Find an exponent $k \in \mathbb{N}$ such that $\frac{x}{\beta^k} \in [0,1)$. Using the transformation T_β derive the β-expansion of $\frac{x}{\beta^k}$ of the form
$$\frac{x}{\beta^k} = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots ,$$
so that
$$x = x_1\beta^{k-1} + x_2\beta^{k-2} + \cdots + x_{k-1}\beta + x_k + \frac{x_{k+1}}{\beta} + \cdots .$$
The β-expansion of x does not depend on the choice of the exponent k for which $\frac{x}{\beta^k} \in [0,1)$, which is – of course – not given uniquely. We write $x = x_1x_2\cdots x_k \bullet x_{k+1}x_{k+2}\cdots$.

The digit string $x_1x_2x_3\cdots$ is said to be *β-admissible* if there exists a number $x \in [0,1)$ so that $x = \bullet x_1x_2x_3\cdots$ is its β-expansion. The set of admissible digit strings can be described using the *Rényi expansion of 1*, denoted by $\mathrm{d}_\beta(1) = t_1t_2t_3\cdots$, where $t_1 = \lfloor\beta\rfloor$ and $\mathrm{d}_\beta(\beta - \lfloor\beta\rfloor) = t_2t_3t_4\cdots$. The Rényi expansion of 1 may or may not be finite (i.e., ending in infinitely many 0's which are often omitted). The infinite Rényi expansion of 1, denoted by $\mathrm{d}^*_\beta(1)$ is defined by
$$\mathrm{d}^*_\beta(1) = \lim_{\varepsilon\to 0+} \mathrm{d}_\beta(1-\varepsilon) ,$$
where the limit is taken over the usual product topology on $\{0,1,\ldots,\lceil\beta\rceil-1\}^{\mathbb{N}}$. It can be shown that
$$\mathrm{d}^*_\beta(1) = \begin{cases} \big(t_1\cdots t_{m-1}(t_m-1)\big)^\omega & \text{if } \mathrm{d}_\beta(1) = t_1\cdots t_m0^\omega \text{ with } t_m \neq 0, \\ \mathrm{d}_\beta(1) & \text{otherwise.} \end{cases}$$

Parry [10] has shown that the digit string $x_1x_2x_3\cdots \in \{0,1,\ldots,\lceil\beta\rceil-1\}^{\mathbb{N}}$ is β-admissible if and only if for all $i = 1,2,3,\ldots$
$$0^\omega \preceq_{\mathrm{lex}} x_ix_{i+1}x_{i+2}\cdots \prec_{\mathrm{lex}} \mathrm{d}^*_\beta(1) , \tag{1}$$
where $\preceq_{\mathrm{lex}}$ is the lexicographical order. The lexicographic order of admissible digit strings corresponds to the standard order on real numbers in $[0,1)$, i.e., if $\mathrm{d}_\beta(x) = x_1x_2x_3\cdots$ and $\mathrm{d}_\beta(y) = y_1y_2y_3\cdots$ are β-expansions of x and y respectively, then $x < y$ if and only if $\mathrm{d}_\beta(x) \prec_{\mathrm{lex}} \mathrm{d}_\beta(y)$.

Using β-admissible digit strings, one can define the set of non-negative β-integers. According to the knowledge of the authors, the β-integers were first defined in [3]. We have
$$\mathbb{Z}^+_\beta := \{a_k\beta^k + \cdots + a_1\beta + a_0 \mid a_k\cdots a_1a_0 0^\omega \text{ is a } \beta\text{-admissible digit string}\} .$$

In other words, a non-negative real number x is a β-integer, if its β-expansion is of the form $x = \sum_{i=0}^{k} a_i\beta^i$, i.e., it has no non-zero digits at the right from the fractional point $\bullet$.

The distances between consecutive β-integers are described in [14]. It is shown that they take values in the set $\{\Delta_i \mid i = 0, 1, 2\cdots\}$, where

$$\Delta_i = \sum_{j=1}^{\infty} \frac{t_{i+j}}{\beta^j}. \tag{2}$$

Since $\Delta_0 = 1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \cdots$ and the digit string $t_2t_3t_4\cdots$ satisfies the condition (1), we have $\Delta_i \leqslant 1$ for all $i = 0, 1, 2, \ldots$.

2.2. Ito-Sadahiro $(-\beta)$-expansions

Consider now the real base $-\beta < -1$ and the transformation $T_{-\beta} : \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big) \to \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ defined by the prescription

$$T_{-\beta}(x) = -\beta x - \Big\lfloor -\beta x + \frac{\beta}{\beta+1}\Big\rfloor.$$

Every number $x \in \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ can be represented in the form

$$x = \frac{x_1}{-\beta} + \frac{x_2}{(-\beta)^2} + \frac{x_3}{(-\beta)^3} + \cdots, \qquad \text{where} \quad x_i = \Big\lfloor -\beta T_{-\beta}^{i-1}(x) + \frac{\beta}{\beta+1}\Big\rfloor. \tag{3}$$

The representation of x in the form (3) is called the $(-\beta)$-*expansion* of x and denoted

$$\mathrm{d}_{-\beta}(x) = x_1x_2x_3\cdots \qquad \text{or} \qquad x = 0\bullet x_1x_2x_3\cdots.$$

It can be easily shown that the digits x_i belong to the set $\{0, 1, \ldots, \lfloor\beta\rfloor\} =: \mathcal{A}_\beta$, and thus the string $x_1x_2x_3\cdots$ belongs to $\mathcal{A}_\beta^{\mathbb{N}}$. Ito and Sadahiro have shown that the order of reals in $\big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ corresponds to the alternate order $\preceq_{\mathrm{alt}}$ of their $(-\beta)$-expansions, i.e.,

$$x < y \quad \Longleftrightarrow \quad \mathrm{d}_{-\beta}(x) = x_1x_2x_3\cdots \prec_{\mathrm{alt}} \mathrm{d}_{-\beta}(y) = y_1y_2y_3\cdots.$$

Let us recall that the alternate order is defined as follows: We say that $x_1x_2x_3\cdots \prec_{\mathrm{alt}} y_1y_2y_3\cdots$, if $(-1)^i(y_i - x_i) > 0$ for the smallest index i satisfying $x_i \neq y_i$.

In order to describe strings that arise as $(-\beta)$-expansions of some $x \in \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$, the so-called $(-\beta)$-*admissible* digit strings, we will use the notation introduced in [7]. Sometimes, we abbreviate the term $(-\beta)$-admissibility by only admissibility when no confusion can occur.

We denote $l_\beta = \frac{-\beta}{\beta+1}$ and $r_\beta = \frac{1}{\beta+1}$ the left and right end-points of the domain I_β of the transformation $T_{-\beta}$, respectively. That is $I_\beta = [l_\beta, r_\beta)$. We also denote

$$\mathrm{d}_{-\beta}(l_\beta) = d_1d_2d_3\cdots.$$

Theorem 1 ([7]). *A string $x_1x_2x_3\cdots$ over the alphabet $\{0,1,\dots,\lfloor\beta\rfloor\}$ is $(-\beta)$-admissible, if and only if for all $i=1,2,3,\dots$,*

$$\mathrm{d}_{-\beta}(l_\beta) \preceq_{alt} x_ix_{i+1}x_{i+2} \prec_{alt} \mathrm{d}^*_{-\beta}(r_\beta)\,,$$

where $\mathrm{d}^*_{-\beta}(r_\beta) = \lim\limits_{\varepsilon\to 0+} \mathrm{d}_{-\beta}(r_\beta-\varepsilon)$.

The relation between $\mathrm{d}^*_{-\beta}(r_\beta)$ and $\mathrm{d}_{-\beta}(l_\beta)$ is described in the same paper.

Theorem 2 ([7]). *Let $\mathrm{d}_{-\beta}(l_\beta) = d_1d_2d_3\cdots$. If $\mathrm{d}_{-\beta}(l_\beta)$ is purely periodic with odd period-length, i.e., $\mathrm{d}_{-\beta}(l_\beta) = (d_1d_2\cdots d_{2l+1})^\omega$, then $\mathrm{d}^*_{-\beta}(r_\beta) = (0d_1d_2\cdots d_{2l}(d_{2l+1}-1))^\omega$. Otherwise, $\mathrm{d}^*_{-\beta}(r_\beta) = 0\mathrm{d}_{-\beta}(l_\beta)$.*

3. $(-\beta)$-expansion of real numbers

Analogically to the case of Rényi β-expansions, we use for obtaining the $(-\beta)$-expansion of an $x\in\mathbb{R}$ a suitable exponent $l\in\mathbb{N}$ such that $\frac{x}{(-\beta)^l}\in\big[\frac{-\beta}{\beta+1},\frac{1}{\beta+1}\big)$. The advantage of $(-\beta)$-expansions is that one can represent both positive and negative real numbers with the same set of digits, without using the minus sign, whereas in the case of Rényi expansions, one can represent only non-negative reals. The disadvantage of $(-\beta)$-expansions is that the choice of the above mentioned exponent $l\in\mathbb{N}$ may influence the representation of x as a $(-\beta)$-expansion.

