Functiones et Approximatio
46.1 (2012), 63–77
doi: 10.7169/facm/2012.46.1.5

EVALUATIONS OF SOME QUADRUPLE EULER SUMS OF EVEN WEIGHT

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Abstract: For positive integers $\alpha_1, \alpha_2, \dots, \alpha_r$ with $\alpha_r \geqslant 2$, the multiple zeta value or r-fold Euler sum is defined by

$$\zeta(\alpha_1, \alpha_2, \dots, \alpha_r) = \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r} n_1^{-\alpha_1} n_2^{-\alpha_2} \cdots n_r^{-\alpha_r}.$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ and r are the weight and depth of $\zeta(\alpha_1, \alpha_2, \dots, \alpha_r)$ respectively. By the general theorem given in [11], the multiple zeta value $\zeta(\alpha_1, \alpha_2, \dots, \alpha_r)$ can be expressed as a rational linear combination of products of multiple zeta values of lower depth if its depth and weight are of different parity. In other words, when the sum of its depth and weight is odd. However, there are still some exceptions for quadruple Euler sums. As conjectured in [6], a quadruple Euler sum with even weight exceeding 14 can be expressed as a rational linear combination of products of multiple zeta values of depth 1, 2 and 3 if and only if it is one of the following forms: $\zeta(1, a, b, a)$, $\zeta(b, 1, a, a)$, $\zeta(b, b, 1, a)$ or $\zeta(a, b, b, a)$ with $a = b$ or $b = 1$. In this paper, we shall evaluate these quadruple Euler sums of even weight by the identities among multiple zeta values with variables and relation obtained from the stuffle formula of two multiple zeta values.

Keywords: Euler sums, Hurwitz zeta function, stuffle formulae, multiple zeta value.

1. Introduction

For a pair (p, q) of positive integers with $q \geqslant 2$, the classical Euler sum is defined by

$$S_{p,q} := \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{k} \frac{1}{j^p}. \tag{1.1}$$

This work was supported by the Department of Mathematics, National Chung Cheng University and by the National Science Council of Taiwan, Republic of China.

2010 Mathematics Subject Classification: primary: 40A25, 40B05; secondary: 11M99, 33E99.

In 1742, Goldbach proposed to Euler the problem of expressing the $S_{p,q}$ in terms of values at positive integers of the Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-s}, \qquad \Re s>1.$$

Euler sowed this problem in the case $p=1$ and gave a general formula for odd weight $p+q$ without any proof in 1775. We now have a clear understanding of this problem [1, 2, 3, 5, 9, 10].

1. $S_{p,q}$ can be expressed in terms of single zeta values when $p=1$ or $(p,q)=(2,4)$ or $(p,q)=(4,2)$ or $p=q$ or $p+q$ is odd.
2. For a positive integer $n \geqslant 2$,

$$S_{1,n}=\frac{n+2}{2}\zeta(n+1)-\frac{1}{2}\sum_{r=2}^{n-1}\zeta(r)\zeta(n+1-r). \tag{1.2}$$

3. For odd weight $w=2m+2n+1$,

$$\begin{aligned} S_{2m,2n+1}=&\frac{1}{2}\zeta(w)+\sum_{r=0}^{n}\binom{w-2r-1}{2m-1}\zeta(2r)\zeta(w-2r)\\ &+\sum_{r=0}^{m}\binom{w-2r-1}{2n}\zeta(2r)\zeta(w-2r). \end{aligned} \tag{1.3}$$

4. For positive integers p,q with $p,q \geqslant 2$,

$$S_{p,q}+S_{q,p}=\zeta(p)\zeta(q)+\zeta(p+q).$$

Formula (4) is called the reflection formula. It plays an important role in our evaluation of Euler sums.

Multiple zeta values are the natural generalizations of classical Euler sums from double sums to more general r-fold Euler sums defined by

$$\zeta(\alpha_1,\alpha_2,\ldots,\alpha_r)=\sum_{1\leqslant n_1<n_2<\cdots<n_r} n_1^{-\alpha_1}n_2^{-\alpha_2}\cdots n_r^{-\alpha_r},$$

where $\alpha_1,\alpha_2,\ldots,\alpha_r$ are positive integers with $\alpha_r \geqslant 2$ for the sake of convergence. The numbers $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_r$ and r are called the weight and depth of $\zeta(\alpha_1,\alpha_2,\ldots,\alpha_r)$ respectively. Triple Euler sums of even weight and quadruple Euler sums of odd weight were evaluated in [13, 14]. Moreover, there is no difficultly in extending the procedure to the evaluation of multiple zeta values of arbitrary depth provided the sum of the weight and depth is odd. On the other hand, Tsumura [11] proved in 2004 that the multiple zeta value $\zeta(\alpha_1,\alpha_2,\ldots,\alpha_r)$ can be expressed as a rational linear combination of products of multiple zeta values of lower depth provided that the sum of weight and depth, $|\alpha|+r$, is odd. However, there are still some exceptions for quadruple Euler sums. In [6], it was asserted that a quadruple Euler sum with even weight exceeding 14 can be expressed as a rational linear combination of products of single, double or triple Euler sums if and only if it is one of the following forms: $\zeta(1,a,b,a)$, $\zeta(b,1,a,a)$, $\zeta(b,b,1,a)$ or $\zeta(a,b,b,a)$ with $a=b$ or $b=1$. Also it was claimed that there is a proof of these assertion. However, we are unable to give such a proof. Instead, we give the evaluation of the quadruple Euler sum in these cases. We note that we have done this for $\zeta(1,1,1,a)$ in [14].

