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BASE CHANGE AND THE BIRCH AND SWINNERTON-DYER CONJECTURE

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Abstract: In this paper we prove that if the Birch and Swinnerton-Dyer conjecture holds for products of abelian varieties attached to Hilbert newforms of parallel weight 2 with trivial central character, then the Birch and Swinnerton-Dyer conjecture holds for products of abelian varieties attached to Hilbert newforms of parallel weight 2 with trivial central character regarded over arbitrary totally real number fields.

Keywords: Birch and Swinnerton-Dyer conjecture; base change; potential modularity.

1. Introduction

Let A be an abelian variety defined over a number field F. Then the Birch and Swinnerton-Dyer conjecture predicts that (see §1 of [T] for details):

Conjecture 1.1 (Birch and Swinnerton-Dyer conjecture). *The L-function $L(s, A_{/F})$ of A over F has a meromorphic continuation to the entire complex plane, and*

1) *The rank $r(A_{/F})$ of A over F is equal the order of vanishing of $L(s, A_{/F})$ at $s = 1$,*
2) $$|Ш(A_{/F})| < \infty,$$
   where $Ш(A_{/F})$ is the Tate-Shafarevich group of $A_{/F}$,
3) $$\lim_{s\to 1} \frac{L(s, A_{/F})}{(s-1)^{r(A_{/F})}} = \frac{|Ш(A_{/F})| \cdot det < a_i, b_j > \cdot V_\infty \cdot V_{\text{bad}}}{|A(F)_{\text{tors}}| \cdot |A'(F)_{\text{tors}}|},$$
   where $A' = Pic^0(A)$, $V_\infty = \text{volume}\, A(F \otimes_{\mathbb{Q}} \mathbb{R})$, $V_{\text{bad}} = \text{volume} \prod_{v\in S} A(F_v)$ and S is the set of bad places of $A_{/F}$, $A(F)_{\text{tors}}$ and $A'(F)_{\text{tors}}$ are the subgroups

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of torsion points of $A(F)$ and $A'(F)$, and $< \, , \, >$ is the height pairing

$$< \, , \, >: A(F) \times A'(F) \to \mathbb{R},$$

and $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ are bases of $A(F)/A(F)_{\mathrm{tors}}$ and $A'(F)/A'(F)_{\mathrm{tors}}$.

Given a totally real number field F and a Hilbert newform f of parallel weight 2 of $\mathrm{GL}(2)/F$ with trivial central character and with the field of coefficients a number field M, it is conjectured that there exists an abelian variety A defined over F, of dimension $[M : \mathbb{Q}]$, such that the L-function $L(s, A)$ is equal to $\prod_{\sigma: M \to \mathbb{C}} L(s, f^\sigma)$ modulo the factors at places dividing the level N of f (we remark that if the field of coefficients M is extended to a finite extension M'/M, then one, as above, obtains an abelian variety A' defined over F which is a product of $[M' : M]$ copies of A; hence obviously if the Birch and Swinnerton-Dyer conjecture holds for A, then the Birch and Swinnerton-Dyer conjecture holds for A'; and in this paper we say that A (or A') is associated to f). This conjecture was proved by Zhang (see Theorem B of [Z]) when f is a Hilbert newform for $\Gamma_0(N)$ (N an ideal of F), and $[F : \mathbb{Q}]$ is odd or $\mathrm{ord}_v(N) = 1$ for at least one finite place of F.

In this paper we prove the following result:

Theorem 1.2. *Let r be a positive integer. Assume that Conjecture 1.1 is true for all totally real number fields F and all abelian varieties of the form $B_{1/F} \times \ldots \times B_{r/F}$, where $B_{1/F}, \ldots, B_{r/F}$ are abelian varieties associated as above to Hilbert newforms of parallel weight 2 with trivial central character of $GL(2)/F$. Then Conjecture 1.1 is true for all abelian varieties of the form $A_{1/F'} \times \ldots \times A_{r/F'}$, where A_{i/F_i} for $i = 1, \ldots, r$, is an abelian variety associated as above to a Hilbert newform of parallel weight 2 with trivial central character of $GL(2)/F_i$, for $i = 1, \ldots, r$, and F'/F is an arbitrary totally real number field containing $F_1, \ldots, F_r$.*

Note the following point: We don't know that the abelian varieties $A_{i/F'}$, for $i = 1, \ldots, r$, correspond to Hilbert newforms, since arbitrary totally real base change for GL_2 is not yet established.

