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ON λ-INVARIANTS OF $\mathbb{Z}_\ell$-EXTENSIONS OVER REAL ABELIAN NUMBER FIELDS OF PRIME POWER CONDUCTORS

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Abstract: For each prime number ℓ less than 10^4, we construct an infinite family of abelian number fields for which Iwasawa λ_ℓ-invariants vanish.

Keywords: Iwasawa invariant, computation.

1. Introduction

For a prime number ℓ and an algebraic number field k, we denote by $\mu_\ell(k)$ and $\lambda_\ell(k)$ the Iwasawa μ-invariant and λ-invariant of the cyclotomic $\mathbb{Z}_\ell$-extension of k respectively. Greenberg conjecture, which is still open, predicts that both $\mu_\ell(k)$ and $\lambda_\ell(k)$ vanish for all prime numbers ℓ and all totally real number fields k. In spite of a large amount of papers about Greenberg conjecture, we lack a systematic knowledge about it. For example, there is no known totally real number field k different with the rational number field $\mathbb{Q}$ such that both $\mu_\ell(k)$ and $\lambda_\ell(k)$ vanish for all prime number ℓ. Similarly, there is no known prime number ℓ such that both $\mu_\ell(k)$ and $\lambda_\ell(k)$ vanish for all totally real number fields k. So we are led to consider the following problems:

Problem 1.1. For a fixed prime number ℓ, find an infinite family of totally real number fields k such that $\mu_\ell(k) = \lambda_\ell(k) = 0$.

Problem 1.2. For a fixed totally real number field k, find an infinite family of prime numbers ℓ such that $\mu_\ell(k) = \lambda_\ell(k) = 0$.

First we explain trivial examples. Let $\ell = 2$. It is well known by genus theory that there exist infinitely many real quadratic fields k with odd class number in which the prime 2 is not decomposed. Then a famous theorem of Iwasawa in [10] immediately shows $\mu_2(k) = \lambda_2(k) = 0$ for such k. Conversely, let k be any real

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quadratic field. Then there exist infinitely many prime numbers ℓ which does not divide the class number of k and is not decomposed in k. Iwasawa's theorem again concludes that $\mu_\ell(k) = \lambda_\ell(k) = 0$ for such ℓ.

We are interested in non-trivial examples. Ozaki-Taya [14] constructs explicitly an infinite family of real quadratic fields k with $\mu_2(k) = \lambda_2(k) = 0$ in which 2 splits. They also construct an infinite family of real quadratic fields k with $\mu_2(k) = \lambda_2(k) = 0$ which have even class numbers. Horie-Nakagawa [12] proved that there are infinitely many real quadratic fields k with class number prime to 3 in which 3 is not decomposed. It follows $\mu_3(k) = \lambda_3(k) = 0$ for such k. Ono [13] extended the result of Horie-Nakagawa to prime numbers less than 5000. Namely, for a prime number ℓ less than 5000, he proved with the aid of computer that there are infinitely many real quadratic fields k with class number prime to ℓ in which ℓ is not decomposed. Of course, $\mu_\ell(k) = \lambda_\ell(k) = 0$ for such k.

In this paper, we construct another type of infinite family of number fields k with $\mu_\ell(k) = \lambda_\ell(k) = 0$, which contributes to Problem 1.1. Our targets in this paper are abelian number fields k and it is known that $\mu_\ell(k) = 0$ by Ferrero-Washington [3]. So we omit the statement $\mu_\ell(k) = 0$ in the following. In a similar, but more general situation, Friedman-Sands [5] investigates the stability of λ_ℓ^--invariants, while our attention concentrates in the vanishing of λ_ℓ-invariants. For a prime number p and an integer m, we denote by $\mathbb{B}_{p,m}$ the m-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. The following are our theorems.