For convenience, we define $S_{p,q,r}$ as

$$S_{p,q,r} := \zeta(p,q,r) + \zeta(p+q,r) + \zeta(p,q+r) + \zeta(p+q+r).$$

Here are our main theorems.

Theorem 1. *For a positive integer $p \geqslant 2$, we have*

$$\begin{aligned} 2\zeta(p,1,1,p) &= 2\zeta(p)\zeta(1,1,p) - \zeta(1,p)^2 + \zeta(2,2p) \\ &\quad + 2\zeta(2,p,p) - 2\zeta(p+1,1,p), \end{aligned}$$

$$\begin{aligned} 2\zeta(1,p,1,p) &= \sum_{\ell=2}^{p-1} (-1)^{p+\ell+1} \zeta(p+1-\ell) \left\{ S_{p,1,\ell} + \zeta(\ell,1,p) \right\} \\ &\quad + \zeta(1,p)^2 - \zeta(2,2p) - 2\zeta(2,p,p) + 2\zeta(p+1,1,p) \\ &\quad + \{(-1)^p - 2\}\zeta(p)\zeta(1,1,p) - C(p), \end{aligned}$$

and

$$\begin{aligned} 4\zeta(1,1,p,p) &= \sum_{\ell=2}^{p-1} (-1)^{p+\ell} \zeta(p+1-\ell) \left\{ S_{p,1,\ell} + \zeta(\ell,1,p) \right\} \\ &\quad - 2\zeta(p+1,1,p) - 2\zeta(1,p+1,p) - 2\zeta(1,1,2p) \\ &\quad - \{(-1)^p - 2\}\zeta(p)\zeta(1,1,p) + C(p). \end{aligned}$$

Here

$$\begin{aligned} C(p) &= (-1)^{p+1}\{\zeta(p)\zeta(p,2) + \zeta(p)\zeta(p+2) + \zeta(p+1)\zeta(1,p)\} \\ &\quad + 2\zeta(p,1,p+1) + \zeta(1,p,p+1) + \zeta(1,p+1,p) \\ &\quad + 3\zeta(p+1,1,p) + \zeta(p,2,p) + \zeta(1,2p+1) \\ &\quad + 2\zeta(p+1,p+1) + \zeta(p,p+2) \\ &\quad + \zeta(p+2,p) + \zeta(2p+2). \end{aligned}$$

Theorem 2. *For an odd integer $p \geqslant 3$, we have*

$$\begin{aligned} 4\zeta(1,p,p,p) &= 2\sum_{\ell=2}^{p-1} (-1)^{\ell} \zeta(p+1-\ell) \left\{ S_{p,p,\ell} + \zeta(\ell,p,p) \right\} \\ &\quad - 2\sum_{\ell=1}^{p-1} (-1)^{\ell} \zeta(p-\ell,p) S_{p,\ell+1} \\ &\quad - \zeta(p)\zeta(p,1,p) + \zeta(2p,1,p) + \zeta(p,p+1,p) \\ &\quad + \zeta(p,1,2p) - 2\zeta(p)\zeta(1,p,p) - 2\widetilde{C}(p), \end{aligned}$$

$$
\begin{aligned}
4\zeta(p,1,p,p) = {} & 6\sum_{\ell=2}^{p-1}(-1)^{\ell+1}\zeta(p+1-\ell)\left\{S_{p,p,\ell}+\zeta(\ell,p,p)\right\} \\
& + 6\sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(p-\ell,p)S_{p,\ell+1} \\
& + 10\zeta(p)\zeta(1,p,p) - 4\zeta(p+1,p,p) - 4\zeta(1,2p,p) \\
& - 4\zeta(1,p,2p) + 3\zeta(p)\zeta(p,1,p) - 3\zeta(2p,1,p) \\
& - 3\zeta(p,p+1,p) - 3\zeta(p,1,2p) + 6\widetilde{C}(p)
\end{aligned}
$$

and

$$
\begin{aligned}
4\zeta(p,p,1,p) = {} & 6\sum_{\ell=2}^{p-1}(-1)^{\ell}\zeta(p+1-\ell)\left\{S_{p,p,\ell}+\zeta(\ell,p,p)\right\} \\
& - 6\sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(p-\ell,p)S_{p,\ell+1} \\
& - 10\zeta(p)\zeta(1,p,p) + 4\zeta(p+1,p,p) \\
& + 4\zeta(1,2p,p) + 4\zeta(1,p,2p) - \zeta(p)\zeta(p,1,p) \\
& + \zeta(2p,1,p) + \zeta(p,p+1,p) + \zeta(p,1,2p) - 6\widetilde{C}(p).
\end{aligned}
$$

Here

$$
\begin{aligned}
\widetilde{C}(p) = {} & \zeta(p+1)\zeta(p,p) + \zeta(p)\zeta(p,p+1) + \zeta(p)\zeta(2p+1) \\
& + \zeta(p,p,p+1) + \zeta(2p,p+1) + \zeta(p,2p+1) \\
& + \zeta(3p+1) + \zeta(1,3p) + \zeta(1,p,2p) \\
& + \zeta(p,1,2p) + \zeta(p+1,2p) + \zeta(1,2p,p) \\
& + \zeta(2p,1,p) + \zeta(2p+1,p) + 2\zeta(p+1,p,p) + \zeta(p,p+1,p).
\end{aligned}
$$