2. Potential modularity

Let F be a totally real number field and J_F be the set of infinite places of F. If f is a Hilbert newform of $\mathrm{GL}(2)/F$ of weight $k = (k_\tau)_{\tau \in J_F}$, where all k_τ have the same parity and $k_\tau \geqslant 2$ (one can replace f below by a cuspidal automorphic representation π of weight k of $\mathrm{GL}(2)/F$ which is discrete series at infinity), then there exists ([TA]) a totally odd λ-adic representation

$$\rho_f := \rho_{f,\lambda} : \Gamma_F \to \mathrm{GL}_2(M_\lambda) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_l),$$

which is unramified outside the primes dividing Nl and satisfies $L(s, \rho_{f,\lambda}) = L(s, f)$ (by fixing a specific isomorphism $\iota : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ one can regard $\rho_{f,\lambda}$ as complex valued). Here $\Gamma_F := \mathrm{Gal}(\bar{F}/F)$, M is the coefficients field of f, λ is a prime ideal of M above some prime number l, N is the level of f, and totally odd means that $\det\rho_f(c) = -1$ for all complex conjugations c. Moreover, each representation $\rho_{f,\lambda}$ is crystalline at any place $v \nmid N$ of F which divides the residue characteristic of λ, and for each embedding $\tau : F \hookrightarrow \bar{\mathbb{Q}}_l$ the τ-Hodge-Tate numbers of $\rho_{f,\lambda}$ are m_τ and $k_0 - m_\tau - 1$, where $k_0 :=\max\{k_\tau | \tau \in J_F\}$, and $m_\tau := (k_0 - k_\tau)/2$.

In this paper we say that a representation

$$\rho : \Gamma_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_l),$$

with F a totally real number field, is modular if there exists a Hilbert newform of weight $k \geqslant 2$ of $\mathrm{GL}(2)/F$ such that $\rho \sim \rho_f$.

We now show the following result:

Theorem 2.1. *For $i = 1, \ldots, r$, let F_i be a totally real number field, and let f_i be a Hilbert newform of weight $k_i \geqslant 2$ of $GL(2)/F_i$. Let l be a rational prime, and F' be a totally real number field which contains F_i for $i = 1, \ldots, r$. Then there exists a totally real number field F'' which contains F' and which is Galois over $\mathbb{Q}$ such that the representations $\rho_{f_i,\lambda_i}|_{\Gamma_{F''}}$, for $i = 1, \ldots, r$, are modular, where $\lambda_i | l$ is a prime of the field of coefficients of f_i.*

Proof: If f_i is a CM-form, then it is well known that $\rho_{f_i,\lambda_i}|_{\Gamma_{F''}}$ is modular for any finite extension F''/F_i (see for example Theorem 7.4 of [G]). Hence it is sufficient to prove Theorem 2.1 when each f_i is a non-CM form. We assume this fact from now on.

We know the following result (this is a particular case of Theorem 4.5.1 of [BGGT]; see also Lemma 1.4.2 of [BGGT] and the remark after Theorem 4.5.1 of [BGGT]; we remark that in the statement below we take ϕ_i representations of $\Gamma_{F'}$, and in Theorem 4.5.1 of [BGGT] ϕ_i was a representation of $\Gamma_{E'}$ satisfying some conjugate self-dual condition, but obviously our $\phi_i|_{\Gamma_{E'}}$ satisfies this condition):

Theorem 2.2. *Suppose that:*

(a) *Let E' be a Galois CM number field, and let F' denote its maximal totally real subfield.*
(b) *Let $l \geqslant 7$ be a rational prime which is unramified in E'.*
(c) *For each $i = 1, \ldots, r$, let $\phi_i : \Gamma_{F'} \to GL_2(\bar{\mathbb{Q}}_l)$ be a totally odd continuous representation.*

For every $i = 1, \ldots, r$, suppose also that:

(1) *ϕ_i is unramified at all but finitely many primes.*
(2) *$k_i = (k_{i\tau})_{\tau \in J_{F'}}$, where all $k_{i\tau}$ have the same parity and $k_{i\tau} \geqslant 2$, and $l > k_{i0}$, where $k_{i0} := \max\{k_{i\tau} | \tau \in J_{F'}\}$.*
(3) *ϕ_i is crystalline at all places $v | l$, and for each embedding $\tau : F' \hookrightarrow \bar{\mathbb{Q}}_l$ the τ-Hodge-Tate numbers of ϕ_i are $(k_{i0} - k_{i\tau})/2$ and $(k_{i0} + k_{i\tau})/2 - 1$.*
(4) *$\bar{\phi}_i|_{\Gamma_{E'(\zeta_l)}}$ is irreducible.*

Then we can find a finite CM extension E''/E', such that E'' is Galois, and for each $i = 1, \ldots, r$, a cuspidal automorphic representation π_i of $GL_2(\mathbb{A}_{E''})$ of weight k_i such that $\phi_i|_{\Gamma_{E''}} \cong \rho_{\pi_i,\gamma_i}$ for some $\gamma_i | l$.

