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ALGEBRAIC INDEPENDENCE OF CERTAIN NUMBERS RELATED TO MODULAR FUNCTIONS

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Dedicated to Professor Georg Johann Rieger
on the occasion of his 80th birthday

Abstract: In previous papers the authors established a method how to decide on the algebraic independence of a set $\{y_1, \dots, y_n\}$ when these numbers are connected with a set $\{x_1, \dots, x_n\}$ of algebraic independent parameters by a system $f_i(x_1, \dots, x_n, y_1, \dots, y_n) = 0$ $(i = 1, 2, \dots, n)$ of rational functions. Constructing algebraic independent parameters by Nesterenko's theorem, the authors successfully applied their method to reciprocal sums of Fibonacci numbers and determined all the algebraic relations between three q-series belonging to one of the sixteen families of q-series introduced by Ramanujan.

In this paper we first give a short proof of Nesterenko's theorem on the algebraic independence of π, $e^{\pi\sqrt{d}}$ and a product of Gamma-values $\Gamma(m/n)$ at rational points m/n. Then we apply the method mentioned above to various sets of numbers. Our algebraic independence results include among others the coefficients of the series expansion of the Heuman-Lambda function, the values $P(q^r), Q(q^r)$, and $R(q^r)$ of the Ramanujan functions P, Q, and R, for $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$ and $r = 1, 2, 3, 5, 7, 10$, and the values given by reciprocal sums of polynomials.

Keywords: algebraic independence, Ramanujan functions, Nesterenko's theorem, complete elliptic integrals, Gamma function.

1. Introduction

In 1916, Ramanujan [19] defined the series

$$S_{2j+1}(x) = \frac{1}{2}\zeta(-2j-1) + \sum_{n=1}^{\infty} \frac{n^{2j+1}x^n}{1-x^n} \qquad (j = 0, 1, 2, \dots),$$

where $\zeta(s)$ is the Riemann zeta function, and studied especially the first three

$$P(x) = -24S_1(x), \qquad Q(x) = 240S_3(x), \qquad R(x) = -504S_5(x)$$

of them. In 1996 Nesterenko [15] proved the following.

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Theorem 1.1 (Nesterenko's Theorem). *For every $x \in \mathbb{C}$ with $0 < |x| < 1$, the set*

$$\{x, P(x), Q(x), R(x)\}$$

contains at least three numbers that are algebraically independent over $\mathbb{Q}$.

Many important corollaries of the theorem are stated in [15], [16] and [17], among others we refer the algebraic independence of π, e^{π}, $\Gamma(1/4)$ (see Lemma 2.2 in Section 2). In this paper we deduce algebraic independence results from Nesterenko's theorem for various sets of numbers applying the following criterion.

Theorem 1.2 (Algebraic independence criterion [13]). *Let $x_1, \dots, x_n \in \mathbb{C}$ be algebraically independent over $\mathbb{Q}$ and let $y_1, \dots, y_n \in \mathbb{C}$ satisfy the system of equations*

$$f_j(x_1, \dots, x_n, y_1, \dots, y_n) = 0 \qquad (1 \leqslant j \leqslant n), \tag{1.1}$$

where $f_j(t_1, \dots, t_n, u_1, \dots, u_n) \in \mathbb{Q}[t_1, \dots, t_n, u_1, \dots, u_n]$ $(1 \leqslant j \leqslant n)$. Assume that

$$\det\left(\frac{\partial f_j}{\partial t_i}(x_1, \dots, x_n, y_1, \dots, y_n)\right) \neq 0. \tag{1.2}$$

Then the numbers $y_1, \dots, y_n$ are algebraically independent over $\mathbb{Q}$.

Corollary 1.1. *Let $x_1, \dots, x_n \in \mathbb{C}$ be algebraically independent over $\mathbb{Q}$ and let $y_j = g_j(x_1, \dots, x_n)$, where $g_j(t_1, \dots, t_n) \in \mathbb{Q}[t_1, \dots, t_n]$ $(j = 1, \dots, n)$. Assume that*

$$\det\left(\frac{\partial g_j}{\partial t_i}(x_1, \dots, x_n)\right) \neq 0\,.$$

Then the numbers $y_1, \dots, y_n$ are algebraically independent over $\mathbb{Q}$.

As far as we know, this criterion first appeared in [13, Lemma 3] (see also [11, Lemma 6], [12, Lemma 3]) and was used together with Nesterenko's theorem to prove the following: Let $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ $(n \geqslant 0)$ denote the Fibonacci numbers. Then, for any distinct positive integers s_1, s_2, s_3, the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s_1}}, \qquad \sum_{n=1}^{\infty} \frac{1}{F_n^{2s_2}}, \qquad \sum_{n=1}^{\infty} \frac{1}{F_n^{2s_3}}$$

are algebraically independent over $\mathbb{Q}$ if and only if at least one of s_i is even (see [10] and [13]). The criterion was applied secondly to the Ramanujan functions $S_{2j+1}(x)$, or the q-series

$$A_{2j+1}(q) = \sum_{n=1}^{\infty} \frac{n^{2j+1} q^{2n}}{1 - q^{2n}} \qquad (j = 0, 1, 2, \dots)$$

(using the notation in [24]). Ramanujan [19] recorded the identity

$$A_7(q) = A_3(q) + 120 A_3(q)^2.$$

Applying the criterion, the authors [11] proved that, for any $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$, the numbers $A_1(q), A_{2i+1}(q), A_{2j+1}(q)$ with $1 \leqslant i < j$ and $(i,j) \neq (1,3)$ are algebraically independent over $\mathbb{Q}$. Furthermore, the authors [12] determined all the algebraic relations among three q-series belonging to one of the sixteen families of q-series studied by Ramanujan [20, Chap. 17] (cf [7]).

This paper is organized as follows. In Section 2, we state two lemmas derived from Nesterenko's theorem, which are used in the proofs of our theorems. In Section 3 we examine algebraic independence properties concerning the coefficients of the series expansion of the Heuman-Lambda function. In Section 4 we prove the algebraic independence of values at algebraic arguments of two classes of series introduced by Ramanujan [21] in connection with Dedekind's eta-function and its third power. The values $P(q^r), Q(q^r), R(q^r)$ with $q \in \overline{\mathbb{Q}}$, $0 < |q| < 1$ for $r = 1, 2, 3, 5, 7, 10$ are discussed in Section 5. In Section 6, we show algebraic independence results for reciprocal sums of polynomials and for other miscellaneous numbers. Finally, in Section 7, we give a method how to check (1.2) when the implicit system (1.1) cannot be solved for $y_1, \dots, y_n$.

All the results obtained in this paper are deduced from Nesterenko's theorem except that in Theorem 6.2 which is proved using Lindemann's theorem. Throughout this paper, we cite for brevity the algebraic independence criterion or its corollary stated above as the AIC.

2. Preliminaries

Let K and E be the complete elliptic integrals of the first and second kind defined by

$$K = K(k) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad E = E(k) := \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt$$

with $k^2 \in \mathbb{C} \setminus (\{0\} \cup [1, \infty))$, where the branch of each integrand is chosen so that it tends to 1 as $t \to 0$. Furthermore, let

$$K' = K'(k) := K(k'), \qquad k^2 + (k')^2 = 1.$$

For each $q \in \mathbb{C}$ with $0 < |q| < 1$, we can choose k such that

$$q = e^{-\pi c}, \qquad c = \frac{K'}{K}. \tag{2.1}$$

Ramanujan [19] gave the expressions

$$P(q^2) = \left(\frac{2K}{\pi}\right)^2 \left(\frac{3E}{K} - 2 + k^2\right),$$

$$Q(q^2) = \left(\frac{2K}{\pi}\right)^4 \left(1 - k^2 + k^4\right),$$

$$R(q^2) = \left(\frac{2K}{\pi}\right)^6 \frac{1}{2}(1+k^2)(1-2k^2)(2-k^2).$$

Nesterenko's theorem combined with them implies the following:

Lemma 2.1. *If* $q = e^{-\pi c}$ *with* $c = K'/K$ *and* $0 < |q| < 1$, *the set*

$$\Big\{q, k, \frac{K}{\pi}, \frac{E}{\pi}\Big\}$$

contains at least three numbers that are algebraically independent over $\mathbb{Q}$.

