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NEW IDENTITIES FOR RAMANUJAN'S CUBIC CONTINUED FRACTION

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Abstract: In this paper, we present some new identities providing relations between Ramanujan's cubic continued fraction $V(q)$ and the other three continued fractions $V(q^9)$, $V(q^{17})$ and $V(q^{19})$. In the process, we establish some new modular equations for the ratios of Ramanujan's theta functions. We also establish some general formulas for the explicit evaluations of ratios of Ramanujan's theta functions.

Keywords: Cubic continued fraction, Modular equation, Theta-function

1. Introduction

In Chapter 16, of his second notebook [13], [5, pp.257-262], S. Ramanujan develops the theory of theta function and his theta function is defined by

$$\begin{aligned} f(a,b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1, \\ &= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \end{aligned}$$

where, $(a;q)_\infty := \prod_{n=1}^{\infty} (1 - aq^{n-1}), \ |q| < 1.$

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Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{1.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.2}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.3}$$

On page 366 of his 'lost' notebook [14], Ramanujan has recorded cubic continued fraction as follows:

$$V(q) := \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+}{} \cdots, \qquad |q|<1, \tag{1.4}$$

and asserted by stating "*and many results analogous to the previous continued fraction.*" Motivated by this fact, H. H. Chan [7] developed the theory of the cubic continued fraction and established relations between $V(q)$ and other three continued fractions $V(-q)$, $V(q^2)$ and $V(q^3)$. In [3], N. D. Baruah established modular equations connecting $V(q)$ with $V(q^5)$ and $V(q^7)$ respectively. In [9], M. S. Mahadeva Naika established several results of cubic continued fraction which are analogous to Rogers–Ramanujan continued fraction [13]. In [8], B. Cho, J. K. Koo and Y. K. Park have extended the results of Chan [7] and Baruah [3] and established relations connecting $V(q)$ with $V(q^{11})$, $V(q^{13})$ and $V(q^{17})$ respectively. In [17], J. Yi and H. S. Sim established some new modular equations by employing the theory of modular forms. For more details on Ramanujan's cubic continued fraction, one can see [1], [2], [11]. Motivated by these works, in this paper, we establish some new modular equations connecting $V(q)$ with $V(q^9)$, $V(q^{17})$ and $V(q^{19})$ respectively.

In [15], Yi introduced two parameterizations $h_{k,n}$ and $h'_{k,n}$ as follows:

$$h_{k,n} := \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})}, \tag{1.5}$$

$$h'_{k,n} := \frac{\varphi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(-e^{-\pi\sqrt{nk}})} \tag{1.6}$$

and established several properties as well as explicit evaluations of $h_{k,n}$ and $h'_{k,n}$ for different positive rational values of n and k.

In [4], Baruah and Nipen Saikia and [16] Yi, Y. Lee and D. H. Paek have defined two parameters $l_{k,n}$ and $l'_{k,n}$ as follows:

$$l_{k,n} := \frac{\psi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(-e^{-\pi\sqrt{nk}})}, \tag{1.7}$$

and

$$l'_{k,n} := \frac{\psi(e^{-\pi\sqrt{n/k}})}{k^{1/4}e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(e^{-\pi\sqrt{nk}})}. \tag{1.8}$$

They have also established several properties as well as explicit evaluations of $l_{k,n}$ and $l'_{k,n}$ for different positive rational values of n and k. In [12], Mahadeva Naika, K. Sushan Bairy and M. Manjunatha have established several new modular equations of degree 4 and established general formulas for explicit evaluations of $h_{4,n}$. In [10], Mahadeva Naika, S. Chandankumar and Bairy have established several new modular equations of degree 9, and also established several general formulas for the explicit evaluations of $l_{9,n}$ and $l'_{9,n}$.

In Section 3, we establish some new modular equations for the ratios of Ramanujan's theta function. In Section 4, we establish some general formulas for the explicit evaluations of ratios of Ramanujan's theta functions by using the results obtained in Section 3. In Section 5, we establish some new relations connecting $V(q)$ with other three continued fractions $V(q^9)$, $V(q^{17})$ and $V(q^{19})$ respectively.

2. Preliminary results

In this section, we collect some relevant identities that are useful in proving our main results.

Lemma 2.1 ([5, Ch.20, Entry 1(iii), p. 345]). *We have*

$$\frac{\varphi(q^{1/3})}{\varphi(q^3)} = 1 + \left(\frac{\varphi^4(q)}{\varphi^4(q^3)} - 1\right)^{1/3}, \tag{2.1}$$

$$\frac{3\varphi(q^9)}{\varphi(q)} = 1 + \left(\frac{9\varphi^4(q^3)}{\varphi^4(q)} - 1\right)^{1/3}. \tag{2.2}$$

Lemma 2.2. *If* $P = \dfrac{\varphi(q)}{\varphi(q^9)}$ *and* $Q = \dfrac{\varphi(q^3)}{\varphi(q^{27})}$, *then*

$$\left(3 - P - \frac{3}{P}\right)\left(3 - Q - \frac{3}{Q}\right) = \left(\frac{Q}{P}\right)^2. \tag{2.3}$$

