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ZEROS OF THE DERIVATIVES OF THE RIEMANN ZETA-FUNCTION

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Abstract: Levinson and Montgomery in 1974 proved many interesting formulae on the zeros of derivatives of the Riemann zeta function $\zeta(s)$. When Conrey proved that at least $2/5$ of the zeros of the Riemann zeta function are on the critical line, he proved the asymptotic formula for the mean square of $\zeta(s)$ multiplied by a mollifier of length $T^{4/7}$ near the $1/2$-line. As a consequence of their papers, we study some aspects of zeros of the derivatives of the Riemann zeta function with no assumption.

Keywords: zeros, derivatives, Riemann zeta function.

1. Introduction

We study properties of zeros of the derivatives of the Riemann zeta function $\zeta(s)$. Levinson and Montgomery [8] achieved several important theorems for the behavior of zeros of $\zeta^{(m)}(s)$ $(m = 1, 2, 3, \cdots)$. If we assume the Riemann hypothesis, $\zeta'(s)$ has no non-real zero in $\operatorname{Re} s < \frac{1}{2}$ and $\zeta^{(m)}(s)$ $(m > 1)$ has at most finitely many zeros in $\operatorname{Re} s < \frac{1}{2}$. Unconditionally, we are able to deduce the following quantitative results by similar methods in [8].

Theorem 1. *We denote $\rho^{(m)} = \beta^{(m)} + i\gamma^{(m)}$ as zeros of $\zeta^{(m)}(s)$. Let $0 < U \leqslant T$. Then, we have*

$$\sum_{\substack{T<\gamma^{(m)}<T+U \\ \beta^{(m)}<\frac{1}{2}}} \left(\frac{1}{2} - \beta^{(m)}\right) \leqslant \sum_{\substack{T<\gamma<T+U \\ \beta>\frac{1}{2}}} \left(\beta - \frac{1}{2}\right) + O(U).$$

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Theorem 2. *Let $T^a \leqslant U \leqslant T$, $a > \frac{1}{2}$. Then, we have*

$$\sum_{\substack{T<\gamma^{(m)}<T+U \\ \beta^{(m)}<\frac{1}{2}}} \left(\frac{1}{2} - \beta^{(m)}\right) = O(U).$$

Theorem 3. *For $T^a \leqslant U \leqslant T$, $a > \frac{1}{2}$, we have*

$$2\pi \sum_{\substack{T<\gamma^{(m)}<T+U \\ \beta^{(m)}>\frac{1}{2}}} \left(\beta^{(m)} - \frac{1}{2}\right) = mU \log\log T + O(U).$$

We note that Theorems 1–3 complement Theorems 3, 4 in [8]

J. B. Conrey proved that at least $2/5$ of the zeros of the Riemann zeta function are simple and on the critical line in [2]. He refined the method of Levinson [7] and used a result of Deshouillers and Iwaniec [4] on averages of Kloosterman sums to obtain the mean square of the Riemann zeta function accompanied with a mollifier of length $T^{4/7}$. The main theorem of Conrey is following:

Theorem A (Conrey). *Let $B(s) = \sum_{k\leqslant y} \frac{b(k)}{k^{s+R/L}}$ be a mollifier of length $y = T^\theta$, where $b(k) = \mu(k)P\left(\frac{\log y/k}{\log y}\right)$, $P(x)$ is a polynomial with $P(0) = 0$, $P(1) = 1$, $0 < R \ll 1$, $L = \log T$, $0 < \theta < \frac{4}{7}$. Let $V(s) = Q\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s)$ for some polynomial $Q(x)$. Then, we have*

$$\int_2^T \left|VB\left(\frac{1}{2} - \frac{R}{L} + it\right)\right|^2 dt \sim c(P,Q,R)T \qquad (T \to \infty),$$

where

$$c(P,Q,R) = |Q(0)|^2 + \frac{1}{\theta}\int_0^1\int_0^1 e^{2Ry}|Q(y)P'(x) + \theta Q'(y)P(x) + \theta RQ(y)P(x)|^2 dxdy.$$

Based on Theorem A, we are able to deduce interesting results about zeros of $\zeta^{(m)}(s)$.