Remark 3. Consider $x=\frac{\beta^2}{\beta+1}\notin I_\beta$. Since $\frac{x}{-\beta}=l_\beta$, we have $\mathrm{d}_{-\beta}(\frac{x}{-\beta}) = d_1d_2d_3\cdots$. On the other hand, we also have $\frac{x}{(-\beta)^3}=\frac{1}{-\beta(\beta+1)}\in I_\beta$. In order to find its expansion $\mathrm{d}_{-\beta}(\frac{x}{(-\beta)^3}) = x_1x_2x_3\cdots$ we compute the first digit

$$x_1 = \Big\lfloor -\beta\frac{x}{(-\beta)^3}+\frac{\beta}{\beta+1}\Big\rfloor = \Big\lfloor \frac{1}{\beta+1}+\frac{\beta}{\beta+1}\Big\rfloor = 1$$

and we have $T_{-\beta}\big(\frac{x}{(-\beta)^3}\big) = -\beta\frac{x}{(-\beta)^3}-1 = \frac{1}{\beta+1}-1 = l_\beta$. Therefore $\mathrm{d}_{-\beta}(\frac{x}{(-\beta)^3}) = 1d_1d_2d_3\cdots$. This however means that

$$\frac{1}{-\beta}+\frac{d_1}{(-\beta)^2}+\frac{d_2}{(-\beta)^3}+\frac{d_3}{(-\beta)^4}+\cdots = \frac{d_1}{(-\beta)^3}+\frac{d_2}{(-\beta)^4}+\frac{d_3}{(-\beta)^5}+\cdots\,.$$

Lemma 4. *Let $a_1a_2a_3\cdots$ be a $(-\beta)$-admissible digit string with $a_1\neq 0$. For fixed $k\in\mathbb{Z}$, denote*

$$z_k = \sum_{i=1}^{\infty} a_i(-\beta)^{k-i}\,.$$

Then

$$z_k \in \begin{cases} \Big[\frac{\beta^{k-1}}{\beta+1},\frac{\beta^{k+1}}{\beta+1}\Big] & \textit{for } k \textit{ odd,}\\ \Big[-\frac{\beta^{k+1}}{\beta+1},-\frac{\beta^{k-1}}{\beta+1}\Big] & \textit{for } k \textit{ even.}\end{cases}$$

Proof. Since the alternate order on admissible strings corresponds to the order of real numbers, we have

$$-\frac{\beta}{\beta+1} \leqslant \frac{a_1}{-\beta} + \frac{a_2}{(-\beta)^2} + \frac{a_3}{(-\beta)^3} + \cdots < \frac{1}{\beta+1} \tag{4}$$

and

$$-\frac{\beta}{\beta+1} \leqslant \frac{a_2}{-\beta} + \frac{a_3}{(-\beta)^2} + \frac{a_4}{(-\beta)^3} + \cdots < \frac{1}{\beta+1}\,. \tag{5}$$

We find the estimate of the value of z_k. Consider k odd. Then multiplying the left inequality of (4) by $(-\beta)^k = -\beta^k$ we have

$$\frac{\beta^{k+1}}{\beta+1} \geqslant a_1(-\beta)^{k-1} + a_2(-\beta)^{k-2} + \cdots = z_k\,.$$

Multiplying the left inequality of (5) by $(-\beta)^{k-1} = \beta^{k-1}$ we obtain

$$-\frac{\beta^k}{\beta+1} \leqslant a_2(-\beta)^{k-2} + a_3(-\beta)^{k-3} + \cdots = z_k - a_1(-\beta)^{k-1} = z_k - a_1\beta^{k-1}\,,$$

which using $a_1 \geqslant 1$ implies

$$z_k \geqslant a_1\beta^{k-1} - \frac{\beta^k}{\beta+1} \geqslant \beta^{k-1} - \frac{\beta^k}{\beta+1} = \frac{\beta^{k-1}}{\beta+1}\,.$$

Analogous arguments can be used to show the statement for k even. ■

Remark 5. It follows from Lemma 4 that the only real numbers which can be represented by two different $(-\beta)$-admissible digit strings are

$$z_k = \frac{(-\beta)^k}{\beta+1}\,, \qquad \text{for } k \in \mathbb{Z}\,. \tag{6}$$

We know that such z_k belongs to the interval $[l_\beta, r_\beta)$ if and only if $k = 1$ or $k \leqslant -1$. We have

$$\mathrm{d}_{-\beta}\big(\tfrac{(-\beta)^k}{\beta+1}\big) = \begin{cases} d_1d_2d_3\cdots & \text{for } k = 1, \\ 0^{-k-1}1d_1d_2d_3\cdots & \text{for } k \leqslant -1. \end{cases}$$

Let us stress that even if the digit string $d_1d_2d_3\cdots$ is $(-\beta)$-admissible, the string $0d_1d_2d_3\cdots$ is not. In order to keep unicity of $(-\beta)$-expansion of all real numbers, we will prefer for numbers z of the form (6) the representation using the string $1d_1d_2d_3\cdots$, since it is natural to require that admissibility of a string w implies admissibility of $0w$. Such a convention, however, has an inconvenient consequence: For bases $-\beta$, satisfying $1 = \frac{(-\beta)^k}{\beta+1}$, the $(-\beta)$-expansion of 1 is not equal to 1. For example, if $\beta = \frac{1+\sqrt{5}}{2}$ is the golden ratio, the preferred $(-\beta)$-expansion of 1 is equal to $110\bullet$.

4. $(-\beta)$-integers

In order to avoid ambiguity in defining the expansion of x, we shall define the $(-\beta)$-integers using admissible digit strings. For a finite digit string $w = a_k a_{k-1} \cdots a_1 a_0$ over the alphabet $\mathcal{A}_\beta = \{0, 1, \ldots, \lfloor \beta \rfloor\}$, we call its evaluation the value

$$\gamma(w) := a_k(-\beta)^k + \cdots + a_1(-\beta) + a_0 \,.$$

We define the evaluation of the empty string ϵ to be $\gamma(\epsilon) = 0$.

Definition 6. *A real number x is called a $(-\beta)$-*integer*, if there exists a finite string $a_k \cdots a_1 a_0$ such that*

$$x = \gamma(a_k \cdots a_1 a_0)$$

and the digit string $a_k \cdots a_1 a_0 0^\omega$ is $(-\beta)$-admissible. The set of all $(-\beta)$-integers is denoted by $\mathbb{Z}_{-\beta}$, i.e.,

$$\mathbb{Z}_{-\beta} = \{a_k(-\beta)^k + \cdots + a_1(-\beta) + a_0 \mid a_k \cdots a_1 a_0 0^\omega \textit{ is } (-\beta)\textit{-admissible}\} \,.$$

As explained in Remark 5, any $(-\beta)$-integer x is the evaluation of an $(-\beta)$-admissible string with prefix 0, which is unique up to the number of prefixed zeros.

Remark 7. Note that since $0 \in I_\beta$ and $T_{-\beta}(0) = 0$, we have $\mathrm{d}_{-\beta}(0) = 0^\omega$. Thus $0 \in \mathbb{Z}_{-\beta}$ for every base $-\beta$. Another trivial property which follows from Theorem 1 describing admissible digit strings is that $-\beta\mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta}$.

Example 8. Let β be the so-called Tribonacci number, i.e., the real root of the polynomial $x^3 - x^2 - x - 1$. Then $\mathrm{d}_{-\beta}(l_\beta) = 101^\omega$. Let us find the first (in absolute value smallest) few $(-\beta)$-integers. Denoting $\mathbb{Z}_{-\beta} = \{z_n \mid n \in \mathbb{Z}\}$, $z_0 = 0$, $z_i < z_{i+1}$ for all i, we have

$$\begin{aligned} 1\bullet &= z_1 \\ 110\bullet &= z_2 \\ 111\bullet &= z_3 \\ 100\bullet &= z_4 \\ 11011\bullet &= z_5 \\ 11000\bullet &= z_6 \\ 11001\bullet &= z_7 \\ 11110\bullet &= z_8 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 11\bullet &= z_{-1} \\ 10\bullet &= z_{-2} \\ 1100\bullet &= z_{-3} \\ 1111\bullet &= z_{-4} \\ 1110\bullet &= z_{-5} \\ 1001\bullet &= z_{-6} \\ 1000\bullet &= z_{-7} \\ 1011\bullet &= z_{-8} \,. \end{aligned}$$

These points are drawn in Figure 1. The distances between consecutive $(-\beta)$-integers take values (cf. Section 5)

$$z_{i+1} - z_i \in \left\{ \Delta_0 = 1,\ \Delta_1 = \beta - 1,\ \Delta_2 = \beta^2 - \beta - 1 = \frac{1}{\beta} \right\} .$$

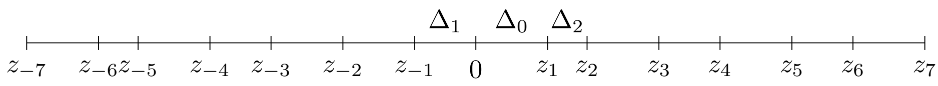

Figure 1. First few $(-\beta)$-integers for the Tribonacci number β.

Example 9. Let β be the smallest Pisot number, i.e., the real root of the polynomial $x^3 - x - 1$. In this case $\mathrm{d}_{-\beta}(l_\beta) = 1001^\omega$. We show that in this case $\mathbb{Z}_{-\beta} = \{0\}$. Since $\lfloor\beta\rfloor = 1$, the digits in admissible strings must belong to the alphabet $\{0, 1\}$. If the digit string $a_k \cdots a_1 a_0 0^\omega \neq 0^\omega$ was $(-\beta)$-admissible, then according to Theorem 1 also the digit string 10^ω must be $(-\beta)$-admissible. However, we have $1001^\omega \not\preceq_{\mathrm{alt}} 10^\omega$, which is a contradiction.

The observation from the previous example is generalized by the following statement, taken from [9].

Proposition 10 ([9]). *Let $\beta > 1$. Then $\mathbb{Z}_{-\beta} = \{0\}$ if and only if $\beta < \frac{1}{2}(1+\sqrt{5})$.*

Let us show that the condition of β being smaller than the golden ratio corresponds to the requirement that the $(-\beta)$-expansion $\mathrm{d}_{-\beta}(l_\beta)$ is of a special form. The following lemma has appeared already in [9], for illustration we give here a proof using another argument.

Lemma 11. *We have $\mathbb{Z}_{-\beta} = \{0\}$ if and only if $10^{2k}1$ is a prefix of $\mathrm{d}_{-\beta}(l_\beta)$ for some $k \geqslant 0$.*

Proof. First realize that a $(-\beta)$-admissible string not equal to 0^ω exists, if and only if the digit string 10^ω is $(-\beta)$-admissible. Since $\mathrm{d}^*_{-\beta}(r_\beta)$ starts always with 0, the alternate inequality $10^\omega \prec_{\mathrm{alt}} \mathrm{d}^*_{-\beta}(r_\beta)$ is always satisfied. It can be seen easily, that the other inequality from Theorem 1, $\mathrm{d}_{-\beta}(l_\beta) \preceq_{\mathrm{alt}} 10^\omega$, is satisfied if and only if no prefix of $\mathrm{d}_{-\beta}(l_\beta)$ has the form $10^{2k}1$ for some $k \geqslant 0$. ∎

5. Distances between $(-\beta)$-integers

From now on, we suppose that the set $\mathbb{Z}_{-\beta}$ is non-trivial, i.e., $\beta \geqslant \frac{1}{2}(1+\sqrt{5})$.