Proof. Using equations (2.1) and (2.2), we arrive at the equation (2.3). ■

Lemma 2.3 ([17, Throrem 2.1]). *Let* $P = \dfrac{f(-q)}{q^{1/12}f(-q^3)}$ *and* $Q = \dfrac{f(-q^{17})}{q^{17/12}f(-q^{51})}$, *then*

$$\begin{aligned}
&\left(\frac{Q}{P}\right)^9-\left(\frac{P}{Q}\right)^9-238\left[\left(\frac{Q}{P}\right)^6+\left(\frac{P}{Q}\right)^6\right]+1853\left[\left(\frac{Q}{P}\right)^3-\left(\frac{P}{Q}\right)^3\right]\\
&-17\left(\frac{Q^8}{P^4}+\frac{3^2P^4}{Q^8}\right)-17\left(\frac{3^2Q^4}{P^8}+\frac{P^8}{Q^4}\right)+34\left(Q^7P-\frac{3^4}{Q^7P}\right)\\
&-34\left(QP^7-\frac{3^4}{QP^7}\right)-442\left(\frac{Q^5}{P}-\frac{3^2P}{Q^5}\right)-442\left(\frac{3^2Q}{P^5}-\frac{P^5}{Q}\right)\\
=&\,(PQ)^8+\left(\frac{3}{PQ}\right)^8-34\left[(PQ)^6+\left(\frac{3}{PQ}\right)^6\right]\\
&+425\left[(PQ)^4+\left(\frac{3}{PQ}\right)^4\right]-2380\left[(PQ)^2+\left(\frac{3}{PQ}\right)^2\right]+8568.
\end{aligned}\tag{2.4}$$

Lemma 2.4 ([17, Theorem 2.3]). *Let*

$$P=\frac{f(-q)}{q^{1/12}f(-q^3)}\qquad and \qquad Q=\frac{f(-q^{19})}{q^{19/12}f(-q^{57})},$$

then

$$\begin{aligned}
&\left(\frac{Q}{P}\right)^{10}-\left(\frac{P}{Q}\right)^{10}-76\left[\left(\frac{Q}{P}\right)^8-\left(\frac{P}{Q}\right)^8\right]+912\left(\left(\frac{Q}{P}\right)^6-\left(\frac{P}{Q}\right)^6\right)\\
&+6650\left[\left(\frac{Q}{P}\right)^4-\left(\frac{P}{Q}\right)^4\right]+9481\left[\left(\frac{Q}{P}\right)^2-\left(\frac{P}{Q}\right)^2\right]-570\left(\frac{3^3Q}{P^7}+\frac{P^7}{Q}\right)\\
&-19\left(\frac{Q^9}{P^3}+\frac{3^3P^3}{Q^9}\right)-19\left(\frac{3^3Q^3}{P^9}+\frac{P^9}{Q^3}\right)-570\left(\frac{Q^7}{P}+\frac{3^3P}{Q^7}\right)\\
&+19\left(Q^8P^4-\frac{3^6}{Q^8P^4}\right)-19\left(Q^4P^8-\frac{3^6}{Q^4P^8}\right)-2166\left(Q^5P+\frac{3^3}{Q^5P}\right)\\
&-2166\left(QP^5+\frac{3^3}{QP^5}\right)=(PQ)^9+\left(\frac{3}{PQ}\right)^9+3211\left[(PQ)^3+\left(\frac{3}{PQ}\right)^3\right].
\end{aligned}\tag{2.5}$$

Lemma 2.5 ([5, Ch. 16, Entry 24(ii), p. 39]). *We have*

$$f^3(-q)=\varphi^2(-q)\psi(q).\tag{2.6}$$

Lemma 2.6 ([5, Ch. 20, Entry 1 (i), p. 345]). *We have*

$$1+\frac{1}{V(q)}=\frac{\psi^4(q)}{q\psi^4(q^3)},\tag{2.7}$$

where $V(q)$ is defined as in the equation (1.4).

Lemma 2.7 ([1, Theorem 5.1]). *If* $P = \dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^3)}$, *then*

$$Q^4 + P^4Q^4 = 9 + P^4. \tag{2.8}$$

Lemma 2.8 ([15, Theorem 2.2(ii)]). *We have*

$$h_{k,n}h_{k,1/n} = 1. \tag{2.9}$$

3. New modular equations for the ratios of Ramanujan's theta functions

In this section, we establish some new modular equations of degree 3 for the ratios of Ramanujan's theta functions.

Theorem 3.1. *If* $P = \dfrac{\varphi(q)}{\varphi(q^3)}$ *and* $Q = \dfrac{\varphi(q^9)}{\varphi(q^{27})}$, *then*

$$\begin{aligned}
&\frac{Q^5}{P^5} + 9\left(\frac{Q^3}{P^3} - \frac{Q^4}{P^4}\right) + 9\left(\frac{5Q}{P} + \frac{9P}{Q}\right) + 9\left(Q^4 + \frac{9}{P^4}\right)\\
&+ 3\left(Q^2 + 3P^2\right)\left(PQ + \frac{9}{P^3Q^3}\right)\\
&= 135 + 3\left(\frac{Q^2}{P^2} + \frac{9P^2}{Q^2}\right) + \left(P^4Q^4 + \frac{3^4}{P^4Q^4}\right)\\
&+ 9\left(P^2Q^2 + \frac{3^2}{P^2Q^2}\right).
\end{aligned} \tag{3.1}$$

Proof. Equations (2.1) and (2.2) can be written as,

$$\begin{aligned}
\frac{\varphi(q^3)}{\varphi(q^{27})} &= 1 + \left(\frac{\varphi^4(q^9)}{\varphi^4(q^{27})} - 1\right)^{1/3},\\
\frac{3\varphi(q^9)}{\varphi(q)} &= 1 + \left(\frac{9\varphi^4(q^3)}{\varphi^4(q)} - 1\right)^{1/3}.
\end{aligned} \tag{3.2}$$

Using the above equation (3.2) in the equation (2.3), we deduce that

$$\begin{aligned}
&3nP^4 - 3P^4 + 6n^2mP^4 - 3m^2P^4 + 6nm^2P^4 - 3n^2P^4\\
&\quad + 3mP^4 + 3Q^4 + m^2P^4Q^4 + mP^4Q^4 + 9n^2 + 9n = 0,
\end{aligned} \tag{3.3}$$

where $m = \left(\dfrac{9}{P^4} - 1\right)^{1/3}$ and $n = \left(Q^4 - 1\right)^{1/3}$.