Theorem 4. *Let $m \geqslant 1$, $\epsilon > 0$. Then we have*

$$\sum_{\substack{\beta^{(m)}>\frac{1}{2} \\ 0<\gamma^{(m)}<T}} 1 \geqslant \mu_m \frac{T\log T}{2\pi}(1 + o_m(1)) \qquad (T \to \infty),$$

where $\rho^{(m)} = \beta^{(m)} + i\gamma^{(m)}$ are zeros of $\zeta^{(m)}(s)$. The coefficient μ_m satisfies $\mu_m \geqslant 1 - \epsilon + O_\epsilon(m^{-1})$ as $m \to \infty$.

It is expected that all the zeros of the Riemann zeta function are simple. (See [3] for a reference.) A related conjecture is that $N_d(T) = N(T)$ for any $T > 0$ where $N(T)$ is the number of zeros $\rho = \beta + i\gamma$ in $0 < \gamma \leqslant T$ with multiplicity, and $N_d(T)$ is the number of distinct zeros in $0 < \gamma \leqslant T$. Regarding this matter, we have the following result.

Theorem 5.

$$\kappa_d = \liminf_{T\to\infty} \frac{N_d(T)}{N(T)} > 0.70.$$

We note that this improves D. W. Farmer's result $\kappa_d \geqslant 0.63952$ in [5].

2. Lemmas

We start with the following.

Lemma 1. *Let* $m = 1, 2, 3, \ldots$, $\chi(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)$ *and* $s = \sigma + it$ $(\sigma, t \in \mathbb{R})$. *Then, we have*

$$\frac{\chi^{(m)}}{\chi}(s) = (-\log|t|)^m + O(\log^{m-1}|t|)$$

for $|t| \geqslant t_0$ *on any fixed vertical strip* $a \leqslant \sigma \leqslant b$.

Proof of Lemma 1. From the Sterling formula, we have

$$\frac{\Gamma'}{\Gamma}(s) = \log|t| + O(1); \qquad \frac{d^m}{ds^m}\left(\frac{\Gamma'}{\Gamma}(s)\right) = O(t^{-m}) \qquad (m = 1, 2, 3, \ldots).$$

Thus we have

$$\frac{\chi'}{\chi}(s) = -\frac{\Gamma'}{\Gamma}(1-s) + \log 2\pi + \frac{\sin\frac{\pi s}{2}}{\cos\frac{\pi s}{2}} = -\log|t| + O(1).$$

Suppose Lemma 1 is true for $m \leqslant k$. Then, we have

$$\begin{aligned}\frac{\chi^{(k+1)}}{\chi}(s) = &\left(\frac{\chi^{(k)}}{\chi}(s)\right)' + \frac{\chi^{(k)}}{\chi}(s)\frac{\chi'}{\chi}(s)\\ =&O(\log^k|t|) + ((-\log|t|)^k + O(\log^{k-1}|t|))(-\log|t| + O(1))\\ =&(-\log|t|)^{k+1} + O(\log^k|t|).\end{aligned}$$

By induction, we have proved the lemma. ■

Lemma 2. *Fix a nonnegative integer* m. *There is* $t_1 > 0$ *such that* $\chi^{(m)}(s)$ *has no zero or pole in* $|\operatorname{Im} s| \geqslant t_1$, $a \leqslant \operatorname{Re} s \leqslant b$.