In order to describe distances between adjacent $(-\beta)$-integers, we will study ordering of finite digit strings in the alternate order. Denote by $\mathcal{S}(k)$ the set of infinite $(-\beta)$-admissible digit strings such that erasing a prefix of length k yields 0^ω, i.e., for $k \geqslant 0$, we have

$$\mathcal{S}(k) = \{a_{k-1}a_{k-2}\cdots a_0 0^\omega \mid a_{k-1}a_{k-2}\cdots a_0 0^\omega \text{ is } (-\beta)\text{-admissible}\},$$

in particular $\mathcal{S}(0) = \{0^\omega\}$. For a fixed k, the set $\mathcal{S}(k)$ is finite. Denote by $\mathrm{Max}(k)$ the string $a_{k-1}a_{k-2}\cdots a_0 0^\omega$ which is maximal in $\mathcal{S}(k)$ with respect to the alternate order and by $\max(k)$ its prefix of length k, i.e., $\mathrm{Max}(k) = \max(k)0^\omega$. Similarly, we define $\mathrm{Min}(k)$ and $\min(k)$. Thus,

$$\mathrm{Min}(k) \preceq_{\mathrm{alt}} r \preceq_{\mathrm{alt}} \mathrm{Max}(k)\,, \qquad \text{for all digit strings } r \in \mathcal{S}(k).$$

Remark 12. For any $a_{k-1}a_{k-2}\cdots a_0 0^\omega \in \mathcal{S}(k)$ its suffix satisfies $a_{j-1}a_{j-2}\cdots a_0 0^\omega \in \mathcal{S}(j)$ for $j \leqslant k$.

Let us start with a simple observation about the alternate order.

Lemma 13. *Let $r^{(1)}$, $r^{(2)}$ be infinite strings over $\mathcal{A}_\beta$ and let w be a finite word over $\mathcal{A}_\beta$ of length $|w|$. Then*

$$r^{(1)} \preceq_{alt} r^{(2)} \implies \begin{cases} wr^{(1)} \preceq_{alt} wr^{(2)} & \text{if } |w| \text{ is even,} \\ wr^{(1)} \succeq_{alt} wr^{(2)} & \text{if } |w| \text{ is odd.} \end{cases}$$

Lemma 14. *Let $wdr^{(1)}$, $wcr^{(2)}$ be $(-\beta)$-admissible strings, where the digits c, d satisfy $d > c$. Then there exists a digit string $r^{(3)}$ such that $w(d-1)r^{(3)}$ is $(-\beta)$-admissible and in the alternate order lies between $wdr^{(1)}$ and $wcr^{(2)}$.*

Proof. If $c = d-1$, it suffices to put $r^{(3)} = r^{(2)}$. If $c \leqslant d-2$, then

$$dr^{(1)} \prec_{\text{alt}} (d-1)r^{(1)} \prec_{\text{alt}} cr^{(2)}\,.$$

If $|w|$ is even, then according to Lemma 13, we obtain

$$wdr^{(1)} \prec_{\text{alt}} w(d-1)r^{(1)} \prec_{\text{alt}} wcr^{(2)}\,.$$

For $|w|$ odd, we obtain

$$wdr^{(1)} \succ_{\text{alt}} w(d-1)r^{(1)} \succ_{\text{alt}} wcr^{(2)}\,.$$

The admissibility of strings $wdr^{(1)}$, $wcr^{(2)}$ implies that $w(d-1)r^{(1)}$ is also $(-\beta)$-admissible. Hence we can put $r^{(3)} = r^{(1)}$. ■

Lemma 15. *Let v be a finite word over the alphabet $\mathcal{A}_\beta$ and let $vdr^{(1)}$, $v(d-1)r^{(2)}$ be $(-\beta)$-admissible strings in $\mathcal{S}(n)$, where $n > |v|$. Then the digit strings $vd\mathrm{Min}(k)$ and $v(d-1)\mathrm{Max}(k)$, for $k = n - |v| - 1$, are $(-\beta)$-admissible strings and no other string from $\mathcal{S}(n)$ lies in between them (with respect to the alternate order).*

Proof. We verify admissibility of $vd\mathrm{Min}(k)$ and $v(d-1)\mathrm{Max}(k)$ by showing that for every suffix w of v the following inequalities are satisfied

$$\begin{aligned} \mathrm{d}_{-\beta}(l_\beta) &\preceq_{\text{alt}} \quad wd\mathrm{Min}(k) \quad \prec_{\text{alt}} \mathrm{d}^*_{-\beta}(r_\beta)\,, \\ \mathrm{d}_{-\beta}(l_\beta) &\preceq_{\text{alt}} w(d-1)\mathrm{Max}(k) \prec_{\text{alt}} \mathrm{d}^*_{-\beta}(r_\beta)\,. \end{aligned}$$

Since $wdr^{(1)} \neq w(d-1)r^{(2)}$ we have one of the following cases.

a) Let $wdr^{(1)} \prec_{\text{alt}} w(d-1)r^{(2)}$. This implies that the length of words wd and $w(d-1)$ is odd. As according to Remark 12 the strings $r^{(1)}$, $r^{(2)}$ belong to $\mathcal{S}(k)$, we obtain

$$\begin{aligned} r^{(1)} \succeq_{\text{alt}} \mathrm{Min}(k) &\implies wdr^{(1)} \preceq_{\text{alt}} wd\mathrm{Min}(k)\,, \\ r^{(2)} \preceq_{\text{alt}} \mathrm{Max}(k) &\implies w(d-1)r^{(2)} \succeq_{\text{alt}} w(d-1)\mathrm{Max}(k)\,. \end{aligned}$$

Now it suffices to use the transitivity of ordering and admissibility of $wdr^{(1)}$ and $w(d-1)r^{(2)}$ to conclude admissibility of $vd\mathrm{Min}(k)$ and $v(d-1)\mathrm{Max}(k)$,

$$\mathrm{d}_{-\beta}(l_\beta) \preceq_{\mathrm{alt}} wdr^{(1)} \preceq_{\mathrm{alt}} wd\mathrm{Min}(k) \prec_{\mathrm{alt}} w(d-1)\mathrm{Max}(k) \preceq_{\mathrm{alt}} w(d-1)r^{(2)} \\ \prec_{\mathrm{alt}} \mathrm{d}^*_{-\beta}(r_\beta).$$

The latter inequality, together with Lemma 14 already implies that $vd\mathrm{Min}(k)$ and $v(d-1)\mathrm{Max}(k)$ are – in the alternate order – adjacent strings in $\mathcal{S}(n)$ with $n = |v| + 1 + k$.

b) Let $wdr^{(1)} \succ_{\mathrm{alt}} w(d-1)r^{(2)}$. In this case words wd and $w(d-1)$ are of even length. We obtain

$$w(d-1)r^{(2)} \preceq_{\mathrm{alt}} w(d-1)\mathrm{Max}(k) \prec_{\mathrm{alt}} wd\mathrm{Min}(k) \preceq_{\mathrm{alt}} wdr^{(1)},$$

and the argumentation is similar. ∎

The following statement is a direct consequence of Lemma 15, taking into account that any pair of $(-\beta)$-integers is evaluation of a pair of strings of the same length, both starting with 0.

Proposition 16. *Let $x < y$ be two consecutive $(-\beta)$-integers. Then there exist a unique non-negative integer $k \in \{0, 1, 2, \dots\}$ and a positive digit $d \in \mathcal{A}_\beta \setminus \{0\}$ such that words $w(d-1)\mathrm{Max}(k)$ and $wd\mathrm{Min}(k)$ are $(-\beta)$-admissible strings and*

$$\begin{aligned} x &= \gamma(w(d-1)\max(k)) < y = \gamma(wd\min(k)) && \text{for } k \text{ even},\\ x &= \gamma(wd\min(k)) < y = \gamma(w(d-1)\max(k)) && \text{for } k \text{ odd}, \end{aligned}$$

where w is a finite string over the alphabet $\mathcal{A}_\beta$ with prefix 0. In particular, the distance $y - x$ between these $(-\beta)$-integers depends only on k and equals to

$$\Delta_k := \Big|(-\beta)^k + \gamma\big(\min(k)\big) - \gamma\big(\max(k)\big)\Big|. \tag{7}$$

From the properties of the transformation $T_{-\beta}$ it follows that the digits d_i of the expansion $\mathrm{d}_{-\beta}(l_\beta)$ satisfy $0 \leqslant d_i \leqslant d_1$ for all i. Imposing more assumptions on the digits d_i will allow us to describe explicitly the maximal and minimal strings in $\mathcal{S}(k)$, and by that also the distances between consecutive $(-\beta)$-integers.

Note that explicit description of strings $\mathrm{Max}(k)$, $\mathrm{Min}(k)$ for any given β is possible, but providing an explicit formula for the general case would require very tedious discussion.

Lemma 17. *Let $\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 d_3 \cdots$, where $0 < d_i$ and $d_1 > d_{2i}$ for all $i = 1, 2, 3, \dots$. Then $\min(0) = \epsilon$, $\max(0) = \epsilon$ and for every $k \geqslant 0$ we have*

$$\begin{aligned} \min(2k) &= d_1 d_2 d_3 \cdots d_{2k-1} d_{2k}\,, \\ \min(2k+1) &= d_1 d_2 d_3 \cdots d_{2k}(d_{2k+1} - 1)\,, \end{aligned} \qquad \textit{and} \qquad \max(k+1) = 0\min(k)\,.$$

Proof. Assumptions on d_i exclude the case that $\mathrm{d}_{-\beta}(l_\beta)$ is purely periodic with odd period length. Therefore by Theorem 2 one has $\mathrm{d}^*_{-\beta}(r_\beta) = 0\mathrm{d}_{-\beta}(l_\beta)$.

Let us show that $\min(2k) = d_1 d_2 \cdots d_{2k}$. First we show that the digit string $d_1 \cdots d_{2k} 0^\omega$ is admissible. For that, we need to verify inequalities

$$d_1 d_2 d_3 \cdots \preceq_{\mathrm{alt}} d_i d_{i+1} \cdots d_{2k} 0^\omega \prec_{\mathrm{alt}} 0 d_1 d_2 d_3 \cdots \tag{8}$$

for all $i = 1, \ldots, 2k$. By the assumption, we have

$$0^\omega \succ_{\mathrm{alt}} d_{2k+1} d_{2k+2} \cdots . \tag{9}$$

For $i = 2r \leqslant 2k$ the string $d_{2r} d_{2r+1} \cdots d_{2k}$ is of odd length, and thus

$$d_{2r} d_{2r+1} \cdots d_{2k} 0^\omega \prec_{\mathrm{alt}} d_{2r} d_{2r+1} \cdots d_{2k} d_{2k+1} d_{2k+2} \cdots \prec_{\mathrm{alt}} 0\mathrm{d}_{-\beta}(l_\beta)\,, \tag{10}$$

where the left inequality is a consequence of (9) and the right inequality follows from admissibility of $\mathrm{d}_{-\beta}(l_\beta)$. On the other hand, since $d_{2r} < d_1$, we have directly

$$\mathrm{d}_{-\beta}(l_\beta) \prec_{\mathrm{alt}} d_{2r} d_{2r+1} \cdots d_{2k} 0^\omega\,.$$

This, together with (10) gives (8) for $i = 2r$.