Solving for m and then cubing both sides, we deduce that

$$\begin{aligned}
&729P^4 + 81Q^4 + 81n^2 + 81n - 729nP^4 + 729n^2P^4 - 999Q^4P^4\\
&+ 30Q^8n^2P^4 + 42nP^4Q^8 + 27Q^4n^2P^4 + P^{12}Q^8n^2 + 5P^{12}nQ^8\\
&- 12P^8n^2Q^8 - 51P^8Q^8n + 9n^2P^{12}Q^4 - 108n^2Q^4 - 54Q^4n - Q^{12}\\
&- 9n^2Q^8 - 36Q^8n + 10P^{12}Q^8 - 111Q^8P^8 - 9P^{12}Q^4 - 81Q^8\\
&+ 108P^8Q^4 + 486nP^4Q^4 + 273P^4Q^8 - 108P^8Q^4n^2 = 0.
\end{aligned} \tag{3.4}$$

Again, solving for n and then cubing both sides, we deduce that

$$\begin{aligned}
&(81P + 81P^3Q^2 - 81Q^4P + 81P^6Q^3 + 135P^5Q^4 + Q^8P^9 + 81P^4Q \\
&+ 27P^2Q^3 + Q^9 - 9Q^8P^5 + 27P^7Q^2 + 45P^4Q^5 + 3Q^6P^3 + 9P^2Q^7 \\
&+ 9P^7Q^6 + 9PQ^8 + 9P^8Q^5 + 3P^6Q^7) \\
\times &(81P - 81Q^4P - 81P^6Q^3 + 135P^5Q^4 + Q^8P^9 - 81P^4Q - 27P^2Q^3 \\
&- Q^9 - 9Q^8P^5 + 27P^7Q^2 + 81P^3Q^2 - 45P^4Q^5 + 3Q^6P^3 - 9P^2Q^7 \\
&+ 9P^7Q^6 + 9PQ^8 - 9P^8Q^5 - 3P^6Q^7) \\
\times &(6561P^2 - 13122P^2Q^4 + 24057P^6Q^4 - 26730Q^8P^6 - 810P^{12}Q^{10} \\
&+ 8019Q^8P^2 + Q^{18} - 1512Q^{12}P^2 + 63P^2Q^{16} + 729P^{14}Q^4 \\
&+ 11745P^{10}Q^8 + 2925P^6Q^{12} - 8748P^{10}Q^4 - 2970Q^{12}P^{10} - 972P^{14}Q^8 \\
&+ 297P^{14}Q^{12} - 168P^6Q^{16} - 18P^{14}Q^{16} + 99P^{10}Q^{16} + P^{18}Q^{16} \\
&- 324Q^{10}P^4 + 117Q^{14}P^4 + 5265P^8Q^{10} - 36P^8Q^{14} + 2187P^8Q^2 \\
&+ 729P^{12}Q^6 + 13365Q^6P^4 + 165P^{12}Q^{14} + 27P^{16}Q^{10} - 18P^{16}Q^{14} \\
&- 7290P^8Q^6 - 13122P^4Q^2) = 0.
\end{aligned} \tag{3.5}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem the second factor vanishes identically. This completes the proof. ■

Remark 1. The equation (3.1) holds for $P = \dfrac{\psi(q)}{q^{1/4}\psi(q^3)}$ and $Q = \dfrac{\psi(q^9)}{q^{9/4}\psi(q^{27})}$.

Theorem 3.2. *If* $P = \dfrac{\psi(q)\psi(q^{17})}{q^{9/2}\psi(q^3)\psi(q^{51})}$ *and* $Q = \dfrac{\psi(q)\psi(q^{51})}{q^{-4}\psi(q^3)\psi(q^{17})}$, *then*

$$\begin{aligned}
&Q^9 - \frac{1}{Q^9} + 34\left(Q^8 + \frac{1}{Q^8}\right) + 272\left(Q^7 - \frac{1}{Q^7}\right) + 238\left(Q^6 + \frac{1}{Q^6}\right) \\
&- 595\left(Q^5 - \frac{1}{Q^5}\right) - 510\left(Q^4 + \frac{1}{Q^4}\right) + 16303\left(Q^3 - \frac{1}{Q^3}\right) \\
&- 5202\left(Q^2 + \frac{1}{Q^2}\right) - 26911\left(Q - \frac{1}{Q}\right) + \left(P^8 + \frac{3^8}{P^8}\right) + 20230 \\
= {}&17\left\{\left(P^2 + \frac{3^2}{P^2}\right)\left[7\left(Q^6 + \frac{1}{Q^6}\right) + 28\left(Q^5 - \frac{1}{Q^5}\right) + 34\left(Q^4 + \frac{1}{Q^4}\right)\right.\right. \\
&\left. -168\left(Q^3 - \frac{1}{Q^3}\right) + 160\left(Q^2 + \frac{1}{Q^2}\right) + 378\left(Q - \frac{1}{Q}\right) - 210\right] \\
&- \left(P^4 + \frac{3^4}{P^4}\right)\left[5\left(Q^4 + \frac{1}{Q^4}\right) + 21\left(Q^3 - \frac{1}{Q^3}\right) - 35\left(Q^2 + \frac{1}{Q^2}\right)\right. \\
&\left.\left. -14\left(Q - \frac{1}{Q}\right) - 30\right] - \left(P^6 + \frac{3^6}{P^6}\right)\left[\left(Q^2 + \frac{1}{Q^2}\right) - 2\left(Q - \frac{1}{Q}\right) - 2\right]\right\}.
\end{aligned} \tag{3.6}$$

