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THE MEROMORPHIC CONTINUATION OF THE ZETA FUNCTION OF SIEGEL MODULAR THREEFOLDS OVER TOTALLY REAL FIELDS

Cristian Virdol

Abstract: In this paper we prove the meromorphic continuation of the zeta function of Siegel modular threefolds over arbitrary totally real number fields.

Keywords: Siegel threefolds, totally real fields, meromorphic continuation.

1. Introduction

Let $S_K := S_{G,K}$ be the Siegel modular threefolds associated to $G := \mathrm{GSp}_4$ and to some open compact subgroup K of $G(\mathbb{A}_{\mathbb{Q},f})$, where $\mathbb{A}_{\mathbb{Q},f}$ is the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$. It is well known that S_K is defined over $\mathbb{Q}$ (see [D]).

In this article we prove the meromorphic continuations of the zeta function of $S_{K/F}$, where F is an arbitrary totally real number field. In order to show this result we use the potential modularity for some l-adic representations of the absolute Galois group of totally real number fields (see [BGGT]).

2. Siegel modular threefolds

Let $G := \mathrm{GSp}_4$ be the symplectic similitudes group over $\mathbb{Q}$ of rank 4. Then

$$\mathrm{GSp}_4(A) = \left\{ g \in \mathrm{GL}_4(A) | {}^t g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} g = \mu(g) \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \text{ for some } \mu(g) \in A^\times \right\},$$

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for all $\mathbb{Q}$-algebras A, where I_2 is the identity matrix of rank 2. Let Sp_4 be the symplectic group over $\mathbb{Q}$ of rank 4. Then

$$\mathrm{Sp}_4(A) = \left\{ g \in \mathrm{GL}_4(A) | {}^t g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\},$$

for all $\mathbb{Q}$-algebras A.

Consider the morphism of $\mathbb{R}$-groups

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$$

given by

$$x + iy \to \begin{pmatrix} xI_2 & yI_2 \\ -yI_2 & xI_2 \end{pmatrix}.$$

The stabilizer of h in $G(\mathbb{R})$ is $K_\infty = Z_\infty K_{\mathbb{R}}$, where Z_∞ is the center of $G(\mathbb{R})$, and $K_{\mathbb{R}}$ is a maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$.

For K, a sufficiently small open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},f})$, let S_K be the smooth toroidal compactification of an open surface S_K^0 that satisfies

$$S_K^0(\mathbb{C}) = G(\mathbb{Q}) \setminus G(\mathbb{A}_{\mathbb{Q}})/K_\infty K,$$

which is a disjoint union of arithmetic quotients of the Siegel upper half plane of degree 2 (see [T]). Hence S_K has dimension 3, and is called a Siegel modular threefold. From [D], we know that S_K is defined over $\mathbb{Q}$.

3. Zeta function of Siegel modular threefolds

Let K be a sufficiently small open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},f})$. Then we have a decomposition (see for example §5 of [HLR])

$$H^i_{et}(S_K, \bar{\mathbb{Q}}_l) = IH^i_{et}(\bar{S}_K, \bar{\mathbb{Q}}_l) \oplus H^i(S_K^\infty, \bar{\mathbb{Q}}_l)$$

where $IH^i_{et}(\bar{S}_K, \bar{\mathbb{Q}}_l)$ is the intersection cohomology of the Baily-Borel compactification $\bar{S}_K$ of S_K^0, and S_K^∞ is the divisor at infinity (a finite set of cusps) such that $\bar{S}_K = S_K^0 \cup S_K^\infty$, and is defined by

$$IH^i_{\mathrm{et}}(\bar{S}_K, \bar{\mathbb{Q}}_l) := \mathrm{Im}(H^i_{\mathrm{et}}(S_K, \bar{\mathbb{Q}}_l) \to H^i_{\mathrm{et}}(S_K^0, \bar{\mathbb{Q}}_l)).$$

We remark that $H^i_{et}(S_K, \bar{\mathbb{Q}}_l) = \{0\}$ unless $0 \leqslant i \leqslant 6$.

If l is a prime number, let $\mathbb{H}_K$ be the Hecke algebra generated by the bi-K-invariant $\bar{\mathbb{Q}}_l$-valued compactly supported functions on $G(\mathbb{A}_{\mathbb{Q},f})$ under convolution. If $\Pi = \Pi_f \otimes \Pi_\infty$ is an automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$, we denote by Π_f^K the space of K-invariants in Π_f. The Hecke algebra $\mathbb{H}_K$ acts on Π_f^K.

We have an action of the Hecke algebra $\mathbb{H}_K$ and an action of the Galois group $\Gamma_{\mathbb{Q}}$:=Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$ on the intersection cohomology $IH^i_{et}(\bar{S}_K, \bar{\mathbb{Q}}_l)$ and these two actions commute. An automorphic representation Π of $G(\mathbb{A}_{\mathbb{Q}})$ is called *cohomological* if $H^*(G(\mathbb{R}), K_\infty, \Pi_\infty) \neq 0$.