Theorem 3. *For an odd integer $p \geqslant 3$ and an even integer $q \geqslant 2$, we have*

$$
\begin{aligned}
2\zeta(1,q,p,q) = {} & \sum_{\ell=2}^{q-1}(-1)^{\ell+1}\zeta(q+1-\ell)\left\{S_{q,p,\ell}+\zeta(\ell,p,q)\right\} \\
& + \sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(p-\ell,q)S_{q,\ell+1} \\
& - \zeta(q)\zeta(p,1,q) + \zeta(1,q)\zeta(p,q) - \zeta(p+1,2q) \\
& - 2\zeta(p+1,q,q) + \zeta(p+q,1,q) + \zeta(q+1,p,q) - C(p,q),
\end{aligned}
$$

$$
\begin{aligned}
2\zeta(q,q,1,p) &= \sum_{\ell=2}^{q-1}(-1)^{\ell+1}\zeta(q+1-\ell)\left\{S_{q,p,\ell}+\zeta(\ell,p,q)\right\} \\
&\quad + 2\sum_{\ell=1}^{q-1}(-1)^{\ell}\zeta(q-\ell,p)S_{q,\ell+1} + \sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(p-\ell,q)S_{q,\ell+1} \\
&\quad + 2\sum_{\ell=1}^{q-1}(-1)^{\ell+1}\zeta(\ell+1)\zeta(q-\ell,q,p) + 2\sum_{\ell=2}^{p-1}(-1)^{\ell}\zeta(p+1-\ell)S_{q,q,\ell} \\
&\quad + 2\zeta(q+1,q,p) + 2\zeta(1,2q,p) + 2\zeta(1,q,p+q) - 2\zeta(q)\zeta(1,q,p) \\
&\quad - \zeta(q)\zeta(p,1,q) + \zeta(1,q)\zeta(p,q) - \zeta(p+1,2q) - 2\zeta(p+1,q,q) \\
&\quad + \zeta(p+q,1,q) + \zeta(q+1,p,q) - C(p,q) - 2\widetilde{C}(p,q),
\end{aligned}
$$

and

$$
\begin{aligned}
2\zeta(p,1,q,q) &= -\sum_{\ell=2}^{q-1}(-1)^{\ell+1}\zeta(q+1-\ell)\left\{S_{q,p,\ell}+\zeta(\ell,p,q)\right\} \\
&\quad - 2\sum_{\ell=1}^{q-1}(-1)^{\ell}\zeta(q-\ell,p)S_{q,\ell+1} - \sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(p-\ell,q)S_{q,\ell+1} \\
&\quad - 2\sum_{\ell=1}^{q-1}(-1)^{\ell+1}\zeta(\ell+1)\zeta(q-\ell,q,p) - 2\sum_{\ell=2}^{p-1}(-1)^{\ell}\zeta(p+1-\ell)S_{q,q,\ell} \\
&\quad + 2\zeta(q)\zeta(q,1,p) + 2\zeta(p)\zeta(1,q,q) - 2\zeta(q,q)\zeta(1,p) + 2\zeta(q+1,p+q) \\
&\quad + 2\zeta(q+1,q,p) + 2\zeta(q+1,p,q) - 2\zeta(2q,1,p) - 2\zeta(p+1,q,q) \\
&\quad - 2\zeta(q+1,q,p) - 2\zeta(1,2q,p) - 2\zeta(1,q,p+q) + 2\zeta(q)\zeta(1,q,p) \\
&\quad + \zeta(q)\zeta(p,1,q) - \zeta(1,q)\zeta(p,q) + \zeta(p+1,2q) + 2\zeta(p+1,q,q) \\
&\quad - \zeta(p+q,1,q) - \zeta(q+1,p,q) + C(p,q) + 2\widetilde{C}(p,q).
\end{aligned}
$$

Here

$$
\begin{aligned}
C(p,q) &= \zeta(q,p,q+1) + \zeta(q+p,q+1) + \zeta(q,p+q+1) + \zeta(p+2q+1) \\
&\quad + \zeta(1,p+2q) + \zeta(1,q,p+q) + \zeta(q,1,p+q) + \zeta(q+1,p+q) \\
&\quad + \zeta(1,p+q,q) + \zeta(p+q,1,q) + \zeta(p+q+1,q) + \zeta(q,p+1,q) \\
&\quad + 2\zeta(q+1,p,q) - \zeta(q+1)\zeta(p,q) - \zeta(q)\{\zeta(q,p+1) + \zeta(p+q+1)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{C}(p,q) &= -\zeta(q+1)\zeta(q,p) + \zeta(p)\zeta(q,q+1) + \zeta(p)\zeta(2q+1) + \zeta(q,q,p+1) \\
&\quad + \zeta(2q,p+1) + \zeta(q,p+q+1) + \zeta(p+2q+1) + \zeta(1,p+2q) \\
&\quad + \zeta(1,q,p+q) + \zeta(q,1,p+q) + \zeta(q+1,p+q) + \zeta(1,2q,p) \\
&\quad + \zeta(2q,1,p) + \zeta(2q+1,p) + 2\zeta(q+1,q,p) + \zeta(q,q+1,p).
\end{aligned}
$$

We prove Theorem 1 in Section 2, Theorem 2 in Section 3, and finally Theorem 3 in Section 4.