Using Lemma 2.1, we deduce

Lemma 2.2. (cf [16, p. 6]) *Let* $-m$ *be the discriminant of an imaginary quadratic field. Then the numbers*

$$\pi, \qquad e^{\pi\sqrt{m}}, \qquad \prod_{n=1}^{m-1} \Gamma\big(n/m\big)^{(-m/n)}$$

are algebraically independent over $\mathbb{Q}$, *where* $(-m/n)$ *is the Kronecker symbol.*

In Section 6 we apply the special case of Lemma 2.2 that, for any $d \in \mathbb{N}$, the numbers π and $e^{\pi\sqrt{d}}$ are algebraically independent over $\mathbb{Q}$.

Proof. Let $c = K'/K = \sqrt{m}$ in (2.1). Then $q = e^{-\pi\sqrt{m}}$. Selberg and Chowla [22] proved that

$$K = \lambda_c \sqrt{\pi} \bigg(\prod_{n=1}^{m-1} \Gamma(n/m)^{(-m/n)} \bigg)^{w/4h},$$

where λ_c is some algebraic number, h is the class number of the field $\mathbb{Q}(\sqrt{-m})$, and w is the number of roots of unity in the field. Since $c^2 \in \mathbb{Q}$, k becomes an algebraic number by a theorem of Abel (cf [23, p. 525]) and E takes the form $E = \pi/(4cK) + \beta_c K$ with $\beta_c \in \overline{\mathbb{Q}}$ (see [19] and [24, p. 195]). Thus we have from Lemma 2.1 with $k \in \overline{\mathbb{Q}}$ that

$$\operatorname{tr.d.}_{\mathbb{Q}} \mathbb{Q}\bigg(q, \pi, \prod_{n=1}^{m-1} \Gamma\big(n/m\big)^{(-m/n)}\bigg) = \operatorname{tr.d.}_{\mathbb{Q}} \mathbb{Q}\bigg(q, \pi, \prod_{n=1}^{m-1} \Gamma\big(n/m\big)^{(-m/n)}, \frac{K}{\pi}, \frac{E}{\pi}\bigg) \geqslant 3.$$

∎

For example, each of the following sets

$$\big\{\pi, e^{\pi}, \Gamma(1/4)\big\}, \qquad \big\{\pi, e^{\pi\sqrt{3}}, \Gamma(1/3)\big\},$$
$$\big\{\pi, e^{\pi\sqrt{3}}, \Gamma(1/6)\big\}, \qquad \big\{e^{\pi\sqrt{3}}, \Gamma(1/3), \Gamma(1/6)\big\}$$

is algebraically independent over $\mathbb{Q}$. For the last two sets, we refer the formula

$$2^{2/3}\pi\Gamma^2(1/6) = 3\Gamma^4(1/3)$$

(cf [2, Table 3, (iv)]). As a result, the three numbers $\Gamma(1/2), \Gamma(1/3), \Gamma(1/6)$ are algebraically dependent over $\mathbb{Q}$, and any two of them are algebraically independent over $\mathbb{Q}$.

3. Heuman-Lambda function

We first state an algebraic independence result for the Weierstrass elliptic function $\wp(z) = \wp(z, g_2, g_3)$ with invariants g_2, g_3. The function admits the series expansion

$$\wp(z) = z^{-2} + \sum_{j=1}^{\infty} b_j z^{2j}$$

around $z = 0$. It is known that $b_j \in \mathbb{Q}[g_2, g_3]$, for example,

$$b_1 = \frac{g_2}{20}, \qquad b_2 = \frac{g_3}{28}, \qquad b_3 = \frac{g_2^2}{1200}, \qquad b_4 = \frac{3g_2g_3}{6160}, \qquad \dots$$

with $b_1^2 = 3b_3$. Applying the AIC (without Nesterenko's theorem), the authors and Tachiya [12] proved the following theorem: *If g_2 and g_3 are algebraically independent over $\mathbb{Q}$, then, for every $(i,j) \neq (1,3)$, $i < j$, the coefficients $b_i = b_i(g_2, g_3)$ and $b_j = b_j(g_2, g_3)$ are algebraically independent over $\mathbb{Q}$.*

In this section we consider the Heuman-Lambda function $\Lambda_0(\varphi, k)$ defined by

$$\Lambda_0(\varphi, k) = \frac{2}{\pi}\big(E(k)F(\varphi, k') + K(k)E(\varphi, k') - K(k)F(\varphi, k')\big),$$

where

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}},$$
$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \vartheta}\, d\vartheta.$$

Let $(t_{2m}(\varphi))_{m \geqslant 0}$ and $(a_{2m}(k))_{m \geqslant 0}$ be defined by

$$t_0(\varphi) = \varphi, \qquad t_{2m}(\varphi) = \frac{2m-1}{2m} t_{2m-2}(\varphi) - \frac{\sin^{2m-1}\varphi \cos\varphi}{2m} \qquad (m \geqslant 1),$$

$$a_0(k) = E, \qquad a_2(k) = \frac{1}{2}(2K - E)k'^2,$$

$$a_{2m}(k) = \frac{(2m-3)!}{2^{2m-2} m!(m-2)!}\big(2mK - (2m-1)E\big)k'^{2m} \qquad (m \geqslant 2).$$

Then we have the series expansion (see [6, formula 904.00])

$$\Lambda_0(\varphi, k) = \frac{2}{\pi}\Big(a_0(k)t_0(\varphi) - \sum_{m=1}^{\infty} a_{2m}(k)t_{2m}(\varphi)\Big) \qquad \Big(0 < \varphi < \frac{\pi}{2},\ k^2 < 1\Big),$$

or

$$\Lambda_0(\varphi, k) = \sum_{m=0}^{\infty} b_{2m}(k)t_{2m}(\varphi) \qquad \Big(0 < \varphi < \frac{\pi}{2},\ k^2 < 1\Big),$$

where

$$b_{2m}(k) = \begin{cases} 2a_0(k)/\pi & \text{if } m = 0, \\ -2a_{2m}(k)/\pi & \text{if } m > 0. \end{cases}$$

Theorem 3.1. *Let* $q = e^{-\pi K'/K} \in \overline{\mathbb{Q}}$ *with* $0 < |q| < 1$. *Then, for any integers* $0 \leqslant m_1 < m_2 < m_3$, *the three numbers* $b_{2m_1}(k), b_{2m_2}(k)$, *and* $b_{2m_3}(k)$ *are algebraically independent over* $\mathbb{Q}$.

Furthermore, any four numbers $b_{2m_1}(k), b_{2m_2}(k), b_{2m_3}(k)$, *and* $b_{2m_4}(k)$ *are algebraically dependent over* $\mathbb{Q}$.