Theorem 1.1. *Let ℓ be a prime number less than 10^4. Then the Iwasawa invariant $\lambda_\ell(\mathbb{B}_{2,m})$ vanishes for all $m \geqslant 0$.*

Theorem 1.2. *Let ℓ be a prime number less than 10^4. Then the Iwasawa invariant $\lambda_\ell(\mathbb{B}_{3,m})$ vanishes for all $m \geqslant 0$.*

Remark 1.1. If ℓ satisfies $\ell^2 \not\equiv 1 \pmod{16}$, then there is only one prime ideal of $\mathbb{B}_{2,m}$ lying above ℓ and the class number of $\mathbb{B}_{2,m}$ is prime to ℓ by [8, Proposition 3]. Hence Iwasawa's theorem shows $\lambda_\ell(\mathbb{B}_{2,m}) = 0$ for all $m \geqslant 0$.

If ℓ satisfies $\ell^2 \not\equiv 1 \pmod 9$, then there is only one prime ideal of $\mathbb{B}_{3,m}$ lying above ℓ and the class number of $\mathbb{B}_{3,m}$ is prime to ℓ by [8, Proposition 2]. Hence Iwasawa's theorem again shows $\lambda_\ell(\mathbb{B}_{3,m}) = 0$ for all $m \geqslant 0$.

Remark 1.2. Friedman [4, Theorem. (B)] describes explicitly the behavior of class numbers of intermediate fields of a multiple $\mathbb{Z}_\ell$-extension using λ_i and ν_i. Our theorems asserts that $\lambda_i = 0$ in some special situations.

2. Preliminaries to Proof

We start with explaining notations. For a finite group G, we denote by $|G|$ the order of G. Let k be an algebraic number field. For a finite algebraic extension K of k, we denote by $[K : k]$ the extension degree of K over k. If K is a Galois extension of k, we denote by $G(K/k)$ the Galois group of K over k. For a prime number ℓ, we denote by $A_\ell(k)$ the ℓ-Sylow subgroup of the ideal class group of k.

We denote by $\overline{\mathbb{Q}}_\ell$ the algebraic closure of the ℓ-adic number field $\mathbb{Q}_\ell$ and suppose the multiplicative valuation $|\ \ |_\ell$ of $\overline{\mathbb{Q}}_\ell$ is normalized so that $|\,\ell\,|_\ell = \ell^{-1}$.

For one more prime number p, we denote by $\mathbb{B}_{p,\infty}$ the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, by $\mathbb{B}_{p,m}$ the m-th layer of $\mathbb{B}_{p,\infty}/\mathbb{Q}$ and by $\lambda_\ell(\mathbb{B}_{p,m})$ the Iwasawa λ-invariant of the cyclotomic $\mathbb{Z}_\ell$-extension $\mathbb{B}_{p,m}\mathbb{B}_{\ell,\infty}/\mathbb{B}_{p,m}$ as mentioned above.

Let p and ℓ be distinct prime numbers. We put $A_{m,n} = A_\ell(\mathbb{B}_{p,m}\mathbb{B}_{\ell,n})$ and $\Gamma = G(\mathbb{B}_{p,\infty}\mathbb{B}_{\ell,\infty}/\mathbb{B}_{p,\infty})$. An element of Γ acts on $A_{m,n}$ canonically. We put

$$A^\Gamma_{m,n} = \{a \in A_{m,n} \mid a^\sigma = a \text{ for any element } \sigma \in \Gamma\}.$$

Then we have

$$|A_{m,n}| \leqslant |A_{m',n'}|$$

and

$$|A^\Gamma_{m,n}| \leqslant |A^\Gamma_{m',n'}|$$

for non-negative integers m, n, m', n' with $m \leqslant m'$ and $n \leqslant n'$ by class field theory and genus theory. Since Leopoldt's conjecture holds for $\mathbb{B}_{p,m}$, there exists integer n_ℓ such that $|A^\Gamma_{m,n_\ell}| = |A^\Gamma_{m,n}|$ for any integer n with $n_\ell \leqslant n$ by [7, Proposition 1]. We put $|A^\Gamma_{m,\infty}| = |A^\Gamma_{m,n_\ell}|$.