For $i = 2r+1 < 2k$ the string $d_{2r+1} \cdots d_{2k}$ is of even length and therefore with the use of (9) and admissibility of $\mathrm{d}_{-\beta}(l_\beta)$ itself we obtain

$$\mathrm{d}_{-\beta}(l_\beta) \preceq_{\mathrm{alt}} d_{2r+1} \cdots d_{2k} d_{2k+1} d_{2k+2} \cdots \prec_{\mathrm{alt}} d_{2r+1} \cdots d_{2k} 0^\omega\,. \tag{11}$$

Since $d_{2r+1} > 0$, we can claim that $d_{2r+1} \cdots d_{2k} 0^\omega \prec_{\mathrm{alt}} 0\mathrm{d}_{-\beta}(l_\beta)$. Together with (11), this confirms validity of (8) for $i = 2r+1$.

It remains to show the minimality of the digit string $d_1 \cdots d_{2k} 0^\omega$ in the set $\mathcal{S}(2k)$. If there exists a string s of length $2k$ such that $d_1 \cdots d_{2k} d_{2k+1} \cdots \preceq_{\mathrm{alt}} s0^\omega \prec_{\mathrm{alt}} d_1 \cdots d_{2k} 0^\omega$, then, from the alternate order, we derive that $d_1 \cdots d_{2k} = s$. Thus $\min(2k) = d_1 d_2 \cdots d_{2k}$.

In order to determine $\min(2k+1)$, we first show that $d_1 d_2 \cdots d_{2k+1} 0^\omega$ is not admissible. As $d_{2k+2} > 0$, we can write that $0^\omega \succ_{\mathrm{alt}} d_{2k+2} d_{2k+3} \cdots$, and this implies $d_1 d_2 \cdots d_{2k+1} 0^\omega \prec_{\mathrm{alt}} \mathrm{d}_{-\beta}(l_\beta)$. Let $\mathrm{Min}(2k+1) = s0^\omega$ for some digit string s of length $2k+1$. Since $\min(2k)0^\omega$ belongs to the set $\mathcal{S}(2k+1)$, we have

$$d_1 \cdots d_{2k} d_{2k+1} d_{2k+2} \cdots \preceq_{\mathrm{alt}} s0^\omega \preceq_{\mathrm{alt}} d_1 \cdots d_{2k} 0^\omega\,,$$

which implies that $s = d_1 d_2 \cdots d_{2k} x$ for some digit $x \in \mathcal{A}_\beta$. Moreover, the digit x is maximal possible, so that the string $s0^\omega$ be admissible. It is easy to see that $x = d_{2k+1} - 1$.

In order to describe maximal strings, realize that the assumption $d_{2i} < d_1$ excludes the possibility that $\mathrm{d}_{-\beta}(l_\beta)$ is purely periodic with odd period-length. Therefore $\mathrm{d}^*_{-\beta}(r_\beta) = 0\mathrm{d}_{-\beta}(l_\beta)$. If for some digit string s the string $s0^\omega$ is admissible and not equal to $\mathrm{d}_{-\beta}(l_\beta)$ then also $0s0^\omega$ is admissible. We immediately obtain $\max(k) = 0\min(k-1)$ for $k \geqslant 1$. ■

Theorem 18. *Let* $\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 d_3 \cdots$, *where* $0 < d_i$ *and* $d_1 > d_{2i}$ *for all* $i = 1, 2, 3, \ldots$. *Then the distances between adjacent* $(-\beta)$*-integers take values*

$$\Delta_0 = 1 \qquad \textit{and} \qquad \Delta_k = \Big|(-1)^k + \sum_{i=1}^{\infty} \frac{d_{k-1+i} - d_{k+i}}{(-\beta)^i}\Big|, \quad k = 1, 2, 3, \ldots$$

Moreover, all the distances are less than 2.

Proof. For the description of distances Δ_k according to (7) we need to evaluate $\gamma(\max(k))$ and $\gamma(\min(k))$. By Lemma 17 we have $\max(k) = 0\min(k-1)$ for every $k \geqslant 1$, and thus $\gamma(\max(k)) = \gamma(\min(k-1))$. For the calculation of $\gamma(\min(k))$ we discuss the cases of even and odd k separately.

According to Lemma 17 we have

$$\gamma(\min(2k)) = \sum_{i=1}^{2k} d_i(-\beta)^{2k-i} = \sum_{i=1}^{+\infty} d_i(-\beta)^{2k-i} - \sum_{i=2k+1}^{+\infty} d_i(-\beta)^{2k-i} = \\ = (-\beta)^{2k} l_\beta - \sum_{i=1}^{+\infty} \frac{d_{2k+i}}{(-\beta)^i}\,.$$

Similarly, we obtain

$$\gamma(\min(2k+1)) = \sum_{i=1}^{2k+1} d_i(-\beta)^{2k+1-i} - 1 = (-\beta)^{2k+1} l_\beta - 1 - \sum_{i=1}^{+\infty} \frac{d_{2k+1+i}}{(-\beta)^i}\,.$$

Therefore, from (7),

$$\Delta_{2k} = \Big|(-\beta)^{2k} + \gamma\big(\min(2k)\big) - \gamma\big(\min(2k-1)\big)\Big| = \\ = \Big|(-\beta)^{2k} + (-\beta)^{2k} l_\beta - (-\beta)^{2k-1} l_\beta + 1 - \sum_{i=1}^{+\infty} \frac{d_{2k+i}}{(-\beta)^i} + \sum_{i=1}^{+\infty} \frac{d_{2k-1+i}}{(-\beta)^i}\Big|\,.$$

Realizing that $(-\beta)^{2k} + (-\beta)^{2k} l_\beta - (-\beta)^{2k-1} l_\beta = 0$, we obtain the formula for Δ_{2k}. The same procedure leads to the description of distances with odd indices.

It remains to show that $\Delta_k < 2$. This follows from the fact that $\sum_{i=1}^{+\infty} \frac{d_{r+i}}{(-\beta)^i} = T_{-\beta}^{r}(l_\beta) \in I_\beta$, and thus the subtraction of two sums in the expression for Δ_k is equal to the difference of two numbers in the interval I_β, which is of length 1. ■

Let us mention that the distances Δ_k for $k \geqslant 1$ can be written in the form

$$\Delta_k = \Big|(-1)^k + T_{-\beta}^{k-1}(l_\beta) - T_{-\beta}^{k}(l_\beta)\Big|\,. \tag{12}$$

Using such expression, one can describe which distances among Δ_k coincide, if $\mathrm{d}_{-\beta}(l_\beta)$ is eventually periodic. In the following corollary, the lengths m of the pre-period and p of the period are considered the smallest possible.

Corllary 19. *Let the digits $d_1, d_2, d_3, \ldots$ of $\mathrm{d}_{-\beta}(l_\beta)$ satisfy the conditions of Theorem 18. Then*

- *If $\mathrm{d}_{-\beta}(l_\beta) = d_1 \cdots d_m (d_{m+1})^\omega$, then $\Delta_{m+k} = \Delta_0$ for all $k \geqslant 1$.*
- *If $\mathrm{d}_{-\beta}(l_\beta) = d_1 \cdots d_m (d_{m+1} d_{m+2})^\omega$, then $\Delta_{m+k} = \Delta_{m+1}$ for all $k \geqslant 1$.*
- *If $\mathrm{d}_{-\beta}(l_\beta) = d_1 \cdots d_m (d_{m+1} \cdots d_{m+p})^\omega$ and p even, $p \geqslant 4$, then $\Delta_{m+p+k} = \Delta_{m+k}$ for all $k \geqslant 1$.*
- *If $\mathrm{d}_{-\beta}(l_\beta) = d_1 \cdots d_m (d_{m+1} \cdots d_{m+p})^\omega$ and p odd, $p \geqslant 3$, then $\Delta_{m+2p+k} = \Delta_{m+k}$ for all $k \geqslant 1$. Moreover, $\Delta_{m+p+k} \neq \Delta_{m+k}$, namely $\Delta_{m+p+k} = 2 - \Delta_{m+k}$.*

Even though the class of numbers β fulfilling the assumption of Theorem 18 is quite large, some interesting cases are omitted; in particular β cannot have finite $\mathrm{d}_{-\beta}(l_\beta)$, i.e., with finitely many non-zero digits. The following theorem gives a result with similar assumption which, moreover, allows $\mathrm{d}_{-\beta}(l_\beta)$ to be finite. We omit its proof because it follows the same ideas as the proof of Lemma 17.

Lemma 20. *Let $\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 \cdots d_m 0^\omega$, where $d_m \neq 0$ and $m \geqslant 1$. If $0 < d_i$ and $d_1 > d_{2i}$ for all $i = 1, 2, 3, 4, \ldots, m$, then*

$$\min(k) = \begin{cases} d_1 \cdots d_k & \text{for even } k < m, \\ d_1 \cdots d_{k-1}(d_k - 1) & \text{for odd } k < m, \\ d_1 \cdots d_m 0^{k-m} & \text{for } k \geqslant m, \end{cases}$$

and

$$\max(k) = \begin{cases} 0 \min(k-1) & \text{for } k \leqslant m, \\ 0 d_1 \cdots d_{m-1}(d_m - 1) & \text{for even } k = m+1, \\ 0 d_1 \cdots d_{m-1}(d_m + 1) & \text{for odd } k = m+1,\ d_m < d_1 - 1, \\ 0 d_1 \cdots d_{m-2}(d_{m-1} - 1)0 & \text{for odd } k = m+1,\ d_m = d_1 - 1, \\ 0 d_1 \cdots d_{m-1}(d_m + 1) \min(1) & \text{for even } k = m+2, \\ 0 d_1 \cdots d_m 0^{k-m-2} 1 & \text{for odd } k \geqslant m+2, \\ 0 d_1 \cdots d_m 0^{k-m-3} 1 \min(1) & \text{for even } k \geqslant m+3. \end{cases}$$

When using Lemma 20 in formula (7), we obtain the following statement describing the distances Δ_k in the case of finite $\mathrm{d}_{-\beta}(l_\beta)$.