Example. Putting $b_{2m} = b_{2m}(k)$ for brevity, we have

$$b_0^2 b_6^2 - 8b_0 b_4^3 + 2b_2^3 b_6 - 3b_2^2 b_4^2 + 6b_0 b_2 b_4 b_6 = 0.$$

Proof of Theorem 3.1. We divide into two cases $m_1 \geqslant 1$ and $m_1 = 0$.
Case 1: $1 \leqslant m_1 < m_2 < m_3$. We define the polynomials

$$\begin{aligned} f_i(t_1, t_2, t_3) &= \frac{(2m_i - 3)!}{2^{2m_i - 1} m_i!(m_i - 2)!} \big(2m_i t_2 - (2m_i - 1)t_3\big) t_1^{2m_i} \\ &= \beta_i \big(2m_i t_2 - (2m_i - 1)t_3\big) t_1^{2m_i} \qquad (i = 1, 2, 3), \end{aligned}$$

where $\beta_1 = 1$ if $m_1 = 1$, and

$$\beta_i = \frac{(2m_i - 3)!}{2^{2m_i - 1} m_i!(m_i - 2)!} \qquad \text{if} \quad m_i \geqslant 2.$$

Applying the determinant rules, we get

$$\det \left(\frac{\partial f_i}{\partial t_j} \right)_{1 \leqslant i, j \leqslant 3}$$

$$= \begin{vmatrix} 2\beta_1 m_1 \big(2m_1 t_2 - (2m_1 - 1)t_3\big) t_1^{2m_1 - 1} & 2\beta_1 m_1 t_1^{2m_1} & -\beta_1 (2m_1 - 1) t_1^{2m_1} \\ 2\beta_2 m_2 \big(2m_2 t_2 - (2m_2 - 1)t_3\big) t_1^{2m_2 - 1} & 2\beta_2 m_2 t_1^{2m_2} & -\beta_2 (2m_2 - 1) t_1^{2m_2} \\ 2\beta_3 m_3 \big(2m_3 t_2 - (2m_3 - 1)t_3\big) t_1^{2m_3 - 1} & 2\beta_3 m_3 t_1^{2m_3} & -\beta_3 (2m_3 - 1) t_1^{2m_3} \end{vmatrix}$$

$$= 4\beta_1\beta_2\beta_3 t_1^{2(m_1+m_2+m_3)-1} \cdot \begin{vmatrix} m_1 \big(2m_1 t_2 - (2m_1 - 1)t_3\big) & m_1 & 1 \\ m_2 \big(2m_2 t_2 - (2m_2 - 1)t_3\big) & m_2 & 1 \\ m_3 \big(2m_3 t_2 - (2m_3 - 1)t_3\big) & m_3 & 1 \end{vmatrix}$$

$$= 8\beta_1\beta_2\beta_3 t_1^{2(m_1+m_2+m_3)-1} (m_2 - m_1)(m_3 - m_1)(m_3 - m_2)(t_3 - t_2).$$

Case 2: $0 = m_1 < m_2 < m_3$. Let $f_1(t_1,t_2,t_3) = 2t_3$ and let $f_2(t_1,t_2,t_3)$, $f_3(t_1,t_2,t_3)$ be defined as in Case 1. Then, we obtain

$$\det\left(\frac{\partial f_i}{\partial t_j}\right)_{1\leqslant i,j\leqslant 3}$$

$$= \begin{vmatrix} 0 & 0 & 2 \\ 2\beta_2 m_2\big(2m_2t_2-(2m_2-1)t_3\big)t_1^{2m_2-1} & 2\beta_2 m_2 t_1^{2m_2} & -\beta_2(2m_2-1)t_1^{2m_2} \\ 2\beta_3 m_3\big(2m_3t_2-(2m_3-1)t_3\big)t_1^{2m_3-1} & 2\beta_3 m_3 t_1^{2m_3} & -\beta_3(2m_3-1)t_1^{2m_3} \end{vmatrix}$$

$$= 2\cdot \begin{vmatrix} 2\beta_2 m_2\big(2m_2t_2-(2m_2-1)t_3\big)t_1^{2m_2-1} & 2\beta_2 m_2 t_1^{2m_2} \\ 2\beta_3 m_3\big(2m_3t_2-(2m_3-1)t_3\big)t_1^{2m_3-1} & 2\beta_3 m_3 t_1^{2m_3} \end{vmatrix}$$

$$= 8\beta_2\beta_3 m_2 m_3 t_1^{2(m_2+m_3)-1}\cdot \begin{vmatrix} 2m_2t_2-(2m_2-1)t_3 & 1 \\ 2m_3t_2-(2m_3-1)t_3 & 1 \end{vmatrix}$$

$$= 16\beta_2\beta_3 m_2 m_3 t_1^{2(m_2+m_3)-1}(m_3-m_2)(t_3-t_2).$$

For $t_1 = k'$, $t_2 = K/\pi$, and $t_3 = E/\pi$ the determinant in both cases does not vanish by Lemma 2.1. Applying the AIC, we may prove the first statement in Theorem 3.1. The second statement follows from the fact that $b_{2m} \in \mathbb{Q}[k', K/\pi, E/\pi]$. ∎

4. Two series introduced by Ramanujan in his lost notebook

Ramanujan [21, pp. 188, 369] introduced the two classes of series

$$T_{2k} := T_{2k}(q) := 1 + \sum_{n=1}^{\infty}(-1)^n\Big\{(6n-1)^{2k}q^{n(3n-1)/2} + (6n+1)^{2k}q^{n(3n+1)/2}\Big\},$$

$$F_{2k} := F_{2k}(q) := \sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)^{2k+1}q^{n(n-1)/2} \qquad (|q|<1),$$

and expressed the functions

$$\frac{T_{2k}(q)}{(q;q)_\infty} =: f_{2k}, \qquad \frac{F_{2k}(q)}{(q;q)_\infty^3} =: U_{2k}$$

with

$$(q;q)_\infty := \prod_{n=1}^{\infty}(1-q^n)$$

as polynomials over $\mathbb{Q}$ in terms of the Ramanujan functions P, Q, and R. We refer Ramanujan's Lost Notebook [1, Chap. 14] for the proofs of Ramanujan's claims which will be used in this section. For example,

$$f_2 = P, \qquad f_4 = 3P^2 - 2Q, \qquad f_6 = 15P^3 - 30PQ + 16R,$$
$$f_8 = 105P^4 - 420P^2Q + 448PR - 132Q^2, \ldots;$$
$$U_0 = 1, \qquad U_2 = P, \qquad U_4 = \frac{1}{3}\big(5P^2 - 2Q\big),$$
$$U_6 = \frac{1}{3}\big(35P^3 - 42PQ + 16R\big), \tag{4.1}$$
$$U_8 = \frac{1}{3}\big(35P^4 - 84P^2Q - 12Q^2 + 64PR\big), \ldots.$$

We note that the formulas $f_2 = P$ and $U_0 = 1$ are proved in [1] by using, respectively, the pentagonal number theorem

$$(q;q)_\infty = 1 + \sum_{n=1}^{\infty} (-1)^n \Big\{ q^{n(3n-1)/2} + q^{n(3n+1)/2} \Big\}$$

and Jacobi's identity

$$(q;q)_\infty^3 = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

In this section we prove the following theorems:

Theorem 4.1. *Let $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$. Then for three distinct positive integers i, j, and k, the numbers $T_{2i}(q)/(q;q)_\infty$, $T_{2j}(q)/(q;q)_\infty$, and $T_{2k}(q)/(q;q)_\infty$ are algebraically independent over $\mathbb{Q}$.*

Theorem 4.2. *Let $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$. Then for three distinct positive integers i, j, and k, the numbers $F_{2i}(q)/(q;q)_\infty^3$, $F_{2j}(q)/(q;q)_\infty^3$, and $F_{2k}(q)/(q;q)_\infty^3$ are algebraically independent over $\mathbb{Q}$.*

We first prove Theorem 4.2. The proof of Theorem 4.1 is similar and much easier. The key to Ramanujan's work on $U_{2k}(q)$ is the differential-recurrence relation

$$U_{2s+2}(q) = P(q)U_{2s}(q) + 8q\frac{dU_{2s}(q)}{dq} \qquad (s \geqslant 0) \tag{4.2}$$

with $U_0 = 1$. From this he deduced the expressions

$$U_{2s} = \sum_{\substack{a,b,c \geqslant 0 \\ a+2b+3c=s}} K_{a,b,c} P^a Q^b R^c \qquad (s \geqslant 1), \tag{4.3}$$

where $K_{a,b,c} \in \mathbb{Q}$, using his differential equations

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3}, \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2}. \tag{4.4}$$