2. Proof of Theorem 1

First we introduce two kinds of multiple zeta values with variables. For positive integers $\alpha_1, \alpha_2, \ldots, \alpha_r$ with $\alpha_r \geqslant 2$ and positive real numbers $x_1, x_2, \ldots, x_r$ no greater than 1, we let

$$H_\alpha(\mathbf{x}) = \sum_{n_r=0}^{\infty} \frac{1}{(n_r+x_r)^{\alpha_r}} \cdots \sum_{n_2=0}^{\widetilde{n}_3} \frac{1}{(n_2+x_2)^{\alpha_2}} \sum_{n_1=0}^{\widetilde{n}_2} \frac{1}{(n_1+x_1)^{\alpha_1}}, \tag{2.1}$$

where for $2 \leqslant j \leqslant r$,

$$\widetilde{n}_j = \begin{cases} n_j, & \text{if } x_{j-1} \leqslant x_j, \\ n_j - 1, & \text{if } x_{j-1} > x_j. \end{cases}$$

With the same conditions on $\alpha_1, \alpha_2, \ldots, \alpha_r$ and $x_1, x_2, \ldots, x_r$, we let

$$G_\alpha(\mathbf{x}) = \sum_{n_r=0}^{\infty} \frac{1}{(n_r+x_r)^{\alpha_r}} \cdots \sum_{n_2=0}^{\widetilde{n}_3} \frac{1}{(n_2+x_2)^{\alpha_2}} \sum_{n_1=0}^{\widehat{n}_r} \frac{1}{(n_1+x_1)^{\alpha_1}}, \tag{2.2}$$

where $\widetilde{n}_j$ for $2 \leqslant j \leqslant r$ is as defined before and

$$\widehat{n}_r = \begin{cases} n_r, & \text{if } x_1 \leqslant x_r, \\ n_r - 1, & \text{if } x_1 > x_r. \end{cases}$$

For example, for $0 < x < y < 1$, we have

$$H_{p,q}(x,x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^q} \sum_{j=0}^{k} \frac{1}{(j+x)^p}$$

and

$$G_{1,p,q}(1,x,y) = \sum_{k=0}^{\infty} \frac{1}{(k+y)^q} \sum_{j=0}^{k} \frac{1}{(j+x)^p} \sum_{\ell=0}^{k-1} \frac{1}{(\ell+1)}.$$

The identities of following theorem served as basic tools in the evaluation of quadruple Euler sums of odd weight. They remain useful in the cases considered here.

Theorem A ([14]). *For each positive integer n and positive real numbers x, y, z with $0 < x < y < z \leqslant 1$, we have*

$$\begin{aligned} G_{1,1,1,2n}(z,x,y,z) - H_{1,1,1,2n}(z,z-x,z-y,1) &= \sum_{\ell=2}^{2n-1} (-1)^{\ell+1}\zeta(2n+1-\ell)H_{1,1,\ell}(x,y,z) \\ &\quad - H_{1,1,2n}(x,y,z)\{\gamma+\psi(z)\} \\ &\quad + H_{1,1,2n}(z-x,z-y,1)\{-\psi(x)+\psi(z)\} \\ &\quad + H_{1,2n}(z-y,1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{z-y}{(j+x)(k+j+y)(k+j+z)} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} G_{1,1,2n+1}(z,x,y,z) + H_{1,1,1,2n+1}(z,z-x,z-y,1) &= \sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(2n+2-\ell)H_{1,1,\ell}(x,y,z) \\ &\quad - H_{1,1,2n+1}(x,y,z)\{\gamma+\psi(z)\} \\ &\quad - H_{1,1,2n+1}(z-x,z-y,1)\{-\psi(x)+\psi(z)\} \\ &\quad - H_{1,2n+1}(z-y,1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{z-y}{(j+x)(k+j+y)(k+j+z)}. \end{aligned} \tag{2.4}$$

Here $\psi(x)$ is the digamma function defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

where $\Gamma(x)$ is the gamma function.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Apply the partial differential operator

$$\frac{-1}{(2n-1)!}\left(\frac{\partial}{\partial x}\right)^{2n-1}$$

to both sides of (2.3) and then set $x = y = z = 1$. We obtain

$$\begin{aligned} 2[\zeta(2n,1,1,2n) + \zeta(1,2n,1,2n)] &= \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}\zeta(2n+1-\ell)\{S_{2n,1,\ell} + \zeta(\ell,1,2n)\} \\ &\quad + \zeta(2n)\zeta(1,1,2n) - C(2n), \end{aligned} \tag{2.5}$$

where $C(p)$ is defined in Theorem 1.

Next, we produce the stuffle formula from the product of $\zeta(1,1,2n)$ and $\zeta(2n)$. Rewrite the product as

$$\left(\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\sum_{j=1}^{k-1}\frac{1}{j}\sum_{\ell=1}^{j-1}\frac{1}{\ell}\right)\left(\sum_{q=1}^{\infty}\frac{1}{q^{2n}}\right).$$

We wish to insert the dummy variable q into the increasing sequence

$$\ell < j < k$$

to form quadruple or triple Euler sums. So we divide the range of q into intervals or a single number as

1. When $q > k$, the corresponding quadruple Euler sum is $\zeta(1,1,2n,2n)$.
2. When $q = k$, the corresponding triple Euler sum is $\zeta(1,1,4n)$.
3. When $j < q < k$, the corresponding quadruple Euler sum is $\zeta(1,1,2n,2n)$.
4. When $q = j$, the corresponding triple Euler sum is $\zeta(1,2n+1,2n)$.
5. When $\ell < q < j$, the corresponding quadruple Euler sum is $\zeta(1,2n,1,2n)$.
6. When $q = \ell$, the corresponding triple Euler sum is $\zeta(2n+1,1,2n)$.
7. When $q < \ell$, the corresponding quadruple Euler sum is $\zeta(2n,1,1,2n)$.