Proof of Lemma 2. By the definition of $\chi(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)$, we know that $\chi(s)$ is meromorphic on the whole complex plane with poles at $s = 1, 3, 5, 7, \cdots$, and zeros at $s = 0, -2, -4, \cdots$. Thus, the Lemma 2 is true for $m = 0$. For the case $m > 0$, we use the Lemma 1

$$\chi^{(m)}(s) = \chi(s)\frac{\chi^{(m)}}{\chi}(s) = \chi(s)(-\log^m|t|)\left(1 + O\left(\log^{-1}|t|\right)\right)$$

for $|\operatorname{Im} s| = |t| \geqslant t_0$. Thus, we prove the lemma. ■

Lemma 3. *Let* $-2 \leqslant a_j \leqslant 2$, $b_j, c_j \geqslant 0$, $0 < p < 1$. *Then, we have*

$$\int_{-2}^{2} \Big| \sum_j \frac{c_j}{x - a_j + ib_j} \Big|^p dx \leqslant \frac{8}{1-p} \Big| \sum_j c_j \Big|^p.$$

For Lemma 3, see [8, Lemma 4.1] or [6, Chap. 4].

3. Proofs of Theorems 1, 2, and 3

Proof of Theorem 1. We begin with the functional equation of the Riemann zeta function $\zeta(1-s) = \chi(1-s)\zeta(s)$. By differentiating m times, we have

$$\zeta^{(m)}(1-s) = \chi^{(m)}(1-s)\zeta(s) + \sum_{j=0}^{m-1} \binom{m}{j} (-1)^{m-j} \chi^{(j)}(1-s)\zeta^{(m-j)}(s).$$

Let $J_m(s)$ be

$$J_m(s) = \zeta(s) + \sum_{j=0}^{m-1} \binom{m}{j} (-1)^{m-j} \frac{\chi^{(j)}}{\chi^{(m)}}(1-s)\zeta^{(m-j)}(s). \tag{3.1}$$

We know that there is $A_m > \frac{1}{2}$ such that $\zeta^{(m)}(s)$ has no zero on $\operatorname{Re} s \geqslant A_m$. Consider the rectangle with vertices $\frac{1}{2} + i(T+U), \frac{1}{2} + iT$, $A_m + iT$, $A_m + i(T+U)$. Since $\zeta^{(m)}(1-s) = \chi^{(m)}(1-s)J_m(s)$, all the zeros of $J_m(s)$ in the rectangle are the same as the zeros of $\zeta^{(m)}(1-s)$, and no poles there by Lemma 2. Now we apply the Littlewood Lemma [10, Chap. 9.9] to get

$$\frac{1}{2\pi} \int_T^{T+U} \log \left| \frac{J_m(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)} \right| dt = \sum_{\substack{T<\gamma^{(m)}<T+U \\ \beta^{(m)}<\frac{1}{2}}} \left(\frac{1}{2} - \beta^{(m)} \right) - \sum_{\substack{T<\gamma<T+U \\ \beta>\frac{1}{2}}} \left(\beta - \frac{1}{2} \right) + O(U/\log T) + O(\log T). \tag{3.2}$$

We consider the integral of the above formula. We note that the simple inequality

$$\Big| 1 + \sum_j z_j \Big| \leqslant 1 + \sum_j |z_j| \ll \exp(\sum_j |z_j|^{m_j})$$

holds for any fixed real $m_j > 0$, where the number of terms in the summations is finite. From this inequality and Lemma 1 together with definition of $J_m(s)$, we readily have

$$\int_T^{T+U} \log \left| \frac{J_m(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)} \right| dt \leqslant C_m \frac{1}{\sqrt{\log T}} \sum_{j=1}^{m} \int_T^{T+U} \left| \frac{\zeta^{(j)}}{\zeta} \left(\frac{1}{2} + it \right) \right|^{\frac{1}{2j}} dt \tag{3.3}$$

for some $C_m > 0$. We need still a claim to complete the proof of Theorem 1.