Let $F_{m,n}$ be the maximal abelian ℓ-extension of $\mathbb{B}_{p,m}$ which is unramified over $\mathbb{B}_{p,m}\mathbb{B}_{\ell,n}$. Then $|A^\Gamma_{m,n}| = (F_{m,n} : \mathbb{B}_{p,m}\mathbb{B}_{\ell,n})$ by genus theory. Let $M^{(\ell)}_m$ be the maximal abelian ℓ-extension of $\mathbb{B}_{p,m}$ unramified outside ℓ. The degree $(M^{(\ell)}_m : \mathbb{B}_{p,m}\mathbb{B}_{\ell,\infty})$ is finite again by the validity of Leopoldt's conjecture for $\mathbb{B}_{p,m}$. More precisely, we have the following lemma which is a direct consequence of applying [2, Lemma 8] to the extension $M^{(\ell)}_m/\mathbb{B}_{p,m}$.

Lemma 2.1. *Let ℓ be an odd prime number with $\ell \neq p$ and $M^{(\ell)}_m$ the maximal abelian ℓ-extension of $\mathbb{B}_{p,m}$ unramified outside ℓ and $R_\ell(\mathbb{B}_{p,m})$ the ℓ-adic regulator of $\mathbb{B}_{p,m}$. Then we have*

$$|G(M^{(\ell)}_m/\mathbb{B}_{p,m}\mathbb{B}_{\ell,\infty})| = \left|\frac{|A_{m,0}|R_\ell(\mathbb{B}_{p,m})}{\ell^{p^m-1}}\right|_\ell^{-1}.$$

Since the degree $[M^{(\ell)}_m : \mathbb{B}_{p,m}\mathbb{B}_{\ell,\infty}]$ is finite, there exists non-negative integer n_0 and an element α in $M^{(\ell)}_m$ such that $M^{(\ell)}_m = \mathbb{B}_{p,m}\mathbb{B}_{\ell,\infty}(\alpha)$ and that

$$[\mathbb{B}_{p,m}\mathbb{B}_{\ell,n_0}(\alpha) : \mathbb{B}_{p,m}\mathbb{B}_{\ell,n_0}] = [M^{(\ell)}_m : \mathbb{B}_{p,m}\mathbb{B}_{\ell,\infty}].$$

Let S be the inertia group of a prime ideal $\mathfrak{L}$ of $\mathbb{B}_{p,m}$ lying above ℓ with respect to the extension $M^{(\ell)}_m/\mathbb{B}_{p,m}$. Let ϕ be the canonical restriction mapping of $G(M^{(\ell)}_m/\mathbb{B}_{p,m})$ to $G(\mathbb{B}_{p,m}\mathbb{B}_{\ell,n_0}(\alpha)/\mathbb{B}_{p,m})$. Since $\phi(S)$ is the inertia group of $\mathfrak{L}$ in $\mathbb{B}_{p,m}\mathbb{B}_{\ell,n_0}(\alpha)$ over $\mathbb{B}_{p,m}$ by [15, p. 395], we have $|A^\Gamma_{m,\infty}| = |A^\Gamma_{m,n_0}|$. The following proposition describes a sufficient condition for Theorems 1.1 and 1.2.

Proposition 2.2. *Assume that there exists a constant* m_p *such that* $|\,A^{\Gamma}_{m_p,\infty}\,| = |\,A^{\Gamma}_{m,\infty}\,|$ *for all* $m \geqslant m_p$. *If* $\lambda_\ell(\mathbb{B}_{p,m_p}) = 0$, *then* $\lambda_\ell(\mathbb{B}_{p,m}) = 0$ *for all* $m \geqslant 0$.