Proof. Using the equation (2.6) in the equation (2.4), we find that

$$\begin{aligned}
&238c^5d^5ab + 238a^5b^5cd - 1853c^4d^4a^2b^2 + 1853a^4b^4c^2d^2 + 8568a^3b^3c^3d^3\\
&- 34a^5b^5c^5d^5 - 24786abcd + 2754a^2b^2p^2p_1^2 + 153a^4b^4pp_1 + 153c^4d^4p_1p\\
&- 2754p^2c^2d^2p_1^2 + 6561pp_1 + 442c^4d^4p_1^2a^2b^2p^2 - 442a^4b^4p^2c^2d^2p_1^2\\
&- 3978a^3b^3pcdp_1 + 3978c^3d^3p_1abp + 17c^5d^5p_1^2abp^2 + 17a^5b^5p^2cdp_1^2\\
&- 21420a^2b^2pc^2d^2p_1 - 2380a^3b^3p^2c^3d^3p_1^2 - 34c^5d^5p_1a^3b^3p + 34c^3d^3p_1a^5b^5p\\
&+ 425a^4b^4pc^4d^4p_1 + 34425abp^2cdp_1^2 - c^6d^6 + a^6b^6 + a^5b^5p^2c^5d^5p_1^2 = 0,
\end{aligned} \tag{3.7}$$

where

$$a = \frac{\varphi^2(-q)}{\varphi^2(-q^3)}, \qquad b = \frac{\psi(q)}{q^{1/4}\psi(q^3)}, \qquad c = \frac{\varphi^2(-q^{17})}{\varphi^2(-q^{51})},$$
$$d = \frac{\psi(q^{17})}{q^{17/4}\psi(q^{51})}, \qquad p = \frac{f(-q)}{q^{1/12}f(-q^3)}, \qquad p_1 = \frac{f(-q^{17})}{q^{17/12}f(-q^{51})}.$$

Collecting the terms containing pp_1 on one side of the equation (3.7) and then cubing both sides, we deduce that

$$\begin{aligned}
&714c_2^8cd^{17}ab + 714a_2^8ab^{17}cd + 19758444939c_2^2cd^5ab + 19758444939a_2^2ab^5cd\\
&+ 2243133abc_2^6cd^{13} + 2243133a_2^6ab^{13}cd + 406552365a_2^4ab^9cd + 406552365c_2^4cd^9ab\\
&- 31093a_2^5b^{10}c_2^8d^{16} + 55006404858c_2^4d^8a_2^3b^6 - 420929877c_2^7d^{14}a_2^2b^4\\
&- 9563678423c_2^6d^{12}a_2^3b^6 - 167688c_2^8d^{16}a_2b^2 + 420929877a_2^7b^{14}c_2^2d^4\\
&+ 167688a_2^8b^{16}c_2d^2 + 141404912a_2^7b^{14}c_2^4d^8 + 3008487501a_2^3b^6c_2^2d^4\\
&+ 31093a_2^8b^{16}c_2^5d^{10} - 821508a_2^8b^{16}c_2^3d^6 - 598879332a_2^6b^{12}c_2d^2\\
&+ 103084180848a_2^5b^{10}c_2^2d^4 + 5661a_2^7b^{14}c_2^6d^{12} + 13164103527c_2^5d^{10}a_2^4b^8\\
&- 13164103527c_2^4d^8a_2^5b^{10} + 9563678423c_2^3d^6a_2^6b^{12} - 141404912c_2^7d^{14}a_2^4b^8\\
&+ 75454602c_2^6d^{12}a_2^5b^{10} + 821508c_2^8d^{16}a_2^3b^6 - 75454602c_2^5d^{10}a_2^6b^{12}\\
&+ 598879332c_2^6d^{12}a_2b^2 - 55006404858c_2^3d^6a_2^4b^8 - 103084180848c_2^5d^{10}a_2^2b^4\\
&- 829854687438c_2^2d^4a_2b^2 + 829854687438a_2^2b^4c_2d^2 - 16524095013a_2b^2c_2^4d^8\\
&- 1198678992966a_2ab^3c_2cd^3 + 3077a_2^2ab^5c_2^8cd^{17} + 3077a_2^8ab^{17}c_2^2cd^5 + 765a_2^4ab^9c_2^8cd^{17}\\
&+ 282429536481abcd - 3008487501c_2^3d^6a_2^2b^4 - 5661c_2^7d^{14}a_2^6b^{12}\\
&+ 16524095013a_2^4b^8c_2d^2 + 2142a_2^8b^{16}c_2^7d^{14} - 2142a_2^7b^{14}c_2^8d^{16} + a_2^8ab^{17}c_2^8cd^{17}\\
&- 3094a_2^7ab^{15}c_2^7cd^{15} + 230516343504c_2^4cd^9a_2^2ab^5 + 14701702c_2^7cd^{15}a_2ab^3\\
&+ 1921729062c_2^5cd^{11}a_2^3ab^7 + 3630060354c_2^6cd^{13}a_2^2ab^5 + 5609286c_2^7cd^{15}a_2^3ab^7\\
&+ 4089169494c_2^5cd^{11}a_2ab^3 + 3630060354c_2^2cd^5a_2^6ab^{13} + 728053975242a_2^3ab^7c_2cd^3\\
&+ 14701702a_2^7ab^{15}c_2cd^3 + 5609286a_2^7ab^{15}c_2^3cd^7 + 4089169494a_2^5ab^{11}c_2cd^3\\
&+ 316208976a_2^6ab^{13}c_2^4cd^9 + 6619358a_2^6ab^{13}c_2^6cd^{13} + 1369962a_2^7ab^{15}c_2^5cd^{11}
\end{aligned}$$