For the proof of Theorem 4.2 we need the explicit values of the coefficients $K_{s,0,0}$, $K_{s,1,0}$, and $K_{s,0,1}$, which can be deduced from the following:

Lemma 4.1. *The coefficients* $K_{s,0,0}, K_{s,1,0}$, *and* $K_{s,0,1}$ *satisfy the recurrence formulas*

$$K_{s+1,0,0} = \frac{2s+3}{3} K_{s,0,0} \qquad (s \geqslant 1), \tag{4.5}$$

$$K_{s-1,1,0} = \frac{2s+7}{3} K_{s-2,1,0} - \frac{2s}{3} K_{s,0,0} \qquad (s \geqslant 2), \tag{4.6}$$

$$K_{s-2,0,1} = \frac{2s+9}{3} K_{s-3,0,1} - \frac{8}{3} K_{s-2,1,0} \qquad (s \geqslant 3), \tag{4.7}$$

where the initial values are given by

$$K_{1,0,0} = 1, \qquad K_{0,1,0} = -\frac{2}{3}, \qquad K_{0,0,1} = \frac{16}{9}. \tag{4.8}$$

Proof. By (4.3), U_{2s} is written as

$$U_{2s} = K_{s,0,0} P^s + K_{s-2,1,0} P^{s-2} Q + K_{s-3,0,1} P^{s-3} R + u_{2s}, \tag{4.9}$$

where $u_{2s} \in \mathbb{Q}[P,Q,R]$ with $\deg_P u_{2s} \leqslant s-4$. Substituting this into (4.2) and using (4.4), we get

$$\begin{aligned} U_{2s+2} = {} & K_{s,0,0} P^{s+1} + K_{s-2,1,0} P^{s-1} Q + K_{s-3,0,1} P^{s-2} R + 8K_{s,0,0} s P^{s-1} \frac{P^2-Q}{12} \\ & + 8K_{s-2,1,0}\Big((s-2)P^{s-3}\frac{P^2-Q}{12} Q + P^{s-2}\frac{PQ-R}{3}\Big) \\ & + 8K_{s-3,0,1}\Big((s-3)P^{s-4}\frac{P^2-Q}{12} R + P^{s-3}\frac{PR-Q^2}{2}\Big) + 8q\frac{du_{2s}}{dq}. \end{aligned} \tag{4.10}$$

We may regard P, Q, and R as independent variables, since they are algebraically independent over $\mathbb{Q}$ by Nesterenko's theorem with $q \in \overline{\mathbb{Q}}$. So we can equate the coefficients of P^{s+1}, $P^{s-1}Q$, and $P^{s-2}R$ on the right-hand sides of (4.9) (with $2s$ replaced by $2s+2$) and (4.10). Thus we obtain (4.5), (4.6), and (4.7), respectively. The initial values (4.8) follow from (4.1). ∎

The following values of $K_{s,0,0}$, $K_{s,1,0}$, and $K_{s,0,1}$ can be obtained from (4.5-4.8) by using the formulas $\Gamma(z+1) = z\Gamma(z)$ and

$$\Gamma\Big(s+\frac{1}{2}\Big) = \frac{1\cdot 3\cdot 5\ldots(2s-1)}{2^s}\sqrt{\pi} \qquad (s = 0,1,2,\ldots).$$

Lemma 4.2. *We have*

$$K_{s,0,0} = 2\,\frac{2^s}{3^s\sqrt{\pi}}\Gamma\Big(s+\frac{3}{2}\Big) = \frac{(2s+1)!}{6^s s!} \qquad (s \geqslant 1),$$

$$K_{s,1,0} = -\frac{8}{45}\,\frac{(s+1)(s+2)2^s}{3^s\sqrt{\pi}}\Gamma\Big(s+\frac{7}{2}\Big) \qquad (s \geqslant 0),$$

$$K_{s,0,1} = \frac{128}{2835}\,\frac{(s+1)(s+2)(s+3)2^s}{3^s\sqrt{\pi}}\Gamma\Big(s+\frac{9}{2}\Big) \qquad (s \geqslant 0).$$

Proof of Theorem 4.2. We apply the AIC by showing that

$$\Delta = \Delta(i,j,k) := \begin{vmatrix} \dfrac{\partial U_{2i}}{\partial P} & \dfrac{\partial U_{2i}}{\partial Q} & \dfrac{\partial U_{2i}}{\partial R} \\[2ex] \dfrac{\partial U_{2j}}{\partial P} & \dfrac{\partial U_{2j}}{\partial Q} & \dfrac{\partial U_{2j}}{\partial R} \\[2ex] \dfrac{\partial U_{2k}}{\partial P} & \dfrac{\partial U_{2k}}{\partial Q} & \dfrac{\partial U_{2k}}{\partial R} \end{vmatrix} \neq 0.$$

Using (4.9), we see by a straightforward computation that

$$\begin{aligned} \Delta &= \begin{vmatrix} iK_{i,0,0}P^{i-1} & K_{i-2,1,0}P^{i-2} & K_{i-3,0,1}P^{i-3} \\ jK_{j,0,0}P^{j-1} & K_{j-2,1,0}P^{j-2} & K_{j-3,0,1}P^{j-3} \\ kK_{k,0,0}P^{k-1} & K_{k-2,1,0}P^{k-2} & K_{k-3,0,1}P^{k-3} \end{vmatrix} + \delta \\ &= \begin{vmatrix} iK_{i,0,0} & K_{i-2,1,0} & K_{i-3,0,1} \\ jK_{j,0,0} & K_{j-2,1,0} & K_{j-3,0,1} \\ kK_{k,0,0} & K_{k-2,1,0} & K_{k-3,0,1} \end{vmatrix} P^{i+j+k-6} + \delta \\ &= CP^{i+j+k-6} + \delta, \end{aligned}$$

where $\delta = \delta(i,j,k) \in \mathbb{Q}[P,Q,R]$ with $\deg_P \delta \leqslant i+j+k-7$ and $C = C(i,j,k) \in \mathbb{Q}$. Note that Δ is a polynomial over $\mathbb{Q}$ in independent variables P, Q, and R. By Lemma 4.2, we have

$$C = \frac{64}{525}\Big(\frac{2}{3}\Big)^{i+j+k} \frac{ijk(i-j)(i-k)(j-k)}{\pi^{3/2}} \Gamma\Big(i+\frac{3}{2}\Big)\Gamma\Big(j+\frac{3}{2}\Big)\Gamma\Big(k+\frac{3}{2}\Big) \neq 0. \quad \blacksquare$$

Proof of Theorem 4.1. The function f_{2k} satisfies the differential equation

$$f_{2k+2}(q) = P(q)f_{2k}(q) + 24q\frac{df_{2k}(q)}{dq}$$

with $f_2(q) = P(q)$, from which it follows that

$$f_{2k} = (2k-1)!!\Big(P^k - \frac{k(k-1)}{3}P^{k-2}Q + \frac{8k(k-1)(k-2)}{45}P^{k-3}R + g_{2k}\Big),$$

where $(2k-1)!! = 1 \cdot 3 \cdots (2k-1)$ and $g_{2k} \in \mathbb{Q}[P,Q,R]$ with $\deg_P g_{2k} \leqslant k-4$