Claim. *For any positive integer j, we have*

$$\int_T^{T+U} \left| \frac{\zeta^{(j)}}{\zeta}\left(\frac{1}{2}+it\right)\right|^{\frac{1}{2j}} dt \ll U\sqrt{\log T}.$$

Proof of Claim. We recall that in [1], the number of zeros of $\zeta^{(k)}(s)$ with $0 < \gamma^{(k)} < T$ is

$$\frac{T}{2\pi}\log\frac{T}{4\pi e} + O_k(\log T). \tag{3.4}$$

Let n be a large positive integer. For $|t-n| \leqslant 1$, $0 < \sigma < 1$, $k = 0, 1, 2, \cdots$, we have

$$\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s) = \sum_{|\gamma^{(k)}-n|<2} \frac{1}{s-\rho^{(k)}} + O(\log n).$$

Then, by this, (3.4) and Lemma 3, we have

$$\begin{aligned}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \left|\frac{\zeta^{(k+1)}}{\zeta^{(k)}}\left(\frac{1}{2}+it\right)\right|^{\frac{1}{2}} dt &\ll \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \left|\sum_{|\gamma^{(k)}-n|<2} \frac{1}{\frac{1}{2}+it-\rho^{(k)}}\right|^{\frac{1}{2}} dt + \sqrt{\log n}\\ &\ll \sqrt{\log n}.\end{aligned}$$

From this, we have

$$\int_T^{T+U} \left|\frac{\zeta^{(k+1)}}{\zeta^{(k)}}\left(\frac{1}{2}+it\right)\right|^{\frac{1}{2}} dt \ll U\sqrt{\log T}.$$

By this and Hölder's inequality, we have

$$\begin{aligned}\int_T^{T+U} \left|\frac{\zeta^{(j)}}{\zeta}\left(\frac{1}{2}+it\right)\right|^{\frac{1}{2j}} dt &= \int_T^{T+U} \left|\left(\frac{\zeta'}{\zeta}\cdot\frac{\zeta''}{\zeta'}\cdots\frac{\zeta^{(j)}}{\zeta^{(j-1)}}\right)\left(\frac{1}{2}+it\right)\right|^{\frac{1}{2j}} dt\\ &\ll U\sqrt{\log T}.\end{aligned}$$

We complete the proof of Claim. ■

By Claim, (3.2) and (3.3), Theorem 1 follows immediately. ■

Proof of Theorem 2. By Selberg [9], if $U \geqslant T^a$, $a > \frac{1}{2}$, then

$$\sum_{\substack{T<\gamma<T+U\\ \beta<\frac{1}{2}}} \left(\frac{1}{2}-\beta\right) = O(U).$$

From this and Theorem 1, Theorem follows. ■

Proof of Theorem 3. We know in [8, Theorem 3] that for $0 < U < T$, we have

$$2\pi \sum_{T \leqslant \gamma^{(m)} \leqslant T+U} \left(\beta^{(m)} - \frac{1}{2}\right) = mU \log\log \frac{T}{2\pi} + U\left(\frac{1}{2}\log 2 - m \log\log 2\right) + O(U^2/(T\log T) + \log T).$$

By this and Theorem 2, we complete the proof of Theorem 3. ■

4. Proofs of Theorems 4 and 5

Proof of Theorem 4. Apply the Littlewood Lemma to deduce

$$\frac{1}{2\pi}\int_2^T \log\left|J_k B\left(\frac{1}{2} - \frac{R}{L} + it\right)\right| dt = \sum_{\substack{\beta^{(k)} < \frac{1}{2} + \frac{R}{L} \\ 0 < \gamma^{(k)} < T}} \left(\frac{1}{2} + \frac{R}{L} - \beta^{(k)}\right) + O(T/L),$$

where $R > 0$, $L = \log T$ and $J_k(s)$ in (3.1), and $B(s)$ in Theorem A. From this, we have

$$\sum_{\substack{\beta^{(k)} \leqslant \frac{1}{2} \\ 0 < \gamma^{(k)} < T}} 1 \leqslant \frac{L}{2\pi R}\int_2^T \log\left|J_k B\left(\frac{1}{2} - \frac{R}{L} + it\right)\right| dt + O(T)$$

By applying Jensen's inequality to this inequality, we have

$$\sum_{\substack{\beta^{(k)} \leqslant \frac{1}{2} \\ 0 < \gamma^{(k)} < T}} 1 \leqslant \frac{TL}{4\pi R} \log\left(\frac{1}{T}\int_2^T \left|J_k B\left(\frac{1}{2} - \frac{R}{L} + it\right)\right|^2 dt\right) + O(T). \tag{4.1}$$