Theorem 21. *Let $\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 \cdots d_m 0^\omega$, where $d_m \neq 0$ and $m \geqslant 1$. If $0 < d_i$ and $d_1 > d_{2i}$ for all $i = 1, 2, 3, 4, \ldots, m$, then the distances between adjacent*

$(-\beta)$-integers take values

$$
\begin{aligned}
\Delta_0 &= 1\,,\\
\Delta_k &= \Big|(-1)^k + \sum_{i=1}^{\infty} \frac{d_{k-1+i} - d_{k+i}}{(-\beta)^i}\Big|\,, \qquad k = 1,\dots,m-1\,,\\
\Delta_m &= \begin{cases} 1 - \frac{d_m}{\beta} & \textit{for } m \textit{ even}\,,\\ \frac{d_m}{\beta} & \textit{for } m \textit{ odd}\,,\end{cases}\\
\Delta_{m+1} &= \begin{cases} \Delta_0 & \textit{for } m \textit{ even, } d_m < d_1 - 1\,,\\ \Delta_1 & \textit{for } m \textit{ even, } d_m = d_1 - 1\,,\\ \Delta_0 & \textit{for } m \textit{ odd}\,,\end{cases}\\
\Delta_k &= \begin{cases} \Delta_0 & \textit{for odd } k \geqslant m+2\,,\\ \Delta_1 & \textit{for even } k \geqslant m+2\,.\end{cases}
\end{aligned}
$$

Moreover, all the distances are less than 2.

6. Infinite words associated with $(-\beta)$-integers

For a positive base β, if the Rényi expansion of unity $\mathrm{d}_\beta(1)$ is eventually periodic, i.e., β is a Parry number, the ordering of distances between consecutive non-negative β-integers can be coded by a right-sided infinite word $\boldsymbol{u}_\beta$ over a finite alphabet. The word $\boldsymbol{u}_\beta$ is a fixed point of a primitive morphism φ_β, see [4] where these morphisms are called canonical substitutions associated with Parry numbers. The set of non-negative β-integers is denoted by $\mathbb{Z}_\beta^+$, and the set of all β-integers is defined symmetrically as $\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+)$. Note that besides the symmetry, there is no natural reason for defining $\mathbb{Z}_\beta$ in this way for $\beta \notin \mathbb{N}$. The distances in the set of non-positive β-integers are then coded by a left-sided infinite word which is invariant under a morphism different from the canonical substitution φ_β, which is by a mirror image connected with φ_β.

Let us now concentrate again on the set of $(-\beta)$-integers which is non-trivial for β greater than or equal to the golden ratio. Note that the set $\mathbb{Z}_{-\beta}$ includes both positive an negative numbers expressed using non-negative digits. Our aim is to provide, analogously to the case of positive base, a morphism fixing a word coding the set of $(-\beta)$-integers. We start by showing that this is possible for every $\beta \geqslant \tau$, irrespectively whether the corresponding $(-\beta)$-shift is sofic or not, when using an infinite alphabet.

If the $(-\beta)$-shift is sofic, one then finds, by a suitable projection of the infinite alphabet, a bidirectional infinite word $\boldsymbol{u}_{-\beta}$ over a finite alphabet fixed by a morphism. The advantage of the negative base number system is that the left and the right side of the word $\boldsymbol{u}_{-\beta}$ are invariant under the same morphism. This is not the case for positive base system. On the other hand, unlike the case of positive base systems, where only two types of canonical substitutions arise according to whether we consider a simple or a non-simple Parry number, for $(-\beta)$-integers, one does not obtain a unified prescription for the morphism dependently on the coefficients d_i of $\mathrm{d}_{-\beta}(l_\beta)$. However, we give examples of classes of numbers β together with prescriptions for the corresponding morphisms.

Let us recall some notions needed in sequel. If $\mathcal{B}$ is a (finite or infinite) alphabet, then $\mathcal{B}^*$ denotes the set of finite words over $\mathcal{B}$. Equipped with the operation of concatenation, $\mathcal{B}^*$ is a free monoid with the empty word ϵ as the neutral element. A *morphism* over $\mathcal{B}$ is a mapping $\psi : \mathcal{B}^* \to \mathcal{B}^*$ such that $\psi(vw) = \psi(v)\psi(w)$ for all pairs of words $v, w \in \mathcal{B}^*$. The action of a morphism can be naturally extended to infinite words, both one-directional and pointed bidirectional by

$$\psi(w_0w_1w_2\cdots) = \psi(w_0)\psi(w_1)\psi(w_2)\cdots ,$$
$$\psi(\cdots w_{-2}w_{-1}|w_0w_1w_2\cdots) = \cdots\psi(w_{-2})\psi(w_{-1})|\psi(w_0)\psi(w_1)\psi(w_2)\cdots ,$$

respectively.

Let us assign to the set $\mathbb{Z}_{-\beta}$ a bidirectional infinite word $\boldsymbol{v}_{-\beta} = (v_n)_{n\in\mathbb{Z}}$ over the infinite alphabet $\mathbb{N}$. We show that $\boldsymbol{v}_{-\beta}$ is invariant under a morphism constructed as the second iteration of an antimorphism Φ. By an *antimorphism* over an alphabet $\mathcal{B}$, we understand a mapping $\varphi : \mathcal{B}^* \to \mathcal{B}^*$ such that $\varphi(wv) = \varphi(v)\varphi(w)$ for any $w, v \in \mathcal{B}^*$.

At first, we define the word $\boldsymbol{v}_{-\beta} = (v_n)_{n\in\mathbb{Z}}$ associated with $(-\beta)$-integers. Let $(z_n)_{n\in\mathbb{Z}}$ be the strictly increasing sequence satisfying

$$z_0 = 0 \qquad \text{and} \qquad \mathbb{Z}_{-\beta} = \{z_n \mid n \in \mathbb{Z}\}\,.$$

According to Proposition 16, for any $n \in \mathbb{Z}$ there exist a unique integer $k \in \mathbb{N}$, a word w with prefix 0, and a letter d such that

$$z_{n+1} - z_n = \Big|\gamma\big(w(d-1)\max(k)\big) - \gamma\big(wd\min(k)\big)\Big|\,. \tag{13}$$

We put $v_n = k$. Thus the n-th letter in the word $\boldsymbol{v}_{-\beta}$ equals to the maximal exponent k for which the coefficients in the $(-\beta)$-expansions of z_n and z_{n+1} differ. The letters of the infinite word $\boldsymbol{v}_{-\beta}$ we have just obtained take values in the infinite alphabet $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The self-similarity of the set $\mathbb{Z}_{-\beta}$, namely the property $-\beta\mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta}$, allows us to find a morphism under which $\boldsymbol{v}_{-\beta}$ is invariant.

Theorem 22. *Let $\boldsymbol{v}_{-\beta}$ be the word associated with $(-\beta)$-integers. There exists an antimorphism $\Phi : \mathbb{N}^* \to \mathbb{N}^*$ such that $\Psi = \Phi^2$ is a non-erasing non-identical morphism and $\Psi(\boldsymbol{v}_{-\beta}) = \boldsymbol{v}_{-\beta}$.*

Proof. Suppose that $x = z_n < y = z_{n+1}$ are two consecutive $(-\beta)$-integers. Obviously, $-\beta x, -\beta y \in -\beta\mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta}$ and $-\beta y < -\beta x$. Let us study which $(-\beta)$-integers lie between $-\beta y$ and $-\beta y$. Our considerations are illustrated by Figures 2 and 3.

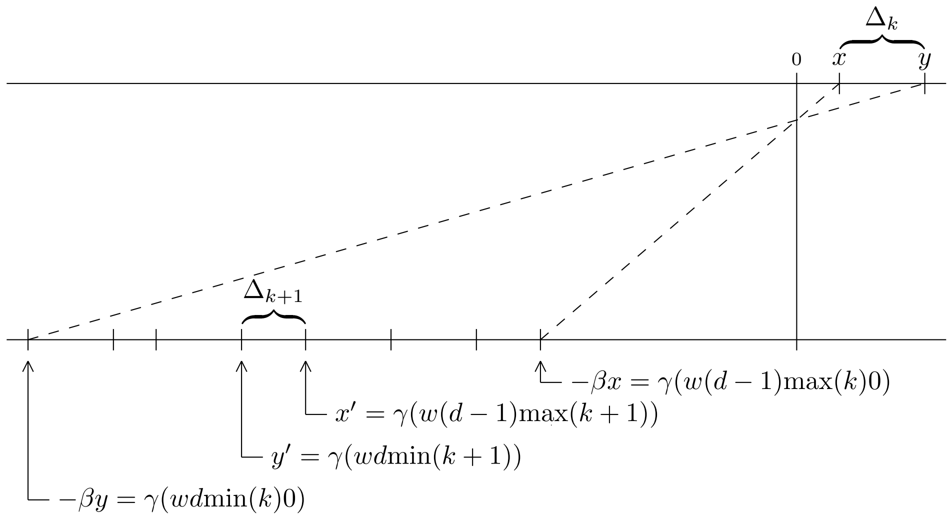

Figure 2. Location of points x, y and $-\beta x, -\beta y$ from (14), and x', y' from (15).

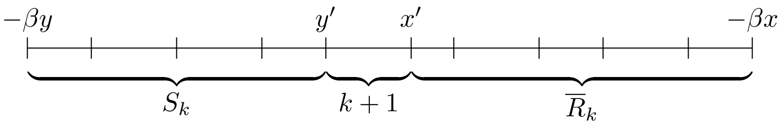

Figure 3. Emplacement of $(-\beta)$-integers $b_0, \dots, b_{n_k}$ of (18) and $c_0, \dots, c_{m_k}$ of (20) for k even.

Suppose at first that the distance $z_{n+1} - z_n$ between x and y is coded by an even number $v_n = k$. Then using (13), we have

$$\begin{aligned} x &= \gamma\big(w(d-1)\max(k)\big), \\ y &= \gamma\big(wd\min(k)\big), \end{aligned} \qquad \text{and} \qquad \begin{aligned} -\beta x &= \gamma\big(w(d-1)\max(k)0\big), \\ -\beta y &= \gamma\big(wd\min(k)0\big). \end{aligned} \tag{14}$$

Let us realize that

$$d\mathrm{Min}(k) \preceq_{\mathrm{alt}} d\mathrm{Min}(k+1) \preceq_{\mathrm{alt}} (d-1)\mathrm{Max}(k+1) \preceq_{\mathrm{alt}} (d-1)\mathrm{Max}(k).$$

Therefore according to Proposition 16, points

$$y' = \gamma\big(wd\min(k+1)\big) \qquad \text{and} \qquad x' = \gamma\big(w(d-1)\max(k+1)\big) \tag{15}$$

are consecutive $(-\beta)$-integers situated between $-\beta y$ and $-\beta x$, see Figure 2. Their distance is coded by the number $k+1$ and the value of their distance is Δ_{k+1}.