Then, we have

$$\begin{aligned}\zeta(1,1,2n)\zeta(2n) &= 2\zeta(1,1,2n,2n)+\zeta(1,2n,1,2n)+\zeta(2n,1,1,2n)\\ &\quad+\zeta(2n+1,1,2n)+\zeta(1,2n+1,2n)+\zeta(1,1,4n),\end{aligned}$$

which we rewrite as

$$\begin{aligned}&2\zeta(1,1,2n,2n)+\zeta(1,2n,1,2n)+\zeta(2n,1,1,2n)\\ &\quad=\zeta(1,1,2n)\zeta(2n)-\zeta(2n+1,1,2n)-\zeta(1,2n+1,2n)-\zeta(1,1,4n).\end{aligned}\tag{2.6}$$

Finally, we produce the stuffle formula from the product of $\zeta(1,2n)$ with itself. Write the product as

$$\left(\sum_{k_1=1}^{\infty}\frac{1}{k_1^{2n}}\sum_{j_1=1}^{k_1-1}\frac{1}{j_1}\right)\left(\sum_{k_2=1}^{\infty}\frac{1}{k_2^{2n}}\sum_{j_2=1}^{k_2-1}\frac{1}{j_2}\right).$$

According to $k_2 > k_1$, $k_2 = k_1$, $j_1 < k_2 < k_1$, $k_2 = j_1$ and $k_2 < j_1$, we get five multiple zeta values of the form $G_\alpha(\mathbf{x})$ or the form $H_\alpha(\mathbf{x})$. Decomposing these multiple zeta values into the usual multiple zeta values, we obtain

$$\begin{aligned}\zeta(1,2n)\zeta(1,2n) &= 4\zeta(1,1,2n,2n)+2\zeta(1,2n,1,2n)+2\zeta(1,2n+1,2n)\\ &\quad+2\zeta(2,2n,2n)+2\zeta(1,1,4n)+\zeta(2,4n),\end{aligned}$$

which we rewrite as

$$\begin{aligned}4\zeta(1,1,2n,2n)+2\zeta(1,2n,1,2n) &= \zeta(1,2n)\zeta(1,2n)-2\zeta(1,2n+1,2n)\\ &\quad-2\zeta(2,2n,2n)-2\zeta(1,1,4n)-\zeta(2,4n).\end{aligned}\tag{2.7}$$

Solving the system of linear equations (2.5), (2.6), (2.7) with $\zeta(1,1,2n,2n)$, $\zeta(1,2n,1,2n)$ and $\zeta(2n,1,1,2n)$ as the unknowns, we deduce the assertion of Theorem 1 for $p=2n$. To get the assertion for $p=2n+1$, we still employ the stuffle formulae (2.6) and (2.7) with $2n$ replaced by $2n+1$. Applying the differential operator

$$\frac{1}{(2n)!}\left(\frac{\partial}{\partial x}\right)^{2n}$$

to both sides of (2-2) and then setting $x=y=z=1$, we get another linear equation in $\zeta(1,1,2n+1,2n+1)$, $\zeta(1,2n+1,1,2n+1)$ and $\zeta(2n+1,1,1,2n+1)$. Our assertion then follows by solving the linear system. ■

The general procedure to produce stuffle formulae from products of two multiple zeta values can be extended as following:

Corollary 1. *For positive integers p, q, r and s with $p,s\geqslant 2$, we have*

$$\begin{aligned}\zeta(p)\zeta(q,r,s) &= \zeta(p,q,r,s)+\zeta(q,p,r,s)+\zeta(q,r,p,s)+\zeta(q,r,s,p)\\ &\quad+\zeta(p+q,r,s)+\zeta(q,p+r,s)+\zeta(q,r,p+s)\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\zeta(q,p)\zeta(r,s) &= \zeta(q,p,r,s)+\zeta(r,s,q,p)+\zeta(q,r,p,s)+\zeta(q,r,s,p)\\ &\quad+\zeta(r,q,p,s)+\zeta(r,q,s,p)+\zeta(q,p+r,s)+\zeta(r,s+q,p)\\ &\quad+\zeta(q+r,p,s)+\zeta(q+r,s,p)+\zeta(r,q,p+s)+\zeta(q,r,p+s)\\ &\quad+\zeta(q+r,p+s).\end{aligned} \tag{2.9}$$

3. Proof of Theorem 2

To prove Theorem 2, we need three independent linear relations among $\zeta(1,p,p,p)$, $\zeta(p,1,p,p)$ and $\zeta(p,p,1,p)$. Two relations come from the stuffle formulae of $\zeta(p)\zeta(p,1,p)$ and $\zeta(p)\zeta(1,p,p)$ and one relation comes from the differentiation of (2-2). The resulting system of linear equations is given as follows.

$$\begin{aligned}2\zeta(p,p,1,p)+2\zeta(p,1,p,p) &= \zeta(p)\zeta(p,1,p)-\zeta(2p,1,p)\\ &\quad-\zeta(p,p+1,p)-\zeta(p,1,2p),\\ \zeta(p,1,p,p)+3\zeta(1,p,p,p) &= \zeta(p)\zeta(1,p,p)-\zeta(p+1,p,p)\\ &\quad-\zeta(1,2p,p)-\zeta(1,p,2p)\end{aligned}$$

and

$$\begin{aligned}2\zeta(1,p,p,p)+\zeta(p,1,p,p)+\zeta(p,p,1,p) &= \sum_{\ell=2}^{p-1}(-1)^{\ell}\zeta(p+1-\ell)\left\{S_{p,p,\ell}+\zeta(\ell,p,p)\right\}\\ &\quad-\sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(p-\ell,p)S_{p,\ell+1}\\ &\quad-\zeta(p)\zeta(1,p,p)-\widetilde{C}(p),\end{aligned}$$

where $\widetilde{C}(p)$ is as given in Theorem 2.