(see [1, Chap. 14]). Hence we have

$$\Delta = \Delta(i,j,k) := \begin{vmatrix} \dfrac{\partial f_{2i}}{\partial P} & \dfrac{\partial f_{2i}}{\partial Q} & \dfrac{\partial f_{2i}}{\partial R} \\ \dfrac{\partial f_{2j}}{\partial P} & \dfrac{\partial f_{2j}}{\partial Q} & \dfrac{\partial f_{2j}}{\partial R} \\ \dfrac{\partial f_{2k}}{\partial P} & \dfrac{\partial f_{2k}}{\partial Q} & \dfrac{\partial f_{2k}}{\partial R} \end{vmatrix}$$

$$= (2i-1)!!(2j-1)!!(2k-1)!!$$

$$\times \begin{vmatrix} i & -\dfrac{i(i-1)}{3} & \dfrac{8i(i-1)(i-2)}{45} \\ j & -\dfrac{j(j-1)}{3} & \dfrac{8j(j-1)(j-2)}{45} \\ k & -\dfrac{k(k-1)}{3} & \dfrac{8k(k-1)(k-2)}{45} \end{vmatrix} P^{i+j+k-6} + \delta$$

$$= \frac{8}{135}(2i-1)!!(2j-1)!!(2k-1)!!ijk(j-k)(j-i)(k-i)P^{i+j+k-6} + \delta,$$

where $\delta = \delta(i,j,k) \in \mathbb{Q}[P,Q,R]$ with $\deg_P \delta \leqslant i+j+k-7$. Therefore $\Delta \neq 0$, and the theorem follows from the AIC. ■

5. Algebraic independence of $P(q^r)$, $Q(q^r)$, and $R(q^r)$

In this section we turn our attention again to the Ramanujan functions P, Q, and R. We already proved in [12, Corollary 2] that for $q \in \overline{\mathbb{Q}}$ with $|q| < 1$ the numbers in each of the sets

$$\{P(q), P(q^2)\}, \qquad \{Q(q), Q(q^2)\}, \qquad \{R(q), R(q^2)\}$$

are algebraically independent over $\mathbb{Q}$. Application of the AIC leads to more general results.

Theorem 5.1. *Let $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$. Then, any three numbers in the set*

$$\{P(q), P(q^2), P(q^5), P(q^{10})\}$$

are algebraically independent over $\mathbb{Q}$ and the four numbers are not. More precisely, putting $P_i = P(q^i)$ $(i = 1, 2, 5, 10)$, we have

$$P_1^2 + 4P_2^2 + 25P_5^2 + 100P_{10}^2 - 6P_1P_2 + 6P_1P_5 - 2P_1P_{10} - 2P_2P_5 + 24P_2P_{10} - 150P_5P_{10} = 0.$$

Theorem 5.2. *Let $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$. Then, in the set*

$$\{Q(q), Q(q^2), Q(q^5), Q(q^{10})\}$$

any two numbers are algebraically independent over $\mathbb{Q}$ and any three are not.

Theorem 5.3. *Let $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$. Then, in the set*

$$\{R(q), R(q^2), R(q^5), R(q^{10})\}$$

any two numbers are algebraically independent over $\mathbb{Q}$ and any three are not.

Theorem 5.4. *Let $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$. Let $X(q^m)$ and $Y(q^n)$ $(m, n \in \{1, 3\})$ be two different numbers in the set $\{Q(q), Q(q^3), R(q), R(q^3)\}$. Then, each of the sets*

$$\{P(q), P(q^3), X(q^m)\}, \qquad \{P(q), X(q^m), Y(q^n)\}, \qquad \{P(q^3), X(q^m), Y(q^n)\}$$

is algebraically independent over $\mathbb{Q}$.

Theorem 5.5. *Let $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$. Let $X(q^m)$ and $Y(q^n)$ $(m, n \in \{1, 7\})$ be two different numbers in the set $\{Q(q), Q(q^7), R(q), R(q^7)\}$. Then, each of the sets*

$$\{P(q), P(q^7), X(q^m)\}, \qquad \{P(q), X(q^m), Y(q^n)\}, \qquad \{P(q^7), X(q^m), Y(q^n)\}$$

is algebraically independent over $\mathbb{Q}$.

We give a detailed proof only for the algebraic independence of P_1, P_2, and P_5 in Theorem 5.1. The remaining cases concerning the set $\{P(q), P(q^2), P(q^5), P(q^{10})\}$ in Theorem 5.1 can be treated similarly. For the proofs of Theorems 5.2-5.5 we shall refer to suitable parameter expressions for the Ramanujan functions P, Q, R. The details of computations for the nonvanishing of the determinant applying the AIC are left to the reader.

Proof of the algebraic independence of P_1, P_2, P_5 in Theorem 5.1. Using the Rogers - Ramanujan continued fraction

$$r = r(q) = q^{1/5} \prod_{j=1}^{\infty} \frac{(1 - q^{5j-4})(1 - q^{5j-1})}{(1 - q^{5j-3})(1 - q^{5j-2})},$$

we define k, z, and y by

$$k = k(q) = r(q) r^2(q^2), \qquad z = z(q) = q \frac{d}{dq} \log k, \qquad y = y(q) = \frac{dz}{dk}. \qquad (5.1)$$

From [9, Theorems 5.3, 5.5, and 5.6], we have the following expressions:

$$P_1 = P(q) = \frac{4(1 + k^2)z}{1 - k^2} + \frac{(1 + k^2)z}{1 + k - k^2} - \frac{4(1 + k^2)z}{1 - 4k - k^2} + 6ky, \qquad (5.2)$$

$$P_2 = \frac{5(1 + k^2)z}{2(1 - k^2)} - \frac{2(1 + k^2)z}{1 + k - k^2} + \frac{(1 + k^2)z}{2(1 - 4k - k^2)} + 3ky, \qquad (5.3)$$

$$P_5 = -\frac{4(1+k^2)z}{5(1-k^2)} + \frac{(1+k^2)z}{1+k-k^2} + \frac{4(1+k^2)z}{5(1-4k-k^2)} + \frac{6}{5}ky, \tag{5.4}$$

$$P_{10} = \frac{(1+k^2)z}{10(1-k^2)} + \frac{2(1+k^2)z}{5(1+k-k^2)} + \frac{(1+k^2)z}{2(1-4k-k^2)} + \frac{3}{5}ky, \tag{5.5}$$

$$\begin{aligned} Q_1 &= Q(q) \\ &= \frac{z^2(k^{12} - 236k^{11} + 1434k^{10} - 740k^9 - 1905k^8 + 3144k^7 + 1196k^6 - 3144k^5)}{(1-k^2)^2(1+k-k^2)^2(1-4k-k^2)^2} \\ &\quad + \frac{z^2(-1905k^4 + 740k^3 + 1434k^2 + 236k + 1)}{(1-k^2)^2(1+k-k^2)^2(1-4k-k^2)^2}, \end{aligned} \tag{5.6}$$

$$R_1 = R(q) = z^3 h(k) p_{1,2}(k) p_{1,5}(k) p_1(k), \tag{5.7}$$

where

$$\begin{aligned} h(k) &:= \frac{1+k^2}{(1-k^2)^3(1+k-k^2)^3(1-4k-k^2)^3}, \\ p_{1,2}(k) &:= k^4 - 22k^3 - 6k^2 + 22k + 1, \\ p_{1,5}(k) &:= k^4 - 4k^3 + 6k^2 + 4k + 1, \\ p_1(k) &:= k^8 + 536k^7 - 268k^6 - 1192k^5 + 470k^4 + 1192k^3 - 268k^2 - 536k + 1. \end{aligned}$$

From (5.2), (5.6), and (5.7) it follows that P_1, Q_1, and R_1 are expressible as rational functions in the three variables k, y, z. For $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$, the numbers P_1, Q_1, and R_1 are algebraically independent over $\mathbb{Q}$ by Nesterenko's theorem, and so are the variables k, y, z. The algebraic relation among P_1, P_2, P_5, and P_{10}, given in Theorem 5.1, can be computed from (5.2), (5.3), (5.4), and (5.5) by using resultants.