Proof. Since $\lambda_\ell(\mathbb{B}_{p,m_p}) = 0$ means the boundness of $\{|A_{m_p,n}|\}_{n=1}^{\infty}$, there exists non-negative integer n_ℓ such that $|A_{m_p,n_\ell}| = |A_{m_p,n}|$ for any integer n with $n_\ell \leqslant n$ and that $|A^{\Gamma}_{m_p,n_\ell}| = |A^{\Gamma}_{m_p,\infty}|$. Let m, n be integers with $m_p \leqslant m$ and $n_\ell \leqslant n$ and assume that $|A_{m_p,n_\ell}| < |A_{m,n}|$. Since A_{m_p,n_ℓ} is isomorphic to $A_{m_p,n}$ as Γ-module, $A_{m,n}$ is isomorphic to the direct sum $A_{m_p,n_\ell} \oplus (A_{m,n}/A_{m_p,n})$ as Γ-module by [15, Lemma 16.15]. Since $|A_{m_p,n_\ell}| = |A_{m_p,n}| < |A_{m,n}|$, Γ-module $A_{m,n}/A_{m_p,n}$ is non-trivial, which implies $(A_{m,n}/A_{m_p,n})^{\Gamma}$ is also non-trivial. Since $A^{\Gamma}_{m,n}$ is isomorphic to $A^{\Gamma}_{m_p,n_\ell} \oplus (A_{m,n}/A_{m_p,n})^{\Gamma}$, we have $|A^{\Gamma}_{m_p,n_\ell}| < |A^{\Gamma}_{m,n}|$. This contradicts $|A^{\Gamma}_{m_p,\infty}| = |A^{\Gamma}_{m,\infty}|$. Hence $|A_{m_p,n_\ell}| = |A_{m,n}|$ for all m, n with $m_p \leqslant m$ and $n_\ell \leqslant n$, from which we conclude that $\lambda_\ell(\mathbb{B}_{p,m}) = 0$ for all $m \geqslant m_p$. The vanishing of $\lambda_\ell(\mathbb{B}_{p,m})$ for $0 \leqslant m < m_p$ is a well-known property of $\mathbb{Z}_\ell$-extensions. ■

Now we consider primitive Dirichlet characters whose values lie in $\overline{\mathbb{Q}}_\ell$. Let ω be the Teichmüller character modulo ℓ and ψ an even character modulo qp^m whose order is p^m, where q is 4 or p according as $p = 2$ or not. Then a generalized Bernoulli number $B_{1,\omega^{-1}\psi} \in \overline{\mathbb{Q}}_\ell$ is defined by

$$B_{1,\omega^{-1}\psi} = \frac{1}{qp^m\ell} \sum_{b=1}^{qp^m\ell} b\,\omega^{-1}\psi(b).$$

It follows that $|B_{1,\omega^{-1}\psi}|_\ell \leqslant 1$ because $\ell \neq p$. (cf. [1]). Let $L_\ell(s,\psi)$ be an ℓ-adic L-function associated to ψ. Then we see that

$$L_\ell(1,\psi) \equiv L_\ell(0,\psi) = -B_{1,\omega^{-1}\psi} \pmod{\ell} \tag{1}$$

by Theorem 5.11 and Corollary 5.13 in [15].

Then we are able to connect the assumption of Proposition 2.2 to a property of Bernoulli numbers.

Proposition 2.3. *Assume that there is a constant* m_p *such that* $|B_{1,\omega^{-1}\psi}|_\ell^{-1} = 1$ *for all* $m \geqslant m_p + 1$ *and all even characters* ψ *modulo* qp^m *with order* p^m. *Then we have* $|A^{\Gamma}_{m_p,\infty}| = |A^{\Gamma}_{m,\infty}|$ *for all* $m \geqslant m_p$.

Proof. Let m be an integer with $m \geqslant m_p + 1$. Theorem 5.24 in [15] says that

$$\left|\frac{|A_{m,0}|R_\ell(\mathbb{B}_{p,m})}{\ell^{p^m-1}}\right|_\ell^{-1} = \left|\prod_{\psi} L_\ell(1,\psi)\right|_\ell^{-1},$$

where ψ runs over all non-trivial even characters modulo qp^m. Since a character modulo qp^m with order p^k is induced from a character modulo qp^{k+1} with order p^k, we have

$$\frac{|G(M_m^{(\ell)}/\mathbb{B}_{p,m}\mathbb{B}_{\ell,\infty})|}{|G(M_{m_p}^{(\ell)}/\mathbb{B}_{p,m_p}\mathbb{B}_{\ell,\infty})|} = \left|\prod_{\psi} L_\ell(1,\psi)\right|_\ell^{-1} = \left|\prod_{\psi} B_{1,\omega^{-1}\psi}\right|_\ell^{-1} = 1$$