4. Further evaluations and proof of Theorem 3

It is interesting to note that neither relations from stuffle formulae nor relations from differentiations of (2.3) or (2.4) are enough to determine the value of $\zeta(1,1,p,p)$, $\zeta(1,p,1,p)$ and $\zeta(1,p,p,p)$, $\zeta(p,1,p,p)$, $\zeta(p,p,1,p)$ uniquely. Besides, the conjecture in the page 13 of [6] is not quite true since we can find more quadruple Euler sums which can be evaluated and which are outside the exceptional cases. For example, quadruple Euler sums of the form:

$$\zeta(1,2n,2m+1,2n), \qquad \zeta(2m,2m,1,2n+1) \qquad \text{and} \qquad \zeta(2n+1,1,2m,2m).$$

In addition, for positive integers p and q with $p \geqslant 2$, that $\zeta(p,q,q,p)$ can be reduced has been proven in [7], but without the reduction formulae. In the following, we proceed to evaluate these quadruple Euler sums.

Proposition 1. *For positive integers p, q, r and s with $p,s \geqslant 2$, we have*

$$\begin{aligned}\zeta(p,q,r,s)+\zeta(s,r,q,p) = {} & \zeta(p)\zeta(q,r,s)+\zeta(s)\zeta(r,q,p)-\zeta(q,p)\zeta(r,s)\\ & +\zeta(q+r,p+s)+\zeta(q+r,p,s)+\zeta(q+r,s,p)\\ & -\zeta(p+q,r,s)-\zeta(s+r,q,p).\end{aligned} \tag{4.1}$$

In particular,

$$2\zeta(p,q,q,p) = 2\zeta(p)\zeta(q,q,p)-\zeta(q,p)^2+\zeta(2q,2p)+2\zeta(2q,p,p)-2\zeta(p+q,q,p).$$

Proof. From Corollary 1, we rewrite (2.8) as

$$\begin{aligned}\zeta(p,q,r,s) = {} & \zeta(p)\zeta(q,r,s)-\zeta(q,p,r,s)-\zeta(q,r,p,s)-\zeta(q,r,s,p)\\ & -\zeta(p+q,r,s)-\zeta(q,p+r,s)-\zeta(q,r,p+s)\end{aligned}$$

and we also have

$$\begin{aligned}\zeta(s,r,q,p) = {} & \zeta(s)\zeta(r,q,p)-\zeta(r,s,q,p)-\zeta(r,q,s,p)-\zeta(r,q,p,s)\\ & -\zeta(s+r,q,p)-\zeta(r,s+q,p)-\zeta(r,q,p+s).\end{aligned}$$

The sum of $\zeta(p,q,r,s)$ and $\zeta(s,r,q,p)$ is equal to

$$\begin{aligned}\zeta(p,q,r,s)+\zeta(s,r,q,p) = {} & \zeta(p)\zeta(q,r,s)+\zeta(s)\zeta(r,q,p)-A(p,q,r,s)\\ & -\zeta(p+q,r,s)-\zeta(q,p+r,s)-\zeta(q,r,p+s)\\ & -\zeta(s+r,q,p)-\zeta(r,s+q,p)-\zeta(r,q,p+s),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}A(p,q,r,s) = {} & \zeta(q,p,r,s)+\zeta(q,r,p,s)+\zeta(q,r,s,p)\\ & +\zeta(r,s,q,p)+\zeta(r,q,s,p)+\zeta(r,q,p,s).\end{aligned}$$

On the other hand, we can rewrite (2.9) as

$$\begin{aligned} A(p,q,r,s) &= \zeta(q,p)\zeta(r,s) - \zeta(q,p+r,s) - \zeta(r,s+q,p) - \zeta(q+r,p,s) \\ &\quad - \zeta(q+r,s,p) - \zeta(r,q,p+s) - \zeta(q,r,p+s) - \zeta(q+r,p+s). \end{aligned} \tag{4.3}$$

Our assertion now follows by substituting of (4.3) into (4.2). ■

We are now ready to give the proof of Theorem 3.

Proof of Theorem 3. Let $p = 2m + 1$ and $q = 2n$. We apply the partial differential operator

$$\frac{-1}{(q-1)!(p-1)!}\left(\frac{\partial}{\partial x}\right)^{q-1}\left(\frac{\partial}{\partial y}\right)^{p-1}$$

to both sides of (2.3), and then set $x = y = z = 1$. we obtain

$$\begin{aligned} &2\zeta(1,q,p,q) + \zeta(q,1,p,q) + \zeta(q,p,1,q) \\ &\quad = \sum_{\ell=2}^{q-1}(-1)^{\ell+1}\zeta(q+1-\ell)\left\{S_{q,p,\ell} + \zeta(\ell,p,q)\right\} \\ &\quad\quad + \sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(p-\ell,q)S_{q,\ell+1} + \zeta(q)\zeta(1,p,q) - C(p,q), \end{aligned} \tag{4.4}$$

where $C(p, q)$ is as given in Theorem 3. Since we have

$$\begin{aligned} \zeta(q,1,p,q) + \zeta(q,p,1,q) &= \zeta(q)\zeta(1,p,q) + \zeta(q)\zeta(p,1,q) - \zeta(1,q)\zeta(p,q) \\ &\quad + \zeta(p+1,2q) + 2\zeta(p+1,q,q) \\ &\quad - \zeta(p+q,1,q) - \zeta(q+1,p,q) \end{aligned} \tag{4.5}$$

from Proposition 1, then combining (4.4) and (4.5) yields the value of $\zeta(1, q, p, q)$.