We let $V_k(s)$ as

$$V_k(s) = \left(1 + \sum_{j=1}^{k} \binom{k}{j} \frac{1}{L^j} \frac{d^j}{ds^j}\right) \zeta(s) = \left(1 + \frac{1}{L}\frac{d}{ds}\right)^k \zeta(s) = Q_k\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s),$$

where $Q_k(x) = (1-x)^k$. Then by Lemma 1 and integration by parts we have

$$\int_2^T \left|J_k B\left(\frac{1}{2} - \frac{R}{L} + it\right)\right|^2 dt \sim \int_2^T \left|V_k B\left(\frac{1}{2} - \frac{R}{L} + it\right)\right|^2 dt. \tag{4.2}$$

By Theorem A, we have

$$\int_2^T \left|V_k B\left(\frac{1}{2} - \frac{R}{L} + it\right)\right|^2 dt \sim c(P, Q_k, R)T,$$

where

$$c(P,Q,R) = 1 + \frac{1}{\theta}\int_0^1\int_0^1 e^{2Ry}|Q(y)P'(x) + \theta Q'(y)P(x) + \theta RQ(y)P(x)|^2 dxdy.$$

By this and (4.1)–(4.2), we have

$$\sum_{\substack{\beta^{(k)}\leqslant \frac{1}{2}\\ 0<\gamma^{(k)}<T}} 1 \leqslant \inf \frac{\log c(P,Q_k,R)}{2R}\frac{TL}{2\pi}(1+o_k(1)), \tag{4.3}$$

where the infimum takes over all polynomials P satisfying $P(0) = 0$, $P(1) = 1$. Since we are taking infimum over certain polynomial, we can choose a continuous function $P(x) = \frac{\sinh \lambda x}{\sinh \lambda}$ since $P(0) = 0$ and $P(1) = 1$. Then we have $\int_0^1 P(x)^2 dx = \frac{1}{2\lambda}\left(1 + O\left(\lambda e^{-2\lambda}\right)\right)$, and $\int_0^1 P'(x)^2 dx = \frac{\lambda}{2}\left(1 + O\left(\lambda e^{-2\lambda}\right)\right)$. Thus, we have

$$\begin{aligned}\inf_P c(P,Q_k,R) \leqslant & 1 + \int_0^1 e^{2Ry}Q_k(y)(Q_k'(y) + RQ_k(y))dy \\ & + \frac{\lambda}{2\theta}\left(1 + O\left(\lambda e^{-2\lambda}\right)\right)\int_0^1 e^{2Ry}Q_k(y)^2 dy \\ & + \frac{\theta}{2\lambda}\left(1 + O\left(\lambda e^{-2\lambda}\right)\right)\int_0^1 e^{2Ry}(Q_k'(y) + RQ_k(y))^2 dy.\end{aligned}$$

By taking

$$\lambda = \theta\sqrt{\frac{\int_0^1 e^{2Ry}(Q_k'(y) + RQ_k(y))^2 dy}{\int_0^1 e^{2Ry}Q_k(y)^2 dy}},$$

we get the minimal value of the right hand side in the previous inequality. Since

$$\begin{aligned}\int_0^1 e^{2Ry}Q_k(y)^2 dy &= \frac{1}{2k}\left(1 + O\left(k^{-1}\right)\right), \\ \int_0^1 e^{2Ry}Q_k'(y)Q_k(y) dy &= \frac{1}{2}\left(1 + O\left(k^{-1}\right)\right), \\ \int_0^1 e^{2Ry}Q_k'(y)^2 dy &= \frac{k}{2}\left(1 + O\left(k^{-1}\right)\right),\end{aligned}$$

we have $\lambda = k\theta\left(1 + O\left(k^{-1}\right)\right)$, and

$$\inf_P c(P,Q_k,R) \leqslant 2 + O\left(k^{-1}\right)$$

as $k \to \infty$. By this together with (4.3) and (3.4), we conclude that for any fixed $R > 0$

$$\sum_{\substack{\beta^{(k)} > \frac{1}{2}\\ 0<\gamma^{(k)}<T}} 1 \geqslant \mu_k \frac{TL}{2\pi}(1+o_k(1)),$$

where $\mu_k = 1 - \frac{\log 2}{2R} + O_R(\frac{1}{k})$ as $k \to \infty$. We complete the proof of Theorem 4. ■