$$+ 230516343504a_2^4ab^9c_2^2cd^5 + 728053975242c_2^3cd^7a_2ab^3 + 1921729062c_2^3cd^7a_2^5ab^{11}$$
$$+ 25749161622c_2^4cd^9a_2^4ab^9 + 291817274c_2^5cd^{11}a_2^5ab^{11} + 212734792746c_2^3cd^7a_2^3ab^7$$
$$+ 316208976c_2^6cd^{13}a_2^4ab^9 + 1369962c_2^7cd^{15}a_2^5ab^{11} + 3517798234878a_2^2ab^5c_2^2cd^5$$
$$+ 765a_2^8ab^{17}c_2^4cd^9 + 51a_2^8ab^{17}c_2^6cd^{13} + 51a_2^6ab^{13}c_2^8cd^{17} - c_2^9d^{18} + a_2^9b^{18} = 0, \tag{3.8}$$

where

$$a_2 = \frac{\varphi^4(-q)}{\varphi^4(-q^3)} \qquad \text{and} \qquad c_2 = \frac{\varphi^4(-q^{17})}{\varphi^4(-q^{51})}.$$

Using the equation (2.8) in the equation (3.8), we obtain the equation (3.6). ∎

Remark 2. The equation (3.6) holds for

$$P = \frac{\varphi(q)\varphi(q^{17})}{\varphi(q^3)\varphi(q^{51})} \qquad \text{and} \qquad Q = \frac{\varphi(q)\varphi(q^{51})}{\varphi(q^3)\varphi(q^{17})}.$$

Theorem 3.3. *If* $P = \dfrac{\psi(q)\psi(q^{19})}{q^5\psi(q^3)\psi(q^{57})}$ *and* $Q = \dfrac{\psi(q)\psi(q^{57})}{q^{-9/2}\psi(q^3)\psi(q^{19})}$, *then*

$$\begin{aligned}
&Q^{10} - \frac{1}{Q^{10}} + \left(P^9 + \frac{3^9}{P^9}\right) + 19\left\{26\left(Q^8 - \frac{1}{Q^8}\right) + 135\left(Q^6 - \frac{1}{Q^6}\right)\right.\\
&+ 1916\left(Q^4 - \frac{1}{Q^4}\right) + 1144\left(Q^2 - \frac{1}{Q^2}\right) - \left(P^7 + \frac{3^7}{P^7}\right)\left[\left(Q^2 - \frac{1}{Q^2}\right) - 4\right]\\
&- 5\left(P^6 + \frac{3^6}{P^6}\right)\left[\left(Q^2 - \frac{1}{Q^2}\right)\right] + 3\left(P^5 + \frac{3^5}{P^5}\right)\left[2\left(Q^4 + \frac{1}{Q^4}\right)\right.\\
&\left.- 14\left(Q^2 + \frac{1}{Q^2}\right) + 11\right] + 2\left(P^4 + \frac{3^4}{P^4}\right)\left[26\left(Q^4 - \frac{1}{Q^4}\right) - \left(Q^2 - \frac{1}{Q^2}\right)\right]\\
&- 5\left(P^2 + \frac{3^2}{P^2}\right)\left[13\left(Q^6 - \frac{1}{Q^6}\right) - 4\left(Q^4 - \frac{1}{Q^4}\right) + 104\left(Q^2 - \frac{1}{Q^2}\right)\right]\\
&- \left(P^3 + \frac{3^3}{P^3}\right)\left[12\left(Q^6 + \frac{1}{Q^6}\right) - 65\left(Q^4 + \frac{1}{Q^4}\right) - 591\right.\\
&\left.+ 200\left(Q^2 + \frac{1}{Q^2}\right)\right] + 3\left(P + \frac{3}{P}\right)\left[2\left(Q^8 + \frac{1}{Q^8}\right) - 29\left(Q^6 + \frac{1}{Q^6}\right)\right.\\
&\left.\left.+ 7\left(Q^4 + \frac{1}{Q^4}\right) - 504\left(Q^2 + \frac{1}{Q^2}\right) + 697\right]\right\} = 0.
\end{aligned} \tag{3.9}$$

The proof of the equation (3.9) is similar to the proof of the equation (3.6) except in the place of the result (2.4), the result (2.5) is used.

Remark 3. The equation (3.9) holds for

$$P = \frac{\varphi(q)\varphi(q^{19})}{\varphi(q^3)\varphi(q^{57})} \qquad \text{and} \qquad Q = \frac{\varphi(q)\varphi(q^{57})}{\varphi(q^3)\varphi(q^{19})}.$$

4. General formulas for the explicit evaluations of ratios of Ramanujan's theta functions

In this section, we establish some general formulas for the explicit evaluations of ratios of Ramanujan's theta functions.