by Lemma 2.1 and (1), where ψ runs over all even characters modulo qp^m with order greater than p^{m_p}. Let $\mathfrak{L}$ be a prime ideal of $\mathbb{B}_{p,m_p}$ lying above ℓ and T the inertia group of $\mathfrak{L}$ in $M_m^{(\ell)}$ over $\mathbb{B}_{p,m_p}$. Let ϕ be the canonical restriction mapping of $G(M_m^{(\ell)}/\mathbb{B}_{p,m})$ onto $G(M_{m_p}^{(\ell)}/\mathbb{B}_{p,m_p})$. Since $\phi(T)$ is the inertia group of $\mathfrak{L}$ in $M_{m_p}^{(\ell)}$ over $\mathbb{B}_{p,m_p}$ by [15, p. 395], we have $|A_{m_p,\infty}^{\Gamma}| = |A_{m,\infty}^{\Gamma}|$. ■

It is proved that a constant m_p in Proposition 2.3 actually exists for a general prime number p (cf. the proof of Theorem 16.12 in [15]). When p is 2 or 3, we are able to give m_p explicitly in the following form, which is a key to proof of theorems. As usual, we denote by $[\,x\,]$ the largest integer not exceeding a real number x.

Proposition 2.4. *Let 2^c be the exact power of 2 dividing $\ell-1$ or ℓ^2-1 according as $\ell \equiv 1 \pmod 4$ or not and put*

$$m_2 = 2c + \left[\frac{1}{2}\log_2(\ell-1)\right] - 2\,.$$

Then we have $|B_{1,\omega^{-1}\psi}|_\ell = 1$ for all $m \geqslant m_2+1$ and for all even characters ψ modulo 2^{m+2} with order 2^m.

Proposition 2.5. *Let 3^c be the exact power of 3 dividing ℓ^2-1 and put*

$$m_3 = 2c + \left[\frac{1}{2}\log_3(\ell-1) + \frac{1}{2}\right] - 1\,.$$

Then we have $|B_{1,\omega^{-1}\psi}|_\ell = 1$ for all $m \geqslant m_3+1$ and for all even characters ψ modulo 3^{m+1} with order 3^m.

We reach Proposition 2.4 by combining [6, Lemma 4.4] and the proof of [6, Proposition 4.7]. The same situation as Proposition 2.5 is treated in [5, p. 1664]. But we follow the argument in [6], which takes a slight different form in the case $p=3$, and give a proof of Proposition 2.5 for completeness and for convenience to readers.

If $2c + [\log_3(\ell-1)] \leqslant m$, then [11, Lemma 1] and the proof of [11, Lemma 2] shows that $|B_{1,\omega^{-1}\psi}|_\ell = 1$. So we assume $m_3+1 \leqslant m \leqslant 2c + [\,\log_3(\ell-1)\,] - 1$ and define a rational function $f_1(T)$ in the rational function field $\mathbb{Q}_\ell(T)$ by

$$f_1(T) = \Bigg(\sum_{\substack{b\equiv 1 \pmod{3^c}\\ 0<b<3^c\ell}} \omega^{-1}(b)T^b\Bigg)(T^{3^c\ell}-1)^{-1}.$$

By specializing the argument in [15, p. 387] to the case $p=3$, we are led to the following fact.

Lemma 2.6. *Suppose that $m \geqslant 2c-1$. If $f_1(\zeta) \not\equiv 0 \pmod{\ell}$ for any primitive 3^{m+1}-th root of unity ζ in $\overline{\mathbb{Q}}_\ell$, then $B_{1,\omega^{-1}\psi} \not\equiv 0 \pmod{\ell}$ for any even character ψ modulo 3^{m+1} with order 3^m.*