Next, let $p = 2n + 1$ and $q = 2m$. We apply the partial differential operator

$$\frac{1}{(q-1)!(q-1)!}\left(\frac{\partial^2}{\partial x\partial y}\right)^{q-1}$$

to both sides of (2.4), and then set $x = y = z = 1$. We obtain

$$\begin{aligned} &\zeta(q,1,q,p) + 2\zeta(1,q,q,p) + \zeta(q,q,1,p) \\ &\quad = \sum_{\ell=1}^{q-1}(-1)^{\ell}\zeta(q-\ell,p)S_{q,\ell+1} + \sum_{\ell=1}^{q-1}(-1)^{\ell+1}\zeta(\ell+1)\zeta(q-\ell,q,p) \\ &\quad\quad + \sum_{\ell=2}^{p-1}(-1)^{\ell}\zeta(p+1-\ell)S_{q,q,\ell} - \widetilde{C}(p,q), \end{aligned}$$

where $\widetilde{C}(p,q)$ is as given in Theorem 3. Finally, consider $\zeta(q)\zeta(1,q,p)$ and rewrite the stuffle formula as follows

$$\begin{aligned}\zeta(q,1,q,p)+2\zeta(1,q,q,p)+\zeta(1,q,p,q)&=\zeta(q)\zeta(1,q,p)-\zeta(q+1,q,p)\\&\quad-\zeta(1,2q,p)-\zeta(1,q,p+q)\end{aligned}$$

Hence, we can reduce $\zeta(q,1,q,p)+2\zeta(1,q,q,p)$ from (4-13) since we have the value of $\zeta(1,q,p,q)$ above, then derive $\zeta(q,q,1,p)$ from (4-12), and therefore $\zeta(p,1,q,q)$ follows by (4.1). ■

5. Applications

The reflection formula for $r=2$ is given by

$$\zeta(p,q)+\zeta(q,p)=\zeta(p)\zeta(q)-\zeta(p+q),$$

which follows from the definition of $\zeta(p,q)$ by elementary considerations. Markett [12] obtained the reflection formula for multiple zeta values of depth 3, namely,

$$\zeta(p,q,r)-\zeta(r,q,p)=\zeta(p)\zeta(q,r)-\zeta(q,p)\zeta(r)-\zeta(p+q,r)+\zeta(q+r,p).$$

Borwein *et. al* [7] considered the general case. They proved that

$$\zeta(s_1,s_2,\ldots,s_k)+(-1)^k\zeta(s_k,\ldots,s_2,s_1)$$

can expressed in terms of multiple zeta values of depth less than k when $s_1\geqslant 2$ and $s_k\geqslant 2$. However, no explicit reduction formula was given.

Here, we shall employ stuffle identities of products of two multiple zeta values to produce an explicit reduction formula for $\zeta(s_1,s_2,\ldots,s_k)+(-1)^k\zeta(s_k,\ldots,s_2,s_1)$. As mentioned before, the depth-length shuffle formula or stuffle identity produced from two multiple zeta values

$$\zeta(\vec{s})=\zeta(s_1,s_2,\ldots,s_m)=\sum_{1\leqslant k_1<k_2\cdots<k_m}k_1^{-s_1}k_2^{-s_2}\cdots k_m^{-s_m}$$

and

$$\zeta(\vec{t})=\zeta(t_1,t_2,\ldots,t_n)=\sum_{1\leqslant j_1<j_2\cdots<j_n}j_1^{-t_1}j_2^{-t_2}\cdots j_n^{-t_n}$$

is the sum over all possible insertions of $j_1,j_2,\ldots,j_n$ into the sequence $1\leqslant k_1<k_2<\cdots<k_m$ which preserve the orders of $j_1,j_2,\ldots,j_n$. On the other hand, this is equivalent to consider the stuffle product $\vec{s}*\vec{t}$ of two strings $\vec{s}$ and $\vec{t}$, which is defined recursively by [4]

$$\begin{aligned}(a,\vec{s})*(b,\vec{t})&=\{(a,\vec{u}):\vec{u}\in\vec{s}*(b,\vec{t})\}\cup\{(b,\vec{u}):\vec{u}\in(a,\vec{s})*\vec{t}\}\\&\quad\cup\{(a+b,\vec{u}):\vec{u}\in\vec{s}*\vec{t}\}\end{aligned}\tag{5.1}$$

with initial conditions $\vec{s} * () = () * \vec{s} = \vec{s}$. The stuffle identity of the product of two multiple zeta values $\zeta(\vec{s})$ and $\zeta(\vec{t})$ is then given by

$$\zeta(\vec{s})\zeta(\vec{t}) = \sum_{\vec{u} \in \vec{s} * \vec{t}} \zeta(\vec{u}).$$

Hence, we have the following result.

Theorem 4. *For positive integers $s_1, s_2, \ldots, s_k$ with $s_1, s_k \geqslant 2$, we have*

$$\begin{aligned} \zeta(s_1, s_2, \ldots, s_k) &+ (-1)^k \zeta(s_k, \ldots, s_2, s_1) \\ &= \sum_{i=1}^{k-1} (-1)^{i-1} \zeta(s_i, \ldots, s_2, s_1) \zeta(s_{i+1}, s_{i+2}, \ldots, s_k) \\ &+ \sum_{i=1}^{k-1} (-1)^i \sum_{\vec{w} \in \vec{u}_i * \vec{v}_i} \zeta(s_i + s_{i+1}, \vec{w}), \end{aligned} \tag{5.2}$$

where $\vec{u}_i$ and $\vec{v}_i$ are given by $(s_{i-1}, \ldots, s_2, s_1)$ and $(s_{i+2}, s_{i+3}, \ldots, s_k)$, respectively.