We want to apply Theorem 2.2 to some rational prime $l \geqslant 7$ and to $\phi_i := \rho_{f_i,\lambda_i}|_{\Gamma_{F'}}$ for $i = 1, \ldots, r$, (one can assume that the totally real field F' from Theorem 2.1 is Galois by replacing it by its Galois closure F'^{gal}; and we assume this fact from now on). We choose a rational prime l unramified in F' which is relatively prime to N_i, for $i = 1, \ldots, r$, where N_i is the level of f_i, and such that $l > k_{i0}$ for each $i = 1, \ldots, r$. Then from above we know that ϕ_i for $i = 1, \ldots, r$, satisfies the conditions (c), (1), (2) and (3) of Theorem 2.2. Since f_i is non-CM, we know from Proposition 3.8 of [D], that for l sufficiently large, the image of the representation $\bar{\rho}_{f_i,\lambda_i}$ contains $\mathrm{SL}_2(\mathbb{F}_l)$. But then for such a prime number l, because F' is totally real, the image of the representation $\bar{\phi}_i = \bar{\rho}_{f_i,\lambda_i}|_{\Gamma_{F'}}$ contains $\mathrm{SL}_2(\mathbb{F}_l)$ (see Proposition 3.5 of [V1]). Hence we can choose l sufficiently large such that for $i = 1, \ldots, r$, the image of $\bar{\phi}_i$ contains $\mathrm{SL}_2(\mathbb{F}_l)$. Thus for $i = 1, \ldots, r$, the representation $\bar{\phi}_i|_{\Gamma_{F'(\zeta_l)}}$ is irreducible. Now one can choose a CM quadratic extension E' of F', which is Galois over $\mathbb{Q}$, such that l is unramified in E', and $E'(\zeta_l)$ linearly disjoint over $F'(\zeta_l)$ from $\bar{\mathbb{Q}}^{\ker\bar{\phi}_i|_{\Gamma_{F'(\zeta_l)}}}$ for $i = 1, \ldots, r$, and hence the representation $\bar{\phi}_i|_{\Gamma_{E'(\zeta_l)}}$ is also irreducible for $i = 1, \ldots, r$. Hence condition (4) of Theorem 2.2 is satisfied. Conditions (a) and (b) are also satisfied from our choices made above. Thus we verified all the conditions of Theorem 2.2, and we conclude that there exists a CM extension E''/E', such that E'' is Galois, and for each $i = 1, \ldots, r$, a cuspidal automorphic representation π_i of $\mathrm{GL}_2(\mathbb{A}_{E''})$ such that $\phi_i|_{\Gamma_{E''}} \cong \rho_{\pi_i,\gamma_i}$ for some $\gamma_i|l$. Let F'' be the maximal totally real subfield of E''. Since $\phi_i|_{\Gamma_{E''}}$ is modular and E''/F'' is quadratic (solvable), one can deduce easily that $\phi_i|_{\Gamma_{F''}}$ is automorphic (see Lemma 1.3 of [BGHT]). Hence we finished the proof of Theorem 2.1. ■

3. The proof of Theorem 1.2

Let F_i for $i = 1, \ldots, r$, be a totally real number field and let f_i be a Hilbert newform of parallel weight 2 with trivial central character of $\mathrm{GL}(2)/F_i$. We denote by A_i the abelian variety defined over F_i conjecturally associated to f_i as above. Let F' be a totally real number field which contains F_i for $i = 1, \ldots, r$. Then from Theorem 2.1 we know that there exists a totally real finite Galois extension F'' of $\mathbb{Q}$ which contains F' such that the representations $\rho_{f_i}|_{\Gamma_{F''}}$, for $i = 1, \ldots, r$, are modular.