We know the following result (see [W1]):

Proposition 3.1. *The representation of* $\Gamma_{\mathbb{Q}} \times \mathbb{H}_K$ *on the intersection cohomology* $IH^i_{et}(\bar{S}_K, \bar{\mathbb{Q}}_l)$ *is isomorphic to*

$$\oplus_{\Pi} \phi^i(\Pi_f) \otimes \Pi_f^K,$$

where $\phi^i(\Pi_f)$ *is a continous representation of the Galois group* $\Gamma_{\mathbb{Q}}$*. The above sum is over cohomological automorphic representations* $\Pi = \Pi_f \otimes \Pi_\infty$ *of* $G(\mathbb{A}_{\mathbb{Q}})$ *that occur in the discrete spectrum of* $G(\mathbb{A}_{\mathbb{Q}})$ *and the* $\mathbb{H}_K$*-representations* Π_f^K *are irreducible and mutually inequivalent.*

We fix an isomorphism $\iota : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and define the L-function

$$L^i(s, S_K) := \prod_{\Pi} L(s, \phi^i(\Pi_f))^{\dim \Pi_f^K}, \qquad \mathrm{Re}(s) >> 0,$$

where

$$L(s, \phi^i(\Pi_f)) := \prod_q L_q(s, \phi^i(\Pi_f)),$$

where q runs over all rational primes and if $\phi^i(\Pi_f))$ is unramified at q we have

$$L_q(s, \phi^i(\Pi_f)) := \prod_q \det(1 - Nq^{-s}\iota(\phi^i(\Pi_f)(\mathrm{Frob}_q)))^{-1}.$$

Here Frob_q is a geometric Frobenius element at the rational prime q.

We define

$$L(s, S_K) := \prod_{i=0}^{6} L^i(s, S_K)^{(-1)^i}.$$

4. Meromorphic continuation

In this section we prove the meromorphic continuation of $L(s, S_{K/F})$ where F is an arbitrary totally real number field. From §3 we get that it is sufficient to prove the meromorphic continuation of each $L(s, \phi^i(\Pi_f)|_{\Gamma_F})$. The representation $\phi^i(\Pi_f)$ that appears in Proposition 3.1 has dimension at most 4 (see [W4]), is unramified outside some finite set of primes S which depends on K, is de Rham at l, is crystalline at l if $l \notin S$, and is totally odd. When $i \neq 3$, the representation $\phi^i(\Pi_f)$ is semisimple (see Theorem 1.1 and §1.7 of [W3]), has dimension at most 2. The representation $\phi^3(\Pi_f)$ has dimension 2 or 4 (see for example [T]) and we denote by $\phi^3(\Pi_f)^{ss}$ its semisimplification (see Theorem I and pages 67-70 of [W4] and §3.2 of [SU] for the properties of $\phi^3(\Pi_f)^{ss}$). Then as above (see Theorems 3.1.3 and 3.1.4 of [SU]) $\phi^3(\Pi_f)^{ss}$ is unramified outside some finite set of primes S which depends on K, is de Rham at l, is crystalline at l if $l \notin S$, is totally odd i.e., $\det\phi^3(\Pi_f)(c) = -1$ where c is the complex conjugation, and is essentially self-dual (see [BGGT] for the terminology).

Theorem 4.1. *If F is a totally real number field, then there exists a totally real finite extension F' of F, which is Galois over $\mathbb{Q}$, such that $\phi^i(\Pi_f)^{ss}|_{\Gamma_{F'}}$ is automorphic, i.e., $\phi^i(\Pi_f)^{ss}|_{\Gamma_{F'}} \cong \rho_{\Pi'_i}$, where Π'_i is an automorphic representation of $GL_{n_i}(\mathbb{A}_{F'})$ and $\rho_{\Pi'_i}$ is the l-adic representation associated to Π'_i.*

Proof. We consider two cases:

(A) $i \neq 3$. We distinguish two subcases (see [W1] and [W2]):

(i) The representation $\phi^i(\Pi_f)^{ss}|_{\Gamma_F}$ is a direct sum of one or two 1-dimensional Hecke characters. Theorem 4.1 is obvious in this case, and the base change is actually arbitrary.

(ii) The representation $\phi^i(\Pi_f)^{ss}|_{\Gamma_F}$ is irreducible of dimension 2, has τ-Hodge-Tate weights 0 and 1 for each embedding $\tau : F \hookrightarrow \bar{\mathbb{Q}}$. Hence from Theorem A of [BGGT] (see the properties of $\phi^i(\Pi_f)^{ss}$ above), we conclude the proof of Theorem 4.1 in this case.

(B) $i = 3$. We distinguish six subcases (see Theorems I, II, III and pages 67–70 of [W4] and §3.2 of [SU]):

(i) The representation $\phi^3(\Pi_f)^{ss}|_{\Gamma_F}$ has dimension 2. Then $\phi^3(\Pi_f)^{ss}|_{\Gamma_F}$ is a sum of two Hecke characters, or $\phi^3(\Pi_f)^{ss}|_{\Gamma_F}$ is irreducible and has distinct τ-Hodge-Tate weights for each embedding $\tau : F \hookrightarrow \bar{\mathbb{Q}}$. Hence from Theorem A of [BGGT] (see the properties of $\phi^3(\Pi_f)^{ss}$ above), we conclude the proof of Theorem 4.1 in this case.