Proof. For $1 \leqslant i \leqslant k-1$, $\vec{u}_i$ and $\vec{v}_i$ are as mentioned above, we consider the product of two multiple zeta values as following

$$\zeta(s_i, \ldots, s_2, s_1)\zeta(s_{i+1}, s_{i+2}, \ldots, s_k),$$

which is equal to

$$\sum_{\vec{w}_1 \in \vec{u}_i * \vec{v}_{i-1}} \zeta(s_i, \vec{w}_1) + \sum_{\vec{w}_2 \in \vec{u}_{i+1} * \vec{v}_i} \zeta(s_{i+1}, \vec{w}_2) + \sum_{\vec{w}_3 \in \vec{u}_i * \vec{v}_i} \zeta(s_i + s_{i+1}, \vec{w}_3)$$

by the stuffle multiplication rule. Note that for $i = 1$ and $i = k-1$, the first and second summations above become $\zeta(s_1, s_2, \ldots, s_k)$ and $\zeta(s_k, \ldots, s_2, s_1)$, respectively. Next, consider the alternating sum for these stuffle formulas, that is,

$$\begin{aligned} &\sum_{i=1}^{k-1} (-1)^{i+1} \zeta(s_i, \ldots, s_2, s_1)\zeta(s_{i+1}, s_{i+2}, \ldots, s_k) \\ &\quad= \sum_{i=1}^{k-1} (-1)^{i+1} \sum_{\vec{w}_1 \in \vec{u}_i * \vec{v}_{i-1}} \zeta(s_i, \vec{w}_1) + \sum_{i=1}^{k-1} (-1)^{i+1} \sum_{\vec{w}_2 \in \vec{u}_{i+1} * \vec{v}_i} \zeta(s_{i+1}, \vec{w}_2) \\ &\quad\quad+ \sum_{i=1}^{k-1} (-1)^{i+1} \sum_{\vec{w} \in \vec{u}_i * \vec{v}_i} \zeta(s_i + s_{i+1}, \vec{w}). \end{aligned} \tag{5.3}$$

As desired, the term

$$\sum_{i=2}^{k-1} (-1)^{i+1} \sum_{\vec{w}_1 \in \vec{u}_i * \vec{v}_{i-1}} \zeta(s_i, \vec{w}_1)$$

is canceled out by

$$\sum_{i=1}^{k-2}(-1)^i \sum_{\vec{w}_2\in\vec{u}_{i+1}*\vec{v}_i} \zeta(s_{i+1},\vec{w}_2).$$

It turns out that

$$\begin{aligned}\sum_{i=1}^{k-1}&(-1)^{i+1}\zeta(s_i,\dots,s_2,s_1)\zeta(s_{i+1},s_{i+2},\dots,s_k)\\ &=\zeta(s_1,s_2,\dots,s_k)+(-1)^k\zeta(s_k,\dots,s_2,s_1)\\ &\quad+\sum_{i=1}^{k-1}(-1)^{i+1}\sum_{\vec{w}\in\vec{u}_i*\vec{v}_i}\zeta(s_i+s_{i+1},\vec{w}).\end{aligned}\tag{5.4}$$

Therefore, our assertion follows by simple rearrangement. ■

Immediately, we can obtain the value of $\zeta(s_1,s_2,\dots,s_k,s_k,\dots,s_2,s_1)$ in terms of lower depth.

Corollary 2. *For positive integers $s_1,s_2,\dots,s_k$ with $s_1\geqslant 2$, we have*

$$\begin{aligned}\zeta(s_1,&s_2,\dots,s_k,s_k,\dots,s_2,s_1)\\ &=\sum_{i=1}^{k-1}(-1)^{i+1}\zeta(s_i,\dots,s_1)\zeta(s_{i+1},\dots,s_k,s_k,\dots,s_2,s_1)\\ &\quad+\frac{(-1)^{k+1}}{2}\left[\zeta^2(s_k,\dots,s_2,s_1)-\sum_{\vec{w}\in\vec{u}_k*\vec{u}_k}\zeta(2s_k,\vec{w})\right]\\ &\quad+\sum_{i=1}^{k-1}(-1)^i\sum_{\vec{w}\in\vec{u}_i*\vec{v}_i}\zeta(s_i+s_{i+1},\vec{w}),\end{aligned}\tag{5.5}$$

where $\vec{u}_i=(s_{i-1},\dots,s_2,s_1)$ and $\vec{v}_i=(s_{i+2},s_{i+3},\dots,s_k,s_k,\dots,s_2,s_1)$.

For example, we have

$$\begin{aligned}\zeta(s_1,s_2,s_3,s_3,s_2,s_1)=&\zeta(s_1)\zeta(s_2,s_3,s_3,s_2,s_1)-\zeta(s_1+s_2,s_3,s_3,s_2,s_1)\\ &-\zeta(s_2,s_1)\zeta(s_3,s_3,s_2,s_1)+\sum_{\vec{w}\in s_1*(s_3,s_2,s_1)}\zeta(s_2+s_3,\vec{w})\\ &+\frac{1}{2}\left[\zeta^2(s_3,s_2,s_1)-\sum_{\vec{w}\in(s_2,s_1)*(s_2,s_1)}\zeta(2s_3,\vec{w})\right].\end{aligned}$$

Acknowledgements. The authors would like to thank the anonymous referees for their valuable suggestions for improving the original version of this paper.

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Received: 5 November 2010