For the description of all $(-\beta)$-integers between $-\beta y$ and $-\beta x$, it suffices to determine all digit strings $s0^\omega$ such that

$$s0^\omega \in \mathcal{S}(k+1) \qquad \text{and} \qquad \min(k+1)0^\omega \preceq_{\mathrm{alt}} s0^\omega \preceq_{\mathrm{alt}} \min(k)00^\omega \tag{16}$$

and to determine all digit strings r such that

$$r0^\omega \in \mathcal{S}(k+1) \qquad \text{and} \qquad \max(k)00^\omega \preceq_{\text{alt}} r0^\omega \preceq_{\text{alt}} \max(k+1)0^\omega \,. \tag{17}$$

Note that similar considerations can be carried out in case that the distance between $(-\beta)$-integers x, y is coded by an odd number k. In the following, we consider both k even and odd.

Let us order the strings $s0^\omega$ satisfying (16) in the alternate order starting with the greatest one and ending with the smallest one. We evaluate $\gamma(wds)$ for all of such strings s. We thus obtain a sequence of consecutive $(-\beta)$-integers, say

$$-\beta y = \gamma\big(wd\min(k)0\big) = b_0,\ b_1,\ b_2,\ \ldots,\ b_{n_k} = \gamma\big(wd\min(k+1)\big) = y' \,. \tag{18}$$

This sequence is increasing if k is even and decreasing if k is odd. The distance between b_{i-1} and b_i, for all $i = 1, 2, \ldots, n_k$, is coded by a number, say s_i, in $\{0, 1, 2, \ldots, k\}$. Denote by S_k the word of length n_k found as the concatenation of $s_1, s_2, \ldots, s_{n_k}$,

$$S_k := s_1 s_2 \cdots s_{n_k} \,. \tag{19}$$

Similarly, we now order the strings $r0^\omega$ satisfying (17) starting with the smallest one $\max(k)00^\omega$ and ending with $\max(k+1)0^\omega$. Evaluating $\gamma\big(w(d-1)r\big)$, we obtain a sequence of consecutive $(-\beta)$-integers, say

$$-\beta x = \gamma\big(w(d-1)\max(k)0\big) = c_0,\ c_1,\ \ldots,\ c_{m_k} = \gamma\big(w(d-1)\max(k+1)\big) = x' \,. \tag{20}$$

This sequence is decreasing for k even and increasing for k odd. Let us denote by $r_i \in \{0, 1, 2, \ldots, k\}$ the number coding the distance between c_{i-1} and c_i for $i = 1, 2, \ldots, m_k$ and denote the concatenation

$$R_k := r_1 r_2 \cdots r_{m_k} \,. \tag{21}$$

Let us stress that words S_k and R_k depend only on k which was assigned to the distance between $x = z_n$ and $y = z_{n+1}$ and that S_k and R_k are independent of w and d occurring in the evaluation of $z_{n+1} - z_n$ in (13). The situation for even k is depicted at Figure 3.

Now we are in the position to define the antimorphism $\Phi : \mathbb{N}^* \to \mathbb{N}^*$. Put for all $\ell \in \mathbb{N}$

$$\Phi(2\ell) = S_{2\ell}\,(2\ell+1)\,\overline{R}_{2\ell} \qquad \text{and} \qquad \Phi(2\ell+1) = R_{2\ell+1}\,(2\ell+2)\,\overline{S}_{2\ell+1} \,. \tag{22}$$

The notation $\overline{w}$ is used for the mirror image of the word w, i.e., $\overline{w} = w_t w_{t-1} \ldots w_2 w_1$ if $w = w_1 w_2 \ldots w_t$.

The word $\Phi(k)$, defined in (22), codes the distances between $-\beta y$ and $-\beta x$ in case that the distance $y - x$ between consecutive $(-\beta)$-integers x, y, $x < y$, is coded by k.

Now applying the same procedure to all pairs of consecutive $(-\beta)$-integers occurring between $-\beta y$ and $-\beta x$, we can find the word coding the sequence of $(-\beta)$-integers between $\beta^2 x$ and $\beta^2 y$, as shown in Figure 4. Clearly, this word – depending only on k – is equal to $\Phi^2(k)$.

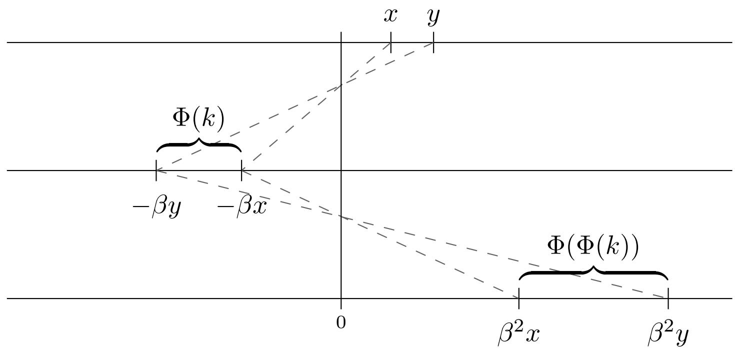

Figure 4. Construction of the morphism $\Psi = \Phi^2$ from the antimorphism Φ.

To conclude, the self-similarity of $\mathbb{Z}_{-\beta}$ guarantees that $\boldsymbol{v}_{-\beta}$ is a fixed point of the morphism $\Psi = \Phi^2$. The prescription (22) guarantees that $\Psi = \Phi^2$ is a non-erasing non-identical morphism. ∎

Theorem 22 shows that a morphism over an infinite alphabet, fixing the word $\boldsymbol{v}_{-\beta}$, exists for every β for which $\boldsymbol{v}_{-\beta}$ can be defined. Such morphism can be explicitly described, whenever strings $\min(k)$ and $\max(k)$ are known, so that we can determine the words S_k and R_k of (22). In determining them, we follow the ideas of the proof of Theorem 22.

Theorem 23. *Let the string* $\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 d_3 \ldots$ *satisfy* $0 < d_i$ *and* $d_1 > d_{2i}$ *for all* $i = 1, 2, 3, \ldots$. *Then the antimorphism from Theorem 22 is of the form*

$$\begin{aligned} \Phi(0) &= 0^{d_1-1}1\,, & \\ \Phi(2\ell) &= 0^{d_{2\ell+1}-1}(2\ell+1)0^{d_1-d_{2\ell}-1}1 & \quad \textit{for } \ell \geqslant 1, \\ \Phi(2\ell+1) &= 0^{d_{2\ell+1}-1}(2\ell+2)0^{d_1-d_{2\ell+2}-1}1 & \quad \textit{for } \ell \geqslant 0. \end{aligned}$$

Proof. We need to determine words S_k and R_k for the prescription for Φ, given in (22). For that we use the explicit form of strings $\min(k)$ and $\max(k)$ from Lemma 17.

By (18) and (20) we have $S_0 = 0^{d_1-1}$ and $R_0 = \epsilon$, thus $\Phi(0) = 0^{d_1-1}1$. Next, let us find the word $S_{2\ell}$ with $\ell \geqslant 1$. Definition (19) of $S_{2\ell}$ requires to find all sequences between

$$\min(2\ell)00^\omega \qquad \text{and} \qquad \min(2\ell+1)0^\omega\,,$$

or equivalently, to determine all $(-\beta)$-integers between

$$\gamma(d_1 d_2 \dots d_{2\ell}0) \qquad \text{and} \qquad \gamma(d_1 d_2 \dots d_{2\ell}(d_{2\ell+1}-1))\,.$$

The words $d_1 d_2 \dots d_{2\ell}0$ and $d_1 d_2 \dots d_{2\ell}(d_{2\ell+1}-1)$ differ only at the last position which in $(-\beta)$-expansion corresponds to the power $(-\beta)^0$. Thus all distances between these consecutive $(-\beta)$-integers are coded by 0, and therefore we have $S_{2\ell} = 0^{d_{2\ell+1}-1}$. Similarly, $S_0 = 0^{d_1-1}$ as $\gamma(\min(0)0) = \gamma(0)$ and $\gamma(\min(1)) = \gamma(d_1-1)$.

Let us determine the words $S_{2\ell+1}$. The complete ordered list of all sequences $s0^\omega$, satisfying (16) for $k = 2\ell+1$ is

$$\begin{array}{rcccccccccc}
\min(2\ell+1)00^\omega & = & d_1 & d_2 & d_3 & \cdots & d_{2\ell} & (d_{2\ell+1}-1) & 0 & 0^\omega \\
 & & d_1 & d_2 & d_3 & \dots & d_{2\ell} & d_{2\ell+1} & (d_1-1) & 0^\omega \\
 & & d_1 & d_2 & d_3 & \dots & d_{2\ell} & d_{2\ell+1} & (d_1-2) & 0^\omega \\
 & & d_1 & d_2 & d_3 & \dots & d_{2\ell} & d_{2\ell+1} & (d_1-3) & 0^\omega \\
 & & & & & \vdots & & & & \\
\min(2\ell+2)0^\omega & = & d_1 & d_2 & d_3 & \dots & d_{2\ell} & d_{2\ell+1} & d_{2\ell+2} & 0^\omega
\end{array}$$

Therefore we obtain the word $S_{2\ell+1} = 10^{d_1-1-d_{2\ell+2}}$.

The equality $\max(k) = 0\min(k-1)$ from Lemma 17 gives us immediately $R_k = S_{k-1}$ for all $k \geqslant 1$. The word R_0 is empty, as $\max(0)0^\omega = \max(1)0^\omega = 0^\omega$. ∎

Let us now study under which conditions one can represent $(-\beta)$-integers by an infinite word over a restricted finite alphabet, so that it is still invariant under a primitive morphism.

Proposition 24. *Let $\boldsymbol{v}$ be an infinite word over the alphabet $\mathbb{N}$, and let $\Psi : \mathbb{N}^* \to \mathbb{N}^*$ be a morphism, such that $\Psi(\boldsymbol{v}) = \boldsymbol{v}$. Let Π be a letter-to-letter morphism $\Pi : \mathbb{N}^* \to \mathcal{B}^*$ which satisfies*

$$\Pi \circ \Psi = \Pi \circ \Psi \circ \Pi\,. \tag{23}$$

Then the infinite word $\boldsymbol{u} = \Pi(\boldsymbol{v})$ is invariant under the morphism $\Pi \circ \Psi$.