We put $g(T) = \sum_{b=0}^{\ell-1} \omega^{-1}(1+3^c b)T^{3^c b}$ and $h(T) = \sum_{b=0}^{\ell-1} \omega^{-1}(1+3^c b)T^b$. Then we have

$$T^{-1}(T^{3^c \ell} - 1)f_1(T) = g(T) = h(T^{3^c}) . \tag{2}$$

Let ζ be a primitive 3^{m+1}-th root of unity in $\overline{\mathbb{Q}}_\ell$ and put $u = m - 2c + 1$, $\theta = \zeta^{3^{u+c}}$, $e = [(\ell-1)/3^u]$ and

$$a_{i,j} = \begin{cases} \omega^{-1}(1 + 3^c(i + 3^u j)) & \text{if } i + 3^u j < \ell, \\ 0 & \text{if } i + 3^u j \geqslant \ell. \end{cases}$$

Then $T^{3^u} - \theta \bmod \ell$ is irreducible over $\mathbb{Z}_\ell[\theta]/\ell\mathbb{Z}_\ell[\theta]$. Since $m \leqslant 2c + [\log_3(\ell-1)] - 1$, we have $u \leqslant [\log_3(\ell-1)]$ and $e \geqslant 1$. We also put $s_i(\theta) = \sum_{j=0}^{e} a_{i,j}\theta^j$ and $r(T) = \sum_{i=0}^{3^u-1} s_i(\theta)T^i$. Then there exists a polynomial $q(T)$ in $\mathbb{Z}_\ell[\theta][T]$ such that

$$h(T) = (T^{3^u} - \theta)q(T) + r(T). \tag{3}$$

We prepare one more auxiliary lemma.

Lemma 2.7. *Let*

$$R = \begin{pmatrix} \overline{a}_{0,0} & \cdots & \overline{a}_{0,e} \\ \overline{a}_{1,0} & \cdots & \overline{a}_{1,e} \\ \vdots & \ddots & \vdots \\ \overline{a}_{3^u-1,0} & \cdots & \overline{a}_{3^u-1,e} \end{pmatrix}$$

be a matrix of size $3^u \times (e+1)$ with $\overline{a}_{i,j} = a_{i,j} + \ell\mathbb{Z}_\ell[\theta]$ in $\mathbb{Z}_\ell[\theta]/\ell\mathbb{Z}_\ell[\theta]$. If $3^u > e$, then the rank of R is greater than or equal to e.

Proof. Note that $a_{i,j} \equiv 1/(1 + 3^c i + 3^{c+u} j) \bmod \ell$ if $a_{i,j} \neq 0$. Remove the last column of R that possibly contains zero entries. Further, remove one row that contains a zero entry and construct the matrix R' of size $(3^u-1) \times e$ or $3^u \times e$. Then the rank of R' is equal to e by [6, Lemma 4.5]. ■

Proof of Proposition 2.5. Let $m_3 + 1 \leqslant m \leqslant 2c + [\log_3(\ell-1)] - 1$ and ζ be a primitive 3^{m+1}-th root of unity in $\overline{\mathbb{Q}}_\ell$. We assume $f_1(\zeta) \equiv 0 \pmod{\ell}$. Then we have $h(\zeta^{3^c}) \equiv 0 \pmod{\ell}$ by (2). Hence we have $r(\zeta^{3^c}) \equiv 0 \pmod{\ell}$ by (3). Since $T^{3^u} - \theta \bmod \ell$ is irreducible over $\mathbb{Z}_\ell[\theta]/\ell\mathbb{Z}_\ell[\theta]$, we have

$$s_i(\theta) \equiv 0 \pmod{\ell} \qquad (0 \leqslant i \leqslant 3^u - 1). \tag{4}$$

From the condition $m_3 + 1 \leqslant m$, it follows that $3^{2u-1} > \ell - 1$, which implies $3^{u-1} > (\ell - 1)/3^u \geqslant e$. Let $\overline{a}_{i,j}$ be the elements in Lemma 2.7 and put $f = \ell - 1 - 3^u e$.