Theorem 4.1. *If* $X = h_{3,n}$ *and* $Y = h_{3,81n}$*, then*

$$\begin{aligned}
&\frac{Y^5}{X^5} + 9\left(\frac{Y^3}{X^3} - \frac{Y^4}{X^4}\right) + 9\left(\frac{5Y}{X} + \frac{9X}{Y}\right) + 27\left(Y^4 + \frac{1}{X^4}\right)\\
&+ 9\left(Y^2 + 3X^2\right)\left(XY + \frac{1}{X^3Y^3}\right) \qquad (4.1)\\
&= 135 + 3\left(\frac{Y^2}{X^2} + \frac{9X^2}{Y^2}\right) + 9\left(X^4Y^4 + \frac{1}{X^4Y^4}\right) + 27\left(X^2Y^2 + \frac{1}{X^2Y^2}\right).
\end{aligned}$$

Proof. Employing the equation (3.1) along with the equation (1.5) with $k = 3$, we obtain the equation (4.1). ■

Corollary 4.1. *We have*

$$h_{3,9} = \frac{\sqrt[3]{4} - \sqrt[3]{2} + 1}{\sqrt{3}}, \qquad (4.2)$$

$$h_{3,1/9} = \frac{1 + \sqrt[3]{2}}{\sqrt{3}}, \qquad (4.3)$$

$$l_{3,9} = \frac{(\sqrt[3]{2} + 1)^2}{\sqrt{3}}, \qquad (4.4)$$

$$l_{3,1/9} = \frac{(\sqrt[3]{4} - 1)}{\sqrt{3}}. \qquad (4.5)$$

Proofs of (4.2) and (4.3). Putting $n = 1/9$ in the equation (3.1) and then using the equation (2.9), we deduce that

$$(h_{3,9}^3 + h_{3,9}^2\sqrt{3} + 3h_{3,9} + \sqrt{3})(-h_{3,9}^3 + h_{3,9}^2\sqrt{3} - 3h_{3,9} + \sqrt{3})(h_{3,9}^4 - 6h_{3,9}^2 + 3)^2 = 0. \qquad (4.6)$$

Since $0 < h_{3,9} < 1$, we find that

$$h_{3,9}^3 - h_{3,9}^2\sqrt{3} + 3h_{3,9} - \sqrt{3} = 0. \qquad (4.7)$$

On solving the above equation, we arrive at the equations (4.2) and (4.3). ■

Proofs of (4.4) and (4.5). Using the equations (2.8), (4.2) and (4.3), we arrive at the equations (4.4) and (4.5). ■

Remark 4. Different proofs of the equations (4.2) and (4.4) can be found in [4], [15] and [16].

Theorem 4.2. *If* $X = h_{3,n}h_{3,289n}$ *and* $Y = \dfrac{h_{3,n}}{h_{3,289n}}$, *then*

$$\begin{aligned}
&Y^9 - \frac{1}{Y^9} + 34\left(Y^8 + \frac{1}{Y^8}\right) + 272\left(Y^7 - \frac{1}{Y^7}\right) + 238\left(Y^6 + \frac{1}{Y^6}\right)\\
&- 595\left(Y^5 - \frac{1}{Y^5}\right) - 510\left(Y^4 + \frac{1}{Y^4}\right) + 16303\left(Y^3 - \frac{1}{Y^3}\right)\\
&- 5202\left(Y^2 + \frac{1}{Y^2}\right) - 26911\left(Y - \frac{1}{Y}\right) + 3^4\left(X^8 + \frac{1}{X^8}\right) + 20230\\
&= 51\left\{\left(X^2 + \frac{1}{X^2}\right)\left[7\left(Y^6 + \frac{1}{Y^6}\right) + 28\left(Y^5 - \frac{1}{Y^5}\right) + 34\left(Y^4 + \frac{1}{Y^4}\right)\right.\right. \qquad (4.8)\\
&\left. -168\left(Y^3 - \frac{1}{Y^3}\right) + 160\left(Y^2 + \frac{1}{Y^2}\right) + 378\left(Y - \frac{1}{Y}\right) - 210\right]\\
&- 3\left(X^4 + \frac{1}{X^4}\right)\left[5\left(Y^4 + \frac{1}{Y^4}\right) + 21\left(Y^3 - \frac{1}{Y^3}\right) - 35\left(Y^2 + \frac{1}{Y^2}\right)\right.\\
&\left.\left. -14\left(Y - \frac{1}{Y}\right) - 30\right] - 9\left(X^6 + \frac{1}{X^6}\right)\left[\left(Y^2 + \frac{1}{Y^2}\right) - 2\left(Y - \frac{1}{Y}\right) - 2\right]\right\}.
\end{aligned}$$

Proof. Employing the equation (3.6) along with the equation (1.5) with $k = 3$ and Remark 2, we obtain the equation (4.8). ∎

Corollary 4.2. *We have*

$$h_{3,17}^2 = \frac{2 + \sqrt{17} - (142 + 34\sqrt{17})^{1/3} + (2 + 2\sqrt{17})^{1/3}}{3}, \qquad (4.9)$$

$$h_{3,1/17}^2 = \frac{\sqrt{17} - 2 + (142 - 34\sqrt{17})^{1/3} + (-2 + 2\sqrt{17})^{1/3}}{3}, \qquad (4.10)$$

$$l_{3,17}^2 = 4 + \sqrt{17} + 2(29 + 7\sqrt{17})^{1/3} + 2(37 + 9\sqrt{17})^{1/3}, \qquad (4.11)$$

$$l_{3,1/17}^2 = -4 + \sqrt{17} - 2(29 - 7\sqrt{17})^{1/3} + 2(-37 + 9\sqrt{17})^{1/3}. \qquad (4.12)$$