Proof. We must verify that $(\Pi \circ \Psi)(\boldsymbol{u}) = \boldsymbol{u}$. We write

$$\begin{aligned}
\boldsymbol{u} &= \Pi(\boldsymbol{v}) = \Pi\big(\Psi(\boldsymbol{v})\big) = \big(\Pi \circ \Psi\big)(\boldsymbol{v}) \\
&= \big(\Pi \circ \Psi \circ \Pi\big)(\boldsymbol{v}) = \big(\Pi \circ \Psi\big)\big(\Pi(\boldsymbol{v})\big) = \big(\Pi \circ \Psi\big)(\boldsymbol{u})\,,
\end{aligned}$$

and hence $\boldsymbol{u} = \Pi(\boldsymbol{v})$ is a fixed point of the morphism $\Pi \circ \Psi$. ∎

Note that since the morphism Ψ fixing the infinite word $\boldsymbol{v}_{-\beta}$ coding the set of $(-\beta)$-integers is a power of an antimorphism Φ, it is sufficient to check that

$$\Pi \circ \Phi = \Pi \circ \Phi \circ \Pi \,. \tag{24}$$

For we have

$$\begin{aligned}\Pi\big(\boldsymbol{v}_{-\beta}\big) &= \Pi\big(\Phi(\Phi(\boldsymbol{v}_{-\beta}))\big) = \Pi\big(\Phi(\Pi(\Phi(\boldsymbol{v}_{-\beta})))\big)\\ &= \Pi\big(\Phi(\Pi(\Phi(\Pi(\boldsymbol{v}_{-\beta}))))\big) = (\Pi \circ \Phi)^2\big(\Pi(\boldsymbol{v}_{-\beta})\big)\,.\end{aligned}$$

Thus the word $\boldsymbol{u}_{-\beta} := \Pi\big(\boldsymbol{v}_{-\beta}\big)$ is fixed by the morphism $\psi = \varphi^2 = (\Pi \circ \Phi)^2$, where $\varphi = \Pi \circ \Phi$ is an antimorphism over the restricted alphabet $\mathcal{B}$.

Let us now consider the cases of bases where the $(-\beta)$-shift is sofic, i.e., such that the $(-\beta)$-expansion $\mathrm{d}_{-\beta}(l_\beta)$ is eventually periodic. We suggest to call such numbers *Ito-Sadahiro numbers*.

These are the cases where the distances Δ_k between consecutive $(-\beta)$-integers take only finitely many values (cf. Corollary 19 and Theorem 21) and thus the set $\mathbb{Z}_{-\beta}$ can be coded by a bidirectional infinite word $\boldsymbol{u}_{-\beta}$ over a finite alphabet $\mathcal{B} \subset \mathbb{N}$. Corollary 19 and Theorem 21 also suggest a suitable projection of the infinite alphabet $\mathbb{N}$ to the restricted alphabet $\mathcal{B}$. By verifying condition (24), one can show that $\boldsymbol{u}_{-\beta}$ is invariant under a primitive morphism. By doing so for eventually periodic $\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 \dots d_m (d_{m+1} \cdots d_{m+p})^\omega$, one finds that a prescription in terms of coefficients d_i cannot be written in one formula. Rather it differs dependently on whether the length of period is shorter or longer, even or odd. Similarly, it is the case for finite $\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 \dots d_m 0^\omega$. The discussion is tedious, that is why we show the procedure on only two classes of numbers β together with the corresponding primitive morphism fixing the word $\boldsymbol{u}_{-\beta}$.

Example 25. Let $\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 \dots d_m (d_{m+1})^\omega$, where $m \in \mathbb{N}$ is minimal possible, i.e., $d_m \neq d_{m+1}$. Assume that for parameters d_i the assumption of Theorem 23 is satisfied. According to Theorem 23, we have for m even

$$\begin{aligned}\Phi(0) &= 0^{d_1-1}1\,,\\ \Phi(1) &= 0^{d_1-1}20^{d_1-d_2-1}1\,,\\ \Phi(2) &= 0^{d_3-1}30^{d_1-d_2-1}1\,,\\ \Phi(3) &= 0^{d_3-1}40^{d_1-d_4-1}1\,,\\ &\;\vdots\\ \Phi(m) &= 0^{d_{m+1}-1}(m+1)0^{d_1-d_m-1}1\,,\\ \Phi(k) &= 0^{d_{m+1}-1}(k+1)0^{d_1-d_{m+1}-1}1 \qquad \text{for all } k \geqslant m+1,\end{aligned}$$

and for m odd

$$\begin{aligned}
\Phi(0) &= 0^{d_1-1}1\,,\\
\Phi(1) &= 0^{d_1-1}20^{d_1-d_2-1}1\,,\\
\Phi(2) &= 0^{d_3-1}30^{d_1-d_2-1}1\,,\\
\Phi(3) &= 0^{d_3-1}40^{d_1-d_4-1}1\,,\\
&\ \vdots\\
\Phi(m) &= 0^{d_m-1}(m+1)0^{d_1-d_{m+1}-1}1\,,\\
\Phi(k) &= 0^{d_{m+1}-1}(k+1)0^{d_1-d_{m+1}-1}1 \qquad \text{for all } k \geqslant m+1.
\end{aligned}$$

For all $m \in \mathbb{N}$, consider the restricted alphabet $\mathcal{B} = \{0,1,2,\ldots,m\}$ and the projection $\Pi : \mathbb{N} \to \mathcal{B}$ defined by

$$\Pi(k) = k \quad \text{for } k = 0,1,\ldots,m \qquad \text{and} \qquad \Pi(k) = 0 \quad \text{for } k \geqslant m+1.$$

It can be verified easily that our choice of Π guarantees validity of condition (24). The antimorphism $\varphi = \Pi \circ \Phi$ associated with β has the form

$$\begin{aligned}
\varphi(0) &= 0^{d_1-1}1\,,\\
\varphi(1) &= 0^{d_1-1}20^{d_1-d_2-1}1\,,\\
\varphi(2) &= 0^{d_3-1}30^{d_1-d_2-1}1\,,\\
\varphi(3) &= 0^{d_3-1}40^{d_1-d_4-1}1\,,\\
&\ \vdots\\
\varphi(m) &= 0^{D+d_1-1}1\,,
\end{aligned}$$

where $D = (-1)^m(d_{m+1} - d_m)$.

According to Corollary 19, distances Δ_k for $k = m+1, m+2, m+3, \ldots$ coincide with $\Delta_0 = 1$. Thus the letters projected by Π to the letter 0 code the same distance between consecutive $(-\beta)$-integers.

Example 26. Let $\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 \cdots d_m 0^\omega$, with $d_m \neq 0$, m even, and assume that $0 < d_i$ and $d_1 > d_{2i}$ for all $i = 1,2,3,4,\ldots,m$. One finds the prescription for the antimorphism Φ from (22). Then one considers the restricted alphabet $\mathcal{B} = \{0,1,2,\ldots,m\}$ and projection $\Pi : \mathbb{N} \to \mathcal{B}$

$$\begin{aligned}
\Pi(k) &= k \quad \text{for } k = 0,1,\ldots,m,\\
\Pi(m+1) &= \begin{cases} 0 & \text{if } d_m < d_1 - 1\,,\\ 1 & \text{if } d_m = d_1 - 1\,,\end{cases}\\
\Pi(k) &= \begin{cases} 0 & \text{for odd } k \geqslant m+2,\\ 1 & \text{for even } k \geqslant m+2.\end{cases}
\end{aligned}$$

Such projection satisfies (24) and yields the antimorphism

$$\begin{aligned}
\varphi(0) &= 0^{d_1-1}1\,,\\
\varphi(1) &= 0^{d_1-1}20^{d_1-d_2-1}1\,,\\
\varphi(2) &= 0^{d_3-1}30^{d_1-d_2-1}1\,,\\
\varphi(3) &= 0^{d_3-1}40^{d_1-d_4-1}1\,,\\
&\vdots\\
\varphi(m) &= 0^{d_1-d_m-1}1.
\end{aligned}$$

Similarly as shown in Examples 25 and 26, one can find for every β with eventually periodic expansion $\mathrm{d}_{-\beta}(l_\beta)$, a bidirectional infinite word $\boldsymbol{u}_{-\beta}$ over a finite alphabet, and a primitive morphism fixing $\boldsymbol{u}_{-\beta}$. The word $\boldsymbol{u}_{-\beta}$ is found by projection of $\boldsymbol{v}_{-\beta}$ over the alphabet $\mathbb{N}$. Another approach for finding $\boldsymbol{u}_{-\beta}$ and the corresponding morphism is used by Steiner in [12]. His method uses the notion of the first return map.

7. $(-\beta)$-integers for two cubic bases and their Rauzy fractals

In this last section we will demonstrate the results for $\mathbb{Z}_{-\beta}$ on two particular cubic Pisot numbers β. The first one is the well-known Tribonacci number β, i.e., the real root of x^3-x^2-x-1. Note that such β does not fulfill the assumptions of either Theorem 18 or 21, and thus an additional inspection of the distances in $\mathbb{Z}_{-\beta}$ is needed. Nevertheless this case is very interesting since it bears a striking similarity to the classical case as demonstrated in the sets $\mathbb{Z}_{-\beta}$ and $\mathbb{Z}_\beta$ as well as on associated Rauzy fractals.

The second case discussed is the cubic number $\beta>1$, root of x^3-2x^2-x+1. Since $\beta = 1+2\cos\frac{2\pi}{7}$, it appears naturally in mathematical models of quasicrystals with 7-fold symmetry [3]. This β is an example of a number covered by Theorem 21, and, moreover, it demonstrates that the similarity of β- and $(-\beta)$-numeration observed in the Tribonacci case is not ubiquitous.

It is then natural to ask: what are the numbers β such that the Rauzy fractals defined by β and $-\beta$ are similar, like in the Tribonacci case? It is evident that the first step in identifying such bases is to decide when the distances between consecutive $(-\beta)$-integers are bounded by 1. This question can be perhaps approached using methods of article [13], which is focused on the study of tile-lengths of $(-\beta)$-integers.

Since we compare the distances between consecutive β-integers with distances between consecutive $(-\beta)$-integers, we differentiate between them by the notation Δ_k^+ and Δ_k^-, respectively.

7.1. Tribonacci case

Let β be the Tribonacci number, i.e., the real root $\beta>1$ of x^3-x^2-x-1. Then $\mathrm{d}_\beta(1) = 1110^\omega$ and according to (2) the distances between consecutive β-integers are

$$\Delta_0^+ = 1\,, \qquad \Delta_1^+ = \beta - 1\,, \qquad \text{and} \qquad \Delta_2^+ = \tfrac{1}{\beta}\,.$$

The infinite word $\boldsymbol{u}_\beta$ coding the set of β-integers is invariant under the canonical substitution

$$\varphi_\beta: \quad 0 \mapsto 01\,, \quad 1 \mapsto 02\,, \quad 2 \mapsto 0\,.$$

Let us inspect the set of $(-\beta)$-integers. According to Section 2.2 we have

- $\mathrm{d}_{-\beta}(l_\beta) = 101^\omega$, admissible digits are $0, 1$,
- since $\mathrm{d}_{-\beta}(l_\beta)$ is not purely periodic, we have $\mathrm{d}^*_{-\beta}(r_\beta) = 0101^\omega$,
- $101^\omega \preceq_{\text{alt}} x_i x_{i+1} x_{i+2} \prec_{\text{alt}} 0101^\omega$ for $i = 1, 2, 3, \ldots$ holds for all $(-\beta)$-admissible digit strings.