Next, we define the following polynomials:

$$\begin{aligned} f_1 &= f_1(z, k, y, \alpha_1, \alpha_2, \alpha_5) \\ &:= (1-k^2)(1+k-k^2)(1-4k-k^2) \\ &\quad \times \Big(\alpha_1 - \frac{4(1+k^2)z}{1-k^2} - \frac{(1+k^2)z}{1+k-k^2} + \frac{4(1+k^2)z}{1-4k-k^2} - 6ky\Big), \\ f_2 &= f_2(z, k, y, \alpha_1, \alpha_2, \alpha_5) \\ &:= 2(1-k^2)(1+k-k^2)(1-4k-k^2) \\ &\quad \times \Big(\alpha_2 - \frac{5(1+k^2)z}{2(1-k^2)} + \frac{2(1+k^2)z}{1+k-k^2} - \frac{(1+k^2)z}{2(1-4k-k^2)} - 3ky\Big), \\ f_5 &= f_5(z, k, y, \alpha_1, \alpha_2, \alpha_5) \\ &:= 5(1-k^2)(1+k-k^2)(1-4k-k^2) \\ &\quad \times \Big(\alpha_5 + \frac{4(1+k^2)z}{5(1-k^2)} - \frac{(1+k^2)z}{1+k-k^2} - \frac{4(1+k^2)z}{5(1-4k-k^2)} - \frac{6}{5}ky\Big). \end{aligned}$$

By definition they vanish if $(\alpha_1, \alpha_2, \alpha_5) = (P_1, P_2, P_5)$. Then,

$$\begin{aligned}
\Delta := &\begin{vmatrix} \dfrac{\partial f_1}{\partial z} & \dfrac{\partial f_1}{\partial k} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial z} & \dfrac{\partial f_2}{\partial k} & \dfrac{\partial f_2}{\partial y} \\ \dfrac{\partial f_5}{\partial z} & \dfrac{\partial f_5}{\partial k} & \dfrac{\partial f_5}{\partial y} \end{vmatrix} \\
= {}& 6k(k^4-1)(k^2-k-1)(k^2+4k-1)\big(1680k^4z + 96k^3z - 234k^4P_1 \\
&+ 462k^3P_1 - 48k^2P_1 - 978k^5P_1 - 96k^5z + 128k^5P_2 + 9P_1 + 2064k^4P_2 \\
&+ 24kP_1 + 96kz - 24P_2 + 4570k^5P_5 - 208kP_2 - 448k^2P_2 + 224k^3P_2 + 15P_5 \\
&+ 400kP_5 + 1360k^2P_5 - 2870k^3P_5 - 3990k^4P_5 - 72z + 192k^6P_1 \\
&+ 768k^2z + 768k^6z - 1664k^6P_2 + 3200k^6P_5 - 96k^7z - 72k^8z - 1970k^7P_5 \\
&- 585k^8P_5 + 30k^9P_5 + 416k^7P_2 + 72k^8P_2 - 48k^9P_2 + 186k^7P_1 \\
&+ 81k^8P_1 + 18k^9P_1\big).
\end{aligned}$$

Substituting the rational expressions (5.2-5.4) into this polynomial, we find

$$\begin{aligned}
\Delta = -144kz(k^2+1)(k^2-k-1)\big(&3k^{12} + 16k^{11} - 22k^{10} - 144k^9 - 3k^8 \\
&- 160k^7 + 44k^6 + 160k^5 - 3k^4 + 144k^3 - 22k^2 - 16k + 3\big) \neq 0,
\end{aligned}$$

which implies the algebraic independence of P_1, P_2, and P_5. ■

For the proofs of Theorems 5.2 and 5.3 we use the same algebraically independent parameters k, y, z in (5.1) as in the proof of Theorem 5.1, and we express the Ramanujan functions at points q^r under consideration using additional identities given by [9, Theorems 5.5 and 5.6].

The parameters applied in the proof of Theorem 5.5 are

$$x = q \prod_{j=1}^{\infty} \frac{(1-q^{7j})^4}{(1-q^j)^4}, \qquad z = \prod_{j=1}^{\infty} \frac{(1-q^j)^7}{1-q^{7j}}, \qquad y = \frac{dz}{dx}.$$

By [8, (3.5),(3.7),(3.10)] we express $P^3(q)$ and $P^3(q^7)$ as rational functions in $\mathbb{Q}(x, y, z)$, whereas we use [8, (3.5), Theorem 3.4] for $Q^3(q)$, $Q^3(q^7)$, $R(q)$, $R(q^7) \in \mathbb{Q}[x, z]$. For $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$ the parameters x, y, z are algebraically independent over $\mathbb{Q}$ by Nesterenko's theorem and the above identities involving $P^3(q), Q^3(q)$, and $R(q)$. Then, applying the AIC on three of the above mentioned identities corresponding to three numbers occasionally chosen in Theorem 5.5, we complete the proof of the theorem.

For the proof of Theorem 5.4 we refer to [4, ch. 33, §4] on the Eisenstein series P, Q, and R, which in [4] are denoted by L, M, and N, respectively. We introduce the parameters

$$x = k^2, \qquad z = \frac{2K}{\pi}, \qquad y = \frac{dz}{dx},$$

where K is the complete elliptic integral of the first kind already mentioned in Section 2. The algebraic independence of these parameters follows from [4, Lemma 4.1, Theorems 4.2-3]. Then we deduce Theorem 5.4 using the expressions [4, (13.17), Lemma 4.1, Theorems 4.2-5] and the AIC.

6. Reciprocal sums of quadratic polynomials and some other numbers

In this section we give algebraic independence results for the numbers stated in the title.

Theorem 6.1. *Let a and b be positive integers with $a^2 - 4b < 0$. Then, any two of the numbers*

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + acn + bc^2} \qquad (c = 1, 2, 3, \dots)$$

are algebraically independent over $\mathbb{Q}$ and any three of them are not.

Example. Let $a = b = 1$. Then $x_c = \sum_{n=0}^{\infty} (n^2 + cn + c^2)^{-1}$ $(c = 1, 2, 3)$ satisfy

$$7x_1^2 - 4x_1 + 6x_2 + 7x_3 + 21(x_1x_3 + 2x_2x_3 - 2x_1x_2) + 1 = 0.$$

Proof of Theorem 6.1. Let $\psi = \Gamma'/\Gamma$. It is known that

$$S := \sum_{n=0}^{\infty} \frac{1}{n^2 + an + b} = \frac{1}{i\sqrt{d}}\big(\psi(\alpha) - \psi(\beta)\big),$$

where $\alpha := (a + i\sqrt{d})/2$, $\beta := (a - i\sqrt{d})/2$, and $-d = a^2 - 4b < 0$. Using the functional equation

$$\psi(z + n) = \frac{1}{z} + \frac{1}{z+1} + \cdots + \frac{1}{z+n-1} + \psi(z) \qquad (z \in \mathbb{C},\ n \in \mathbb{N} \cup \{0\})$$

with $\beta = (1 - \alpha) + (a - 1)$, we have

$$\begin{aligned} \psi(\alpha) - \psi(\beta) &= \psi(\alpha) - \Big(\frac{1}{1-\alpha} + \frac{1}{2-\alpha} + \cdots + \frac{1}{a-1-\alpha} + \psi(1-\alpha)\Big) \\ &= -\pi\cot(\pi\alpha) - \sum_{k=1}^{a-1} \frac{1}{k-\alpha}. \end{aligned}$$