Proofs of (4.9) and (4.10). Putting $n = 1/17$ in the equation (4.8) and then using the equation (2.9), we find that

$$\begin{aligned}
&(h_{3,17}^{12} - 4h_{3,17}^{10} + 5h_{3,17}^8 + 24h_{3,17}^6 - 5h_{3,17}^4 - 4h_{3,17}^2 - 1)\\
&\quad \times (h_{3,17}^4 + h_{3,17}^2 - 1)^2(h_{3,17}^8 - 16h_{3,17}^6 - 22h_{3,17}^4 + 16h_{3,17}^2 + 1)^2 = 0.
\end{aligned} \qquad (4.13)$$

We observe that the first factor of the equation (4.13) vanishes for the specific value of $q = e^{-\pi\sqrt{17/3}}$, but the other two factors does not vanish. Since $0 < h_{3,17} < 1$, hence we deduce that

$$x^3 + 8x - 4x^2 + 16 = 0, \qquad (4.14)$$

where $x = h_{3,17}^2 - \dfrac{1}{h_{3,17}^2}$.

On solving the above equation, we find that

$$h_{3,17}^2 - \frac{1}{h_{3,17}^2} = \frac{2}{3}\left[(-37+9\sqrt{17})^{1/3} - (37+9\sqrt{17})^{1/3} + 2\right]. \tag{4.15}$$

On solving the above equation, we arrive at the equations (4.9) and (4.10). ■

Proofs of (4.11) and (4.12). By using the equation (2.8) along with the equations (4.9) and (4.10) respectively, we arrive at the equations (4.11) and (4.12). ■

Theorem 4.3. *If* $X = h_{3,n}h_{3,361n}$ *and* $Y = \dfrac{h_{3,n}}{h_{3,361n}}$, *then*

$$\begin{aligned}
&Y^{10} - \frac{1}{Y^{10}} + 3^4\sqrt{3}\left(X^9 + \frac{1}{X^9}\right) + 19\left\{1916\left(Y^4 - \frac{1}{Y^4}\right) + 135\left(Y^6 - \frac{1}{Y^6}\right)\right.\\
&+ 26\left(Y^8 - \frac{1}{Y^8}\right) + 1144\left(Y^2 - \frac{1}{Y^2}\right) - 3^3\sqrt{3}\left(X^7 + \frac{1}{X^7}\right)\left[\left(Y^2 - \frac{1}{Y^2}\right) - 4\right]\\
&- 135\left(X^6 + \frac{1}{X^6}\right)\left[\left(Y^2 - \frac{1}{Y^2}\right)\right] + 27\sqrt{3}\left(X^5 + \frac{1}{X^5}\right)\left[2\left(Y^4 + \frac{1}{Y^4}\right)\right.\\
&\left.- 14\left(Y^2 + \frac{1}{Y^2}\right) + 11\right] + 18\left(X^4 + \frac{1}{X^4}\right)\left[26\left(Y^4 - \frac{1}{Y^4}\right) - \left(Y^2 - \frac{1}{Y^2}\right)\right]\\
&- 15\left(X^2 + \frac{1}{X^2}\right)\left[13\left(Y^6 - \frac{1}{Y^6}\right) - 4\left(Y^4 - \frac{1}{Y^4}\right) + 104\left(Y^2 - \frac{1}{Y^2}\right)\right]\\
&- 3\sqrt{3}\left(X^3 + \frac{1}{X^3}\right)\left[12\left(Y^6 + \frac{1}{Y^6}\right) - 65\left(Y^4 + \frac{1}{Y^4}\right) - 591\right.\\
&\left.+ 200\left(Y^2 + \frac{1}{Y^2}\right)\right] + 3\sqrt{3}\left(X + \frac{1}{X}\right)\left[2\left(Y^8 + \frac{1}{Y^8}\right) - 29\left(Y^6 + \frac{1}{Y^6}\right)\right.\\
&\left.\left.+ 7\left(Y^4 + \frac{1}{Y^4}\right) - 504\left(Y^2 + \frac{1}{Y^2}\right) + 697\right]\right\} = 0.
\end{aligned} \tag{4.16}$$

Proof. Employing the equation (3.9) along with the equation (1.5) with $k = 3$ and Remark 3, we obtain the equation (4.16). ■

Corollary 4.3. *We have*

$$h_{3,19}^4 = (26 - 15\sqrt{3})(2\sqrt{19} + 5\sqrt{3}), \tag{4.17}$$

$$h_{3,1/19}^4 = (26 + 15\sqrt{3})(2\sqrt{19} - 5\sqrt{3}), \tag{4.18}$$

$$l_{3,19}^4 = (26 + 15\sqrt{3})(2\sqrt{19} + 5\sqrt{3}), \tag{4.19}$$

$$l_{3,1/19}^4 = (26 - 15\sqrt{3})(2\sqrt{19} - 5\sqrt{3}). \tag{4.20}$$

Proofs of (4.17) and (4.18). Putting $n = 1/19$ in the equation (4.16) and then using the equation (2.9), we find that

$$\begin{aligned}&(h_{3,19}^8 + (450 + 260\sqrt{3})h_{3,19}^4 - 1351 - 780\sqrt{3})\\ \times\,&(-h_{3,19}^8 - (4 - 4\sqrt{3})h_{3,19}^4 + 2 - \sqrt{3})^2\\ \times\,&(-h_{3,19}^8 - (18 - 12\sqrt{3})h_{3,19}^4 + 7 - 4\sqrt{3})^2 = 0.\end{aligned} \tag{4.21}$$