(ii) $\phi^3(\Pi_f)^{ss}|_{\Gamma_F} \cong \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$, where χ_1, χ_2, χ_3 and χ_4 are Hecke characters, and thus Theorem 4.1 is obvious in this case, and the base change is actually arbitrary.

(iii) $\phi^3(\Pi_f)^{ss}|_{\Gamma_F} \cong \chi_1 \oplus \chi_2 \oplus \sigma$, where χ_1 and χ_2 are Hecke characters and σ is an l-adic irreducible representation of dimension 2 which is totally odd and has distinct τ-Hodge-Tate weights for each embedding $\tau : F \hookrightarrow \bar{\mathbb{Q}}$. Hence from Theorem A of [BGGT] applied to σ (actually to the weakly compatible system of l-adic representations σ) we conclude the proof of Theorem 4.1 in this case.

(iv) $\phi^3(\Pi_f)^{ss}|_{\Gamma_F} \cong \sigma_1 \oplus \sigma_2$, where σ_1 and σ_2 are l-adic irreducible representations of dimension 2 which are totally odd and each has distinct τ-Hodge-Tate weights for each embedding $\tau : F \hookrightarrow \bar{\mathbb{Q}}$. Then from Theorem 2.1 of [V1] or Theorem 6.1 of [V2] (see their proofs) we conclude the proof of Theorem 4.1 in this case.

(v) $\phi^3(\Pi_f)^{ss}|_{\Gamma_F} \cong \chi \oplus \sigma$, where χ is a Hecke character and σ is an l-adic irreducible representation of dimension 3, which is actually impossible (for details see §3.2, case A), iii) of [SU]).

(vi) $\phi^3(\Pi_f)^{ss}|_{\Gamma_F}$ is irreducible and has τ-Hodge-Tate weights 0, 1, 2 and 3 for each embedding $\tau : F \hookrightarrow \bar{\mathbb{Q}}$. Hence from Theorem A of [BGGT] applied to the weakly compatible system of l-adic representations $\phi^3(\Pi_f)^{ss}|_{\Gamma_F}$ we conclude the proof of Theorem 4.1 in this final subcase. ∎

Let F be a totally real number field. From Theorem 4.1 we deduce that there exists a totally real field F', Galois over F, such that $\phi^i(\Pi_f)^{ss}|_{\Gamma_{F'}} \cong \rho_{\Pi'_i}$, where Π'_i is an automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_{F'})$.

From Theorem 15.10 of [CR] we know that there exist some subfields $F_j \subseteq F'$, such that $\mathrm{Gal}(F'/F_j)$ are solvable, and some integers m_j, such that the trivial representation

$$1_F : \mathrm{Gal}(F'/F) \to \bar{\mathbb{Q}}^\times,$$

can be written as

$$1_F = \sum_{j=1}^{u} m_j \mathrm{Ind}_{\mathrm{Gal}(F'/F_j)}^{\mathrm{Gal}(F'/F)} 1_{F_j}.$$

Then

$$\begin{aligned} L(s, \phi^i(\Pi_f)^{ss}|_{\Gamma_F}) &= \prod_{j=1}^{u} L\left(s, \phi^i(\Pi_f)^{ss}|_{\Gamma_F} \otimes \mathrm{Ind}_{\Gamma_{F_j}}^{\Gamma_F} 1_{F_j}\right)^{m_j} \\ &= \prod_{j=1}^{u} L\left(s, \mathrm{Ind}_{\Gamma_{F_j}}^{\Gamma_F} (\phi^i(\Pi_f)^{ss}|_{\Gamma_{F_j}})\right)^{m_j} \\ &= \prod_{j=1}^{u} L\left(s, \phi^i(\Pi_f)^{ss}|_{\Gamma_{F_j}}\right)^{m_j}. \end{aligned}$$

Since $\phi^i(\Pi_f)^{ss}|_{\Gamma_{F'}}$ is automorphic and $\mathrm{Gal}(F'/F_j)$ is solvable, one can deduce easily that $\phi^i(\Pi_f)|_{\Gamma_{F_j}}$ is automorphic. Hence the function $L(s, \phi^i(\Pi_f)^{ss}|_{\Gamma_F})$ has a meromorphic continuation to the entire complex plane because each function $L(s, \phi^i(\Pi_f)^{ss}|_{\Gamma_{F_j}})$ has a meromorphic continuation to the entire complex plane. Since $L(s, \phi^i(\Pi_f)|_{\Gamma_F})$ is equal to $L(s, \phi^i(\Pi_f)^{ss}|_{\Gamma_F})$ up to finitely many Euler factors (see for example [HLR]) we get that the function $L(s, \phi^i(\Pi_f)|_{\Gamma_F})$ has a meromorphic continuation to the entire complex plane. ■

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Address: Cristian Virdol: Department of Mathematics for Industry, Kyushu University, Japan.

E-mail: virdol@imi.kyushu-u.ac.jp

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