First suppose $f \geqslant 3^{u-1}$, which implies $f > e$. We put

$$R_1 = \begin{pmatrix} \overline{a}_{0,0} & \cdots & \overline{a}_{0,e} \\ \overline{a}_{1,0} & \cdots & \overline{a}_{1,e} \\ \vdots & \ddots & \vdots \\ \overline{a}_{e+1,0} & \cdots & \overline{a}_{e+1,e} \end{pmatrix}.$$

By [6, Lemma 4.5], the rank of R_1 is equal to $e + 1$. Hence we have $\theta \equiv 0 \pmod{\ell}$ by (4), which is a contradiction. Next suppose $f < 3^{u-1}$, which implies $3^u - f > e + 1$. We put

$$R_2 = \begin{pmatrix} \overline{a}_{f,0} & \cdots & \overline{a}_{f,e} \\ \overline{a}_{f+1,0} & \cdots & \overline{a}_{f+1,e} \\ \vdots & \ddots & \vdots \\ \overline{a}_{f+e+1,0} & \cdots & \overline{a}_{f+e+1,e} \end{pmatrix}.$$

From the definition of $a_{i,j}$, we have $\overline{a}_{f+1,e} = \cdots = \overline{a}_{f+e+1,e} = 0$. By Lemma 2.7 and [6, Lemma 4.5], the rank of R_2 is equal to $e + 1$ if $\overline{a}_{f,e} \neq 0$ or e if $\overline{a}_{f,e} = 0$. In both cases, we have $\theta \equiv 0 \pmod{\ell}$ by (4), which is a contradiction. Hence $f_1(\zeta) \not\equiv 0 \pmod{\ell}$ and Lemma 2.6 yield the conclusion. ■

3. Proof of Theorems

We combine Propositions 2.2, 2.3, 2.4 and 2.5 to establish the following theorem, which is a criterion for Theorems 1.1 and 1.2.

Theorem 3.1. *Let p be 2 or 3, ℓ a prime number with $p \neq \ell$ and m_p the integer defined in Propositions 2.4 or 2.5. If $\lambda_\ell(\mathbb{B}_{p,m_p}) = 0$, then $\lambda_\ell(\mathbb{B}_{p,m}) = 0$ for all $m \geqslant 0$.*

We show with the aid of computer that $\lambda_\ell(\mathbb{B}_{2,m_2}) = 0$ and $\lambda_\ell(\mathbb{B}_{3,m_3}) = 0$ for all $\ell < 10^4$. Theorems 1.1 and 1.2 follow from these computational results. We explain briefly computational procedures to show $\lambda_\ell(\mathbb{B}_{p,m_p}) = 0$

We first note that $\lambda_2(\mathbb{B}_{2,m}) = \lambda_3(\mathbb{B}_{3,m}) = 0$ for all $m \geqslant 0$, which is a direct consequence of the fact $\lambda_\ell(\mathbb{Q}) = 0$ for all prime number ℓ. Next, we exclude ℓ which satisfies $\ell^2 \not\equiv 1 \pmod{16}$ if $p = 2$ and $\ell^2 \not\equiv 1 \pmod{9}$ if $p = 3$ by Remark 1.1. For the remaining ℓ, we apply the technique in Ichimura-Sumida [9].

Let $\Delta_m = G(\mathbb{B}_{p,m}/\mathbb{Q})$ and ψ be a character of Δ_m with values in $\overline{\mathbb{Q}}_\ell$, namely a character modulo qp^m. Then an idempotent $e_\psi \in \mathbb{Z}_\ell[\Delta_m]$ is defined by

$$e_\psi = \frac{1}{|\Delta_m|} \sum_{\sigma \in \Delta_m} \mathrm{Tr}(\psi(\sigma))\sigma^{-1}$$

and $\lambda_\ell(\mathbb{B}_{p,m})$ is decomposed as

$$\lambda_\ell(\mathbb{B}_{p,m}) = \sum_{\psi} \lambda_{\ell,\psi}(\mathbb{B}_{p,m}),$$

where Tr is the trace map from $\mathbb{Q}_\ell(\psi(\Delta_m))$ to $\mathbb{Q}_\ell$ and ψ runs over all representatives of $\mathbb{Q}_\ell$-conjugacy classes of characters of Δ_m. Since Δ_m is canonically isomorphic to $G(\mathbb{B}_{p,m}\mathbb{B}_{\ell,\infty}/\mathbb{B}_{\ell,\infty})$, Δ_m acts on $A_{m,n} = A_\ell(\mathbb{B}_{p,m}\mathbb{B}_{\ell,n})$ canonically and $\lambda_{\ell,\psi}(\mathbb{B}_{p,m})$ is defined as an integer satisfying

$$|\, e_\psi(A_{m,n}) \,| = \lambda_{\ell,\psi}(\mathbb{B}_{p,m})n + \nu \qquad (n >> 0)$$

with a constant integer ν. Now we summarize a condition for $\lambda_\ell(\mathbb{B}_{p,m_p}) = 0$ as the following lemma.