From Theorem 15.10 of [CR] we know that there exists some subfields $E_j \subseteq F''$, such that $\mathrm{Gal}(F''/E_j)$ are solvable, each E_j contains F', and some integers n_j, such that the trivial representation

$$1_{F'} : \mathrm{Gal}(F''/F') \to \bar{\mathbb{Q}}^{\times},$$

can be written as

$$1_{F'} = \sum_{j=1}^{u} n_j \mathrm{Ind}_{\mathrm{Gal}(F''/E_j)}^{\mathrm{Gal}(F''/F')} 1_{E_j} \qquad \text{(a virtual sum)}, \tag{3.1}$$

from which we get that

$$L(s, A_{1/F'} \times \ldots \times A_{r/F'}) = \prod_{j=1}^{u} L(s, A_{1/E_j} \times \ldots \times A_{r/E_j})^{n_j}.$$

Since $\rho_{f_i}|_{\Gamma_{F''}}$ is modular, and $\mathrm{Gal}(F''/E_j)$ is solvable, from Langlands base change for solvable extensions ([L]), one can deduce easily that $\rho_{f_i}|_{\Gamma_{E_j}}$ is modular, and hence the abelian variety A_{i/E_j} corresponds to a Hilbert newform of parallel weight 2 and trivial central character of $\mathrm{GL}(2)/E_j$, and $\rho_{f_i}|_{\Gamma_{E_j}}$ corresponds to one piece of the l-adic Tate module of A_{i/E_j}. Hence the function $L(s, A_{1/F'} \times \ldots \times A_{r/F'})$ has a meromorphic continuation to the entire complex plane because from our assumptions in Theorem 1.2 the functions $L(s, A_{1/E_j} \times \ldots \times A_{r/E_j})$ have meromorphic continuations to the entire complex plane.

We know (Theorem 1 of [M]):

Theorem 3.1. *Let L/K be an extension of number fields, and A be an abelian variety defined over L. Then the Birch and Swinnerton-Dyer conjecture holds for A if and only if the Birch and Swinnerton-Dyer conjecture holds for $Res_{L/K}A$.*

We know (Theorem 7.3 of [M1]):

Theorem 3.2. *Let A and B be two isogeneous abelian varieties defined over a number field L. Then the Birch and Swinnerton-Dyer conjecture holds for A if and only if the Birch and Swinnerton conjecture holds for B.*

The following theorem is obvious:

Theorem 3.3. *Let A and B be two abelian varieties defined over a number field L. If the Birch and Swinnerton-Dyer conjecture holds for A and B, then Birch and Swinnerton conjecture holds for $A \times B$. If the Birch and Swinnerton-Dyer conjecture holds for B and $A \times B$, then the Birch and Swinnerton conjecture holds for A.*

In order to simplify the notations we denote $A_{/F'} := A_{1/F'} \times \ldots \times A_{r/F'}$. From our assumptions in Theorem 1.2 we know that the Birch and Swinnerton-Dyer holds for $A_{/E_j}$, and thus from Theorem 3.1, we deduce that the Birch and Swinnerton-Dyer conjecture holds for $\mathrm{Res}_{E_j/F'}A_{/E_j}$. Now from (3.1) we get that the abelian varieties $A_{/F'} \times \prod_{j'} -n_{j'}\mathrm{Res}_{E_{j'}/F'}A_{/E_{j'}}$ and $\prod_{j''} n_{j''}\mathrm{Res}_{E_{j''}/F'}A_{/E_{j''}}$ are isogenous (for details see the proof of Theorem 2.3 of [DO]), where j' has the property that $n_{j'}$ is negative, and j'' has the property that $n_{j''}$ is positive. Because the Birch and Swinnerton-Dyer conjecture holds for each $\mathrm{Res}_{E_{j''}/F'}A_{/E_{j''}}$, from Theorem 3.3 we get that the Birch and Swinnerton-Dyer conjecture holds for $\prod_{j''} n_{j''}\mathrm{Res}_{E_{j''}/F'}A_{/E_{j''}}$. But the varieties $A_{/F'} \times \prod_{j'} -n_{j'}\mathrm{Res}_{E_{j'}/F'}A_{/E_{j'}}$ and $\prod_{j''} n_{j''}\mathrm{Res}_{E_{j''}/F'}A_{/E_{j''}}$ are isogenous, and hence from Theorem 3.2 we get that the Birch and Swinnerton-Dyer conjecture holds for $A_{/F'} \times \prod_{j'} -n_{j'}\mathrm{Res}_{E_{j'}/F'}A_{/E_{j'}}$. Now because the Birch and Swinnerton-Dyer conjecture holds for $\prod_{j'} -n_{j'}\mathrm{Res}_{E_{j'}/F'}A_{/E_{j'}}$ and $A_{/F'} \times \prod_{j'} -n_{j'}\mathrm{Res}_{E_{j'}/F'}A_{/E_{j'}}$, from Theorem 3.3 we get that the Birch and Swinnerton-Dyer conjecture holds for $A_{/F'}$, and we conclude the proof of Theorem 1.2. ■

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