Here,

$$\cot(\pi\alpha) = \begin{cases} -\tan\left(\dfrac{i\pi}{2}\sqrt{d}\right) = -i\,\dfrac{e^{\pi\sqrt{d}} - 1}{e^{\pi\sqrt{d}} + 1} & \text{if } a \equiv 1 \pmod 2, \\ \cot\left(\dfrac{i\pi}{2}\sqrt{d}\right) = -i\,\dfrac{e^{\pi\sqrt{d}} + 1}{e^{\pi\sqrt{d}} - 1} & \text{if } a \equiv 0 \pmod 2, \end{cases}$$

and

$$\sum_{k=1}^{a-1}\frac{1}{k-\alpha}=\frac{1}{2}\sum_{k=1}^{a-1}\frac{2k-a}{k^2-ak+b}+\frac{i\sqrt{d}}{2}\sum_{k=1}^{a-1}\frac{1}{k^2-ak+b}=\frac{i\sqrt{d}}{2}\sum_{k=1}^{a-1}\frac{1}{k^2-ak+b}.$$

Thus we obtain

$$S=\frac{\pi}{\sqrt{d}}\,\frac{e^{\pi\sqrt{d}}+(-1)^a}{e^{\pi\sqrt{d}}-(-1)^a}-\frac{1}{2}\sum_{k=1}^{a-1}\frac{1}{k^2-ak+b}. \tag{6.1}$$

We note that the two numbers $x_1=\pi/\sqrt{d}$ and $x_2=e^{\pi\sqrt{d}}$ are algebraically independent over $\mathbb{Q}$ by Lemma 2.2. Now let c_1, c_2 be distinct positive integers. We put

$$S_i:=\sum_{n=0}^{\infty}\frac{1}{n^2+ac_in+bc_i^2},\qquad r_i:=\frac{1}{2}\sum_{k=1}^{ac_i-1}\frac{1}{k^2-ac_ik+bc_i^2}\qquad (i=1,2).$$

We divide into three cases: *Case 1.* $ac_1c_2\equiv 1 \pmod 2$, *Case 2.* $ac_1\equiv 0 \pmod 2$ and $ac_2\equiv 1 \pmod 2$, *Case 3.* $a\equiv 0 \pmod 2$.
Case 1. $ac_1c_2\equiv 1 \pmod 2$. Using (6.1) for odd a we have

$$S_1=\frac{x_1(x_2^{c_1}-1)}{c_1(x_2^{c_1}+1)}-r_1,\qquad S_2=\frac{x_1(x_2^{c_2}-1)}{c_2(x_2^{c_2}+1)}-r_2. \tag{6.2}$$

We define two polynomials

$$\begin{aligned} f_1(t_1,t_2,u_1,u_2)&:=c_1(t_2^{c_1}+1)u_1-t_1(t_2^{c_1}-1)+c_1r_1(t_2^{c_1}+1),\\ f_2(t_1,t_2,u_1,u_2)&:=c_2(t_2^{c_2}+1)u_2-t_1(t_2^{c_2}-1)+c_2r_2(t_2^{c_2}+1), \end{aligned}$$

which satisfy

$$f_i(x_1,x_2,y_1,y_2)=0,\qquad y_i=S_i\qquad (i=1,2).$$

To apply the AIC we introduce the determinant

$$\Delta:=\det\left(\frac{\partial f_i}{\partial t_j}\right)_{1\leqslant i,j\leqslant 2}=\begin{vmatrix} 1-t_2^{c_1} & c_1^2t_2^{c_1-1}u_1-c_1t_1t_2^{c_1-1}+c_1^2r_1t_2^{c_1-1}\\ 1-t_2^{c_2} & c_2^2t_2^{c_2-1}u_2-c_2t_1t_2^{c_2-1}+c_2^2r_2t_2^{c_2-1}\end{vmatrix}.$$

Computing Δ and substituting $(t_1,t_2,u_1,u_2)=(x_1,x_2,y_1,y_2)$ with $y_i=S_i$ given by (6.2), we obtain

$$\Delta=\frac{2x_1(c_1x_2^{c_1}-c_2x_2^{c_2}-c_1x_2^{c_1+2c_2}+c_2x_2^{c_2+2c_1})}{x_2(x_2^{c_1}+1)(x_2^{c_2}+1)},$$

which does not vanish, since x_1 and x_2 are algebraically independent over $\mathbb{Q}$.

Case 2. Let $ac_1 \equiv 0 \pmod 2$ and $ac_2 \equiv 1 \pmod 2$. We use (6.1) for S_1 and S_2, respectively. The polynomials f_1 and f_2 in this case are given by

$$f_1(t_1,t_2,u_1,u_2) := c_1(t_2^{c_1}-1)u_1 - t_1(t_2^{c_1}+1) + c_1r_1(t_2^{c_1}-1),$$
$$f_2(t_1,t_2,u_1,u_2) := c_2(t_2^{c_2}+1)u_2 - t_1(t_2^{c_2}-1) + c_2r_2(t_2^{c_2}+1).$$

The corresponding determinant Δ with $(t_1,t_2,u_1,u_2) = (x_1,x_2,y_1,y_2)$ as above is

$$\Delta = \frac{2x_1(-c_1x_2^{c_1} - c_2x_2^{c_2} + c_1x_2^{c_1+2c_2} + c_2x_2^{c_2+2c_1})}{x_2(x_2^{c_1}-1)(x_2^{c_2}+1)} \neq 0.$$

Case 3. Let $a \equiv 0 \pmod 2$. The values of S_1 and S_2 are again given by (6.1). Defining

$$f_1(t_1,t_2,u_1,u_2) := c_1(t_2^{c_1}-1)u_1 - t_1(t_2^{c_1}+1) + c_1r_1(t_2^{c_1}-1),$$
$$f_2(t_1,t_2,u_1,u_2) := c_2(t_2^{c_2}-1)u_2 - t_1(t_2^{c_2}+1) + c_2r_2(t_2^{c_2}-1),$$

we get

$$\Delta = \frac{2x_1(-c_1x_2^{c_1} + c_2x_2^{c_2} + c_1x_2^{c_1+2c_2} - c_2x_2^{c_2+2c_1})}{x_2(x_2^{c_1}-1)(x_2^{c_2}-1)} \neq 0.$$

In any case we find $\Delta \neq 0$, which implies by the AIC that the two numbers $y_1 = S_1$ and $y_2 = S_2$ are algebraically independent over $\mathbb{Q}$. The second statement of the theorem follows from (6.1). ∎

In the following we state some algebraic independence results without proofs.

- Gun, Murty, and Rath [14, Theorem 4.1(1)] deduced the transcendence of each of the sums

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + b\alpha^2} \qquad (b \in \mathbb{N},\ \alpha \in \mathbb{Q} \setminus \{0\}) \tag{6.3}$$

  from Lemma 2.2 by showing the expression

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + b\alpha^2} = -\frac{1}{2b\alpha^2} - \frac{\pi}{2\alpha\sqrt{b}}\Big(\frac{1+e^{2\pi\alpha\sqrt{b}}}{1-e^{2\pi\alpha\sqrt{b}}}\Big).$$

  Using the AIC, we can prove that, for a fixed $b \in \mathbb{N}$, any two sums in (6.3) with distinct $\alpha_1, \alpha_2 \in \mathbb{Q} \setminus \{0\}$ are algebraically independent over $\mathbb{Q}$.
- Ramanujan [18] (see also [3, p. 231, Corollary (i),(ii)]) proved that for any $n \in \mathbb{N}$

$$\prod_{k=1}^{\infty} \Big(1 + \Big(\frac{2n}{n+k}\Big)^3\Big) = \frac{\Gamma^3(n+1)\sinh(\pi n\sqrt{3})}{\Gamma(3n+1)\pi n\sqrt{3}},$$
$$\prod_{k=1}^{\infty} \Big(1 + \Big(\frac{2n+1}{n+k}\Big)^3\Big) = \frac{\Gamma^3(n+1)\cosh(\pi(n+1/2)\sqrt{3})}{\Gamma(3n+2)\pi}.$$

Then in the set

$$\Big\{ \prod_{k=1}^{\infty} \Big(1 + \Big(\frac{2m}{m+k}\Big)^3\Big), \prod_{k=1}^{\infty} \Big(1 + \Big(\frac{2n+1}{n+k}\Big)^3\Big) ; m, n = 1, 2, 3, \dots \Big\}$$

any two numbers are algebraically independent over $\mathbb{Q}$ and any three of them are not.