Proof of Theorem 5. Let $H(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. Then, we have the Riemann ξ-function $\xi(s) = H(s)\zeta(s)$. Consider the function $f(s) = H(s)^{-1}\{(g+g_0)\xi(s) + \frac{g_1}{L}\xi'(s)\}$, where $L = \log T$, $g, ig_0, g_1 \in \mathbb{R}$. (We use the same notations as in [2].) We apply the Littlewood lemma as before to obtain

$$\sum_{\substack{fB(\beta+i\gamma)=0\\ \beta\geqslant\frac{1}{2}-\frac{R}{L}\\ 0<\gamma\leqslant T}} \left(\beta - \frac{1}{2} + \frac{R}{L}\right) = \frac{1}{2\pi}\int_1^T \log\left|fB\left(\frac{1}{2} - \frac{R}{L} + it\right)\right| dt + O(T/L) \quad (4.4)$$

with the mollifier $B(s)$ introduced in Theorem A. For the error term $O(T/L)$, we need some condition of g_j that will be discussed at the end of the proof. We note that simple zeros of $\xi(s)$ are not zeros of $f(s)$, besides multiple zeros of $\xi(s)$ are zeros whose multiplicities decrease by one. From symmetry of zeros of $\xi(s)$ to 1/2, the left hand side of (4.4) is not less than $\frac{R}{L}(N(T) - N_d(T))$. By Jensen's inequality, we can deduce that

$$N(T) - N_d(T) \leqslant \frac{TL}{4\pi R}\log\left(\frac{1}{T}\int_2^T \left|fB\left(\frac{1}{2} - \frac{R}{L} + it\right)\right|^2 dt\right) + O(T)$$

or

$$\kappa_d \geqslant 1 - \frac{1}{2R}\log\left(\limsup_{T\to\infty}\frac{1}{T}\int_2^T \left|fB\left(\frac{1}{2} - \frac{R}{L} + it\right)\right|^2 dt\right).$$

All we need is to get an asymptotic formula for the mean square of fB. We have

$$\begin{aligned} f(s) &= (g+g_0)\zeta(s) + \frac{g_1}{L}\left(\frac{H'}{H}(s)\zeta(s) + \zeta'(s)\right) \\ &= \left(Q_1\left(\frac{\log\frac{s}{2\pi}}{2L} + \frac{1}{L}\frac{d}{ds}\right)\zeta(s)\right)\left(1 + O\left(|t|^{-1}\right)\right), \end{aligned}$$

where $Q_1(x) = g + g_0 + g_1 x$. Using the last two equations, integration by parts leads us to

$$\kappa_d \geqslant 1 - \frac{1}{2R}\log\left(\limsup_{T\to\infty}\frac{1}{T}\int_2^T \left|VB\left(\frac{1}{2} - \frac{R}{L} + it\right)\right|^2 dt\right),$$

where $V(s) = Q(-\frac{1}{L}\frac{d}{ds})\zeta(s)$, and $Q(x) = Q_1(\frac{1}{2} - x) = g + g_0 + \frac{1}{2}g_1 - g_1 x$. By Theorem A, we have

$$\kappa_d \geqslant 1 - \frac{1}{2R}\log(c(P,Q,R)).$$

The condition $Q(0) = 1$ is required when we apply the Littlewood lemma to derive (4.4). Then we have $Q(x) = 1 - g_1 x$. In [2, p.10] Conrey made an optimal choice of this case. If we choose $g_1 = 1.02$, $R = 1.2$, $\theta = \frac{4}{7}$, we have $\frac{\log c}{R} = 0.598...$. Therefore, we conclude that $\kappa_d > 0.70$. ∎

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