By Proposition 16 one can find the distances in the set $\mathbb{Z}_{-\beta}$ by evaluating expressions

$$\big|(-\beta)^l + \gamma\big(\min(l)\big) - \gamma\big(\max(l)\big)\big| \qquad \text{for } l = 0, 1, 2, \ldots.$$

From the admissibility rule we get the following $\min(l)$ and $\max(l)$: $\min(0) = \epsilon$, $\min(1) = 1$, $\max(0) = \epsilon$ and for every $k \geqslant 1$ we have

$$\begin{aligned} \min(2k) &= 10(11)^{k-1}\,, \\ \min(2k+1) &= 10(11)^{k-1}0\,, \end{aligned} \qquad \text{and} \qquad \max(k) = 0\min(k-1)\,.$$

Having these extremal strings, it can be shown that the distances between adjacent $(-\beta)$-integers take values

$$\Delta_0^- = 1\,, \qquad \Delta_1^- = \beta - 1\,, \qquad \text{and} \qquad \Delta_2^- = \tfrac{1}{\beta}\,.$$

Therefore in this case the distances between consecutive elements in $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ coincide. In order to obtain the morphism under which the infinite word $\boldsymbol{u}_{-\beta}$ is invariant, we use Theorem 22, equation (22) and the projection

$$\Pi(k) = k \quad \text{for } k = 0, 1, 2, \qquad \text{and} \qquad \Pi(k) = 0 \quad \text{for } k \geqslant 3,$$

since for such k, $\Delta_k^- = \Delta_0^-$. By this, we obtain the antimorphism

$$\varphi: \quad 0 \mapsto 01\,, \quad 1 \mapsto 02\,, \quad 2 \mapsto 0\,.$$

The similarity between the sets $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ can be also observed on their Rauzy fractals in Figures 5 and 6. The pictures represent the set

$$\mathbb{Z}'_\beta = \{z' \mid z \in \mathbb{Z}_\beta\}\,, \qquad \text{respectively} \qquad \mathbb{Z}'_{-\beta} = \{z' \mid z \in \mathbb{Z}_{-\beta}\}\,,$$

where z' is the field conjugate of $z \in \mathbb{Q}(\beta)$, i.e.,

$$\text{if} \quad z = a + b\beta + c\beta^2\,,\ a, b, c \in \mathbb{Q}\,, \quad \text{then } z' = a + b\beta' + c{\beta'}^2\,,$$

where β' is the conjugate of β. Since β' is a complex number, $|\beta'| < 1$, the sets $\mathbb{Z}'_\beta$, $\mathbb{Z}'_{-\beta}$ are bounded in the complex plane. The closures of these sets are the so-called Rauzy fractals.

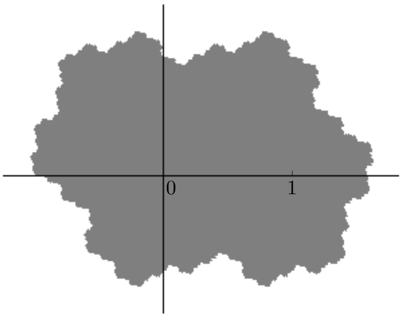

Figure 5. Rauzy fractal for $\mathbb{Z}_\beta$, β Tribonacci number.

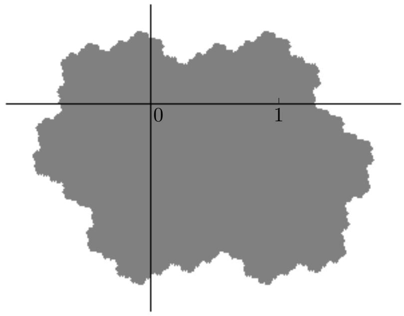

Figure 6. Rauzy fractal for $\mathbb{Z}_{-\beta}$, β Tribonacci number.

Apparently, the two Rauzy fractals coincide up to a translation in the complex plane. The reason for this is the following. The morphism φ^2 under which $\boldsymbol{u}_{-\beta}$ is invariant and the morphism φ_β^2 fixing $\boldsymbol{u}_\beta$ are conjugated. Indeed,

$$\varphi^2 : \begin{array}{l} 0 \mapsto 0201 \\ 1 \mapsto 001 \\ 2 \mapsto 01 \end{array} \qquad \varphi_\beta^2 : \begin{array}{l} 0 \mapsto 0102 \\ 1 \mapsto 010 \\ 2 \mapsto 01 \end{array}$$

and obviously

$$01\varphi^2(a) = \varphi_\beta^2(a)01 \qquad \text{for} \quad a \in \{0,1,2\}.$$

7.2. β root of $x^3 - 2x^2 - x + 1$

Let $\beta > 1$ be the root of $x^3 - 2x^2 - x + 1$. Then $\mathrm{d}_\beta(1) = 2(01)^\omega$ and the distances between consecutive β-integers are

$$\Delta_0^+ = 1\,, \qquad \Delta_1^+ = \beta - 2\,, \qquad \text{and} \qquad \Delta_2^+ = 1 - \frac{1}{\beta}\,,$$

and the infinite word $\boldsymbol{u}_\beta$ is fixed by the canonical substitution

$$\varphi_\beta : \quad 0 \mapsto 001\,, \quad 1 \mapsto 2\,, \quad 2 \mapsto 01\,.$$

The properties of $(-\beta)$-integers are simple to derive, since $\mathrm{d}_{-\beta}(l_\beta) = 210^\omega$ and thus β fulfills the assumptions of Theorem 21 and Example 26. The distances between adjacent $(-\beta)$-integers are

$$\Delta_0^- = 1\,, \qquad \Delta_1^- = \beta - 1\,, \qquad \text{and} \qquad \Delta_2^- = 1 - \frac{1}{\beta}\,.$$

Clearly, the distances between consecutive elements in $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ are different. Indeed, $\Delta_1^- = \Delta_1^+ + 1 > 1$ which cannot happen in the case of Rényi expansions. The infinite word $\boldsymbol{u}_{-\beta}$ is invariant under the morphism

$$\varphi^2 : \quad 0 \mapsto 02101\,, \quad 1 \mapsto 021101\,, \quad 2 \mapsto 021\,.$$

The dissimilarity of $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ can be also observed in their Rauzy fractals in Figures 7 and 8. The base β has now two real conjugates, $\beta' \simeq -0.8019$ and $\beta'' \simeq 0.5550$. We consider the sets

$$\{(x', x'') \mid x \in \mathbb{Z}_\beta\}, \qquad \text{respectively} \qquad \{(x', x'') \mid x \in \mathbb{Z}_{-\beta}\},$$

where x', x'' are images of $x \in \mathbb{Q}(\beta)$ under the isomorphisms

$$x = a + b\beta + c\beta^2 \ \mapsto \begin{cases} x' = a + b\beta' + c\beta'^2, \\ x'' = a + b\beta'' + c\beta''^2, \end{cases}$$

where $a, b, c \in \mathbb{Q}$.

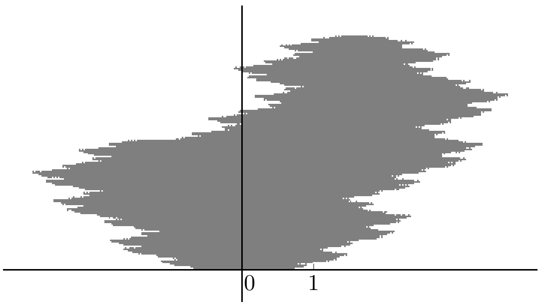

Figure 7. Rauzy fractal for $\mathbb{Z}_\beta$, β root of $x^3 = 2x^2 + x - 1$.

Figure 8. Rauzy fractal for $\mathbb{Z}_{-\beta}$, β root of $x^3 = 2x^2 + x - 1$.

References

[1] S. Akiyama, *Pisot number system and its dual tiling*, In "Physics and Theoretical Computer Science", ed. by J.P. Gazeau et al., IOS Press (2007), 133–154.

[2] P. Arnoux, S. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Belg. Math. Soc. Simon Stevin **8** (2001), 181–207.

[3] Č. Burdík, Ch. Frougny, J.P. Gazeau, R. Krejcar, *Beta-Integers as Natural Counting Systems for Quasicrystals*, J. Phys. A: Math. Gen. **31** (1998), 6449–6472.

[4] S. Fabre, *Substitutions et β-systèmes de numération*, Theoret. Comput. Sci. **137** (1995), 219–236.

[5] Ch. Frougny and A.C. Lai. *On negative bases*, In 'Proceedings of DLT 09', Lectures Notes in Computer Science **5583** (2009), 252–263.

[6] Ch. Frougny and B. Solomyak, *Finite β-expansions*, Ergodic Theory Dynam. Systems **12** (1994), 713–723.

[7] S. Ito and T. Sadahiro, *$(-\beta)$-expansions of real numbers*, Integers **9** (2009), 239–259.

[8] C. Kalle, W. Steiner, *Beta-expansions, natural extensions and multiple tilings associated with Pisot units*, to appear in Trans. Amer. Math. Soc., (2010).

[9] Z. Masáková, E. Pelantová, T. Vávra, *Arithmetics in number systems with a negative base*, Theor. Comp. Sci. **412** (2011), 835–845.

[10] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hung. **11** (1960), 401–416.

[11] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung. **8** (1957), 477–493.

[12] W. Steiner, *On the structure of $(-\beta)$-integers*, RAIRO – Theoretical Informatics and Applications **46** (2012), 181–200.

[13] W. Steiner, *On the Delone property of $(-\beta)$-integers*, in Proceedings WORDS 2011, EPTCS **63** (2011), 247–256.

[14] W.P. Thurston, *Groups, tilings, and finite state automata*, AMS Colloquium Lecture Notes, American Mathematical Society, Boulder, 1989.

Address: Petr Ambrož, Daniel Dombek, Zuzana Masáková, Edita Pelantová: Department of Mathematics FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Praha 2, Czech Republic.

E-mail: petr.ambroz@fjfi.cvut.cz, D.Dombek@seznam.cz, zuzana.masakova@fjfi.cvut.cz, edita.pelantova@fjfi.cvut.cz

Received: 2 November 2011