- Any two numbers in the set

$$\big\{\pi\alpha \coth(\pi\alpha) \,|\, \alpha \in \mathbb{Q} \setminus \{0\}\big\}$$

are algebraically independent over $\mathbb{Q}$ and any three of them are not. In particular, any two of the following continued fractions given by Ramanujan ([4, p. 59, Entry 34])

$$1 + \cfrac{n^2}{1 + \cfrac{1^2(n^2+1^2)}{3 + \cfrac{2^2(n^2+2^2)}{5 + \cfrac{3^2(n^2+3^2)}{7 + \dots}}}} = \frac{\pi n}{2} \coth\Big(\frac{\pi n}{2}\Big) \qquad (n = 1, 2, 3 \dots)$$

are algebraically independent over $\mathbb{Q}$. 1979 Bundschuh in [5] remarked that the number

$$\sum_{n=2}^{\infty} \frac{1}{n^4 - 1} = \frac{7}{8} - \frac{\pi}{4} \coth(\pi)$$

is transcendental if π and e^{π} are algebraically independent over $\mathbb{Q}$.

In the AIC, the ring $\mathbb{Q}[t_1, \dots, t_n, u_1, \dots, u_n]$ may be replaced by $\overline{\mathbb{Q}}[t_1, \dots, t_n, u_1, \dots, u_n]$. As an example we have the following result, which relies on [14, Theorem 4.2(1)] and Lindemann's theorem.

Theorem 6.2. *For $r \geqslant 2$, let $P_1(x), \dots, P_r(x), Q(x)$ be polynomials with algebraic coefficients satisfying $r = \deg Q$. Suppose that $\deg P_j \leqslant r - 1$ for $j = 1, \dots, r$ and that Q has simple zeros $\alpha_1^2, \dots, \alpha_r^2$ such that $\alpha_1, \dots, \alpha_r$ are linearly independent over $\mathbb{Q}$. Moreover, we assume that*

$$\det\Big(P_i(\alpha_j^2)\Big)_{1 \leqslant i,j \leqslant r} \neq 0.$$

Then the r numbers

$$y_j := \sum_{n=1}^{\infty} \frac{P_j(\pi^2 n^2)}{Q(\pi^2 n^2)} \qquad (j = 1, \dots, r)$$

are algebraically independent over $\mathbb{Q}$.

Gun, Murty, and Rath [14, Theorem 4.2(1)] proved the transcendence of y_j. (Note that we have removed some unnecessary conditions from [14, Theorem 4.2(1)] and that we have corrected misprints.)

Corollary 6.1. *Suppose that* $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}$ *with* $r \geqslant 2$ *are linearly independent over* $\mathbb{Q}$. *Then the* r *numbers*

$$y_j := \sum_{n=1}^{\infty} \frac{(\pi^2 n^2)^{j-1}}{(\pi^2 n^2 - \alpha_1^2) \dots (\pi^2 n^2 - \alpha_r^2)} \qquad (j = 1, \dots, r)$$

are algebraically independent over $\mathbb{Q}$.

7. Remarks on the application of the AIC

In this paper, we applied the Corollary of the AIC with some modifications in the proofs of Theorems 5.1 - 5.5, which are necessarily because the expressions of the parameters are rational functions, but not polynomials. It may happen that the system of equations (1.1) is not solvable for $y_1, \dots, y_n$. How shall we then check the nonvanishing of the determinant (1.2)? To overcome this point we use resultants. We start with the equations (1.1) and set

$$f_{n+1}(t_1, \dots, t_n, u_1, \dots, u_n) := \det \left(\frac{\partial f_j}{\partial t_i} \right)_{1 \leqslant i,j \leqslant n} \in \mathbb{Q}[t_1, \dots, t_n, u_1, \dots, u_n].$$

We may assume that f_{n+1} is a nonzero polynomial, since otherwise we cannot apply the AIC. We compute recursively the following $n(n+1)/2$ resultants:

$$\begin{aligned} &f_{i,0} := f_i \qquad (i = 1, \dots, n+1), \\ &f_{i,j} := \operatorname{Res}_{u_j}\left(f_{1,j-1}, f_{i+1,j-1}\right) \qquad \text{for } j = 1, \dots, n \text{ and } i = 1, \dots, n+1-j . \end{aligned} \tag{7.1}$$

It is clear that $f_{1,n} \in \mathbb{Q}[t_1, \dots, t_n]$. The AIC works successfully if $f_{1,n}$ turns out to be a nonzero polynomial. Indeed, if $f_{n+1}(x_1, \dots, x_n, y_1, \dots, y_n) = 0$, then $f_{1,n}(x_1, \dots, x_n) = 0$ by the definition of the iterated resultants (Note that $f_{n+1} = f_{n+1,0}$). But this contradicts the algebraic independence over $\mathbb{Q}$ of $x_1, \dots, x_n$, and therefore that of $y_1, \dots, y_n$ follows.

Example. Let $x_1, x_2, x_3 \in \mathbb{C}$ be algebraically independent over $\mathbb{Q}$ and let $y_1, y_2, y_3 \in \mathbb{C}$ be any solution of the system $f_j(x_1, x_2, x_3, y_1, y_2, y_3) = 0$ $(j = 1, 2, 3)$, where

$$\begin{aligned} f_{1,0} &= f_1(t_1, t_2, t_3, u_1, u_2, u_3) = t_1 u_1 + t_2 u_2 + t_3 u_3 - 1, \\ f_{2,0} &= f_2(t_1, t_2, t_3, u_1, u_2, u_3) = (t_1 u_1)^2 + (t_2 u_2)^2 + (t_3 u_3)^2 - 2, \\ f_{3,0} &= f_3(t_1, t_2, t_3, u_1, u_2, u_3) = (t_1 u_1)^4 + (t_2 u_2)^4 + (t_3 u_3)^4 - 3. \end{aligned}$$

We put

$$f_{4,0} = f_4(t_1,t_2,t_3,u_1,u_2,u_3) := \det\left(\frac{\partial f_i}{\partial t_j}\right)_{1\leqslant i,j\leqslant 3} = \begin{vmatrix} u_1 & u_2 & u_3 \\ 2t_1u_1^2 & 2t_2u_2^2 & 2t_3u_3^2 \\ 4t_1^3u_1^4 & 4t_2^3u_2^4 & 4t_3^3u_3^4 \end{vmatrix}$$

$$= 8u_1u_2u_3(t_1u_1-t_2u_2)(t_1u_1-t_3u_3)(t_3u_3-t_2u_2)(t_1u_1+t_2u_2+t_3u_3).$$

We then compute recursively the resultants (7.1) for $j=1,2,3$ and $i=1,\dots,4-j$, namely,

$$f_{1,1} = \operatorname{Res}_{u_1}\big(f_{1,0},f_{2,0}\big), \qquad f_{2,1} = \operatorname{Res}_{u_1}\big(f_{1,0},f_{3,0}\big), \qquad f_{3,1} = \operatorname{Res}_{u_1}\big(f_{1,0},f_{4,0}\big),$$
$$f_{1,2} = \operatorname{Res}_{u_2}\big(f_{1,1},f_{2,1}\big), \qquad f_{2,2} = \operatorname{Res}_{u_2}\big(f_{1,1},f_{3,1}\big),$$
$$f_{1,3} = \operatorname{Res}_{u_3}\big(f_{1,2},f_{2,2}\big) = 2^{90}\cdot 5^{18}\cdot t_1^{288}t_2^{144}t_3^{60} \not\equiv 0,$$

which imply the algebraic independence of y_1,y_2,y_3 over $\mathbb{Q}$.

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