Lemma 3.2. *We have $\lambda_\ell(\mathbb{B}_{p,m_p}) = 0$ if and only if $\lambda_{\ell,\psi}(\mathbb{B}_{p,m}) = 0$ for all integer m with $1 \leqslant m \leqslant m_p$ and for all representatives ψ of $\mathbb{Q}_\ell$-conjugacy classes of characters of Δ_m with order p^m.*

We are in the situation (A) or (C) in [9] and note that $|B_{1,\omega^{-1}\psi}|_\ell = 1$ implies $\lambda_{\ell,\psi}(\mathbb{B}_{p,m}) = 0$ (cf. (1), (2) and (3) in [9]). It is easy to calculate $B_{1,\omega^{-1}\psi}$. Our calculation shows that there are seven pairs (ℓ,ψ) in the case $p = 2$ and four pairs (ℓ,ψ) in the case $p = 3$ which does not satisfy $|B_{1,\omega^{-1}\psi}|_\ell = 1$ in the range $\ell < 10^4$. For all these (ℓ,ψ), p^m divides $\ell - 1$, namely the condition (C1) in [9] holds. We applied Ichimura-Sumida criterion for these eleven pairs and verified that the condition $(H_{P_i,n}) = (H_{i,n})$ in [9] holds for $n = 2$, namely $\lambda_{\ell,\psi}(\mathbb{B}_{p,m}) = 0$. We show numerical data for these (ℓ,ψ). Readers should replace χ in [9] with ψ in our notation.

In this section, we write $\zeta_k = \exp(2\pi\sqrt{-1}/k)$. Let σ be the generator of Δ_m induced by $\zeta_{2^{n+2}} \mapsto \zeta_{2^{n+2}}^5$ or $\zeta_{3^{n+1}} \mapsto \zeta_{3^{n+1}}^4$ according as $p = 2$ or $p = 3$. Let g_ℓ be the minimal primitive root of ℓ and η_m the primitive p^m-th root of unity in $\mathbb{Q}_\ell$ satisfying

$$\eta_m \equiv g_\ell^{\frac{\ell-1}{p^m}} \pmod{\ell}.$$

We denote by ψ_m the character of Δ_m satisfying $\psi_m(\sigma) = \eta_m$. We show numerical data about $\psi = \psi_m^k$ with $|B_{1,\omega^{-1}\psi}|_\ell \neq 1$ in the following tables in which $P_\psi(T)$ denotes the Iwasawa polynomial associated with ψ and ℓ^* means the prime number ℓ in [9, Corollary 2]. The program written by TC running on two computers with 64 bit Xeon processor have done the calculations in a month.

Table 1: $p = 2$

ℓ	ψ	Case	$P_\psi(T)$ mod ℓ^2	ℓ^*
31	ψ_1	(C)	$T + 186$	1429969
193	ψ_6^{25}	(A)	$T + 33389$	5521195777
257	ψ_7^{97}	(A)	$T + 12593$	52145949697
521	ψ_3	(A)	$T + 204753$	18101857409
641	ψ_7^{17}	(A)	$T + 223068$	1213630714369
3617	ψ_5^{23}	(A)	$T + 11965036$	60569710224641
4513	ψ_5^{17}	(A)	$T + 15930890$	235307606264321

Table 2: $p = 2$

ℓ	ψ	Case	$P_\psi(T)$ mod ℓ^2	ℓ^*
73	ψ_1	(C)	$T + 2263$	56018449
109	ψ_3^{14}	(A)	$T + 2289$	1888152283
487	ψ_4^{61}	(C)	$T + 39934$	280668166291
1621	ψ_4^{55}	(A)	$T + 2207802$	16560570765169

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