Since $0 < h_{3,19} < 1$, we deduce that

$$h_{3,19}^8 + (450 - 260\sqrt{3})h_{3,19}^4 - 1351 + 780\sqrt{3} = 0. \tag{4.22}$$

Solving the above equation, we obtain (4.17) and (4.18). ■

Proofs of (4.19) and (4.20). Using the equations (2.8), (4.17) and (4.18), we arrive at the equations (4.19) and (4.20). ■

5. Modular relations for Ramanujan's cubic continued fraction

In this section, by using the modular equations established in Section 3 we find modular relations connecting $V(q)$ with other three continued fractions $V(q^9)$, $V(q^{17})$ and $V(q^{19})$.

Theorem 5.1. *If $v = V(q)$ and $w = V(q^9)$, then*

$$\begin{aligned}&w^9 - (36v^3 + 256v^9 - 5 + 288v^6)w^8 - (90v^3 - 640v^9 - 576v^6 - 10)w^7\\ &- (648v^6 + 640v^9 + 81v^3 - 11)w^6 - (-10 - 352v^9 + 72v^3 - 396v^6)w^5\\ &- (144v^6 - 11 + 99v^3 + 160v^9)w^4 - (-88v^9 + 81v^3 - 10 - 81v^6)w^3\\ &- (40v^9 + 45v^6 - 5 + 36v^3)w^2 - (-9v^6 - 1 - 10v^9 + 9v^3)w - v^9 = 0.\end{aligned} \tag{5.1}$$

Proof. Using the Remark 1, we deduce that

$$\begin{aligned}&- 1944w^9Q^2v^6 + 396w^9Q^2v^3 - 432w^{12}Q^2v^6 - 120w^{12}Q^2v^3 - 4848v^{12}Q^2w^6\\ &- 468v^9Q^2w^3 - 232v^{12}Q^2w^3 - 9288v^9Q^2w^6 - 15808v^{12}w^9Q^2 - 18144v^9w^9Q^2\\ &- 256Q^2v^{12}w^{12} - 576Q^2v^9w^{12} + P^2 + 423Q^2v^3w^6 - 93Q^2v^3w^3 - 4050Q^2v^6w^6\\ &- 351Q^2v^6w^3 + 5064P^2v^6w^9 - 16832P^2v^{12}w^9 - 1352P^2v^{12}w^3 - 10016P^2v^9w^9\\ &- 12648P^2v^9w^6 - 380P^2w^9v^3 + 40P^2w^{12}v^3 + 240P^2w^{12}v^6 + 256P^2v^{12}w^{12}\\ &+ 448P^2v^9w^{12} + 241P^2v^3w^3 - 165P^2v^3w^6 + 2898P^2v^6w^6 - 11856P^2v^{12}w^6\\ &- 2204P^2v^9w^3 - 14P^2v^3 + 4P^2w^3 + 6P^2w^6 + 51P^2v^6 - 1875P^2v^6w^3 - v^{12}Q^2\\ &+ 4P^2w^9 - 23P^2v^{12} + 18Q^2w^3 + 27w^6Q^2 - 9w^{12}Q^2 - 20P^2v^9 + P^2w^{12} = 0,\end{aligned} \tag{5.2}$$

where

$$P = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \text{and} \qquad Q = \frac{\psi(q^9)}{q^{9/4}\psi(q^{27})}.$$

By collecting the terms containing P^2 on one side of the equation (5.2), then squaring both sides along with the equation (2.7), we find that

$$\begin{aligned}
&(-10w^3 + 81v^3w^3 - 11w^6 - w + 648v^6w^6 + 81v^3w^6 - 81v^6w^3 - 5w^2\\
&- 88v^9w^3 + 640v^9w^6 + 9v^3w + 36v^3w^2 + 99v^3w^4 + v^9 - w^9 - 11w^4 - 5w^8\\
&- 10w^5 + 45v^6w^2 + 288v^6w^8 + 72v^3w^5 - 396v^6w^5 + 36w^8v^3 - 576v^6w^7\\
&+ 144v^6w^4 - 9wv^6 + 90w^7v^3 - 10w^7 - 10v^9w + 40v^9w^2 + 160v^9w^4\\
&+ 256v^9w^8 - 352v^9w^5 - 640v^9w^7)\\
\times &(-5w^3 + 81v^3w^3 + 35w^6 + 4698v^6w^6 - 567v^3w^6 - 324v^6w^3 + w^2\\
&- 23409v^{12}w^6 - 111v^9w^3 - 13851v^6w^9 + 110808v^{12}w^9 + 119376v^9w^9\\
&+ 567v^{12}w^3 - 16023v^9w^6 - 23409w^{12}v^6 + 567w^{12}v^3 - 729w^9v^3\\
&+ 300672v^{12}w^{12} + 128184v^9w^{12} - 18v^3w^2 - 189v^3w^4 - 14w^{12} + 55w^9 + 15w^4\\
&- 14w^8 - 28w^5 + 99v^6w^2 - 198v^6w^8 + 369v^3w^5 - 207w^8v^3 + 594v^6w^7\\
&- 54v^6w^4 + 621w^7v^3 - 21w^7 - 7104v^9w^{15} + 165888v^{12}w^{15} - 46656v^{15}w^9\\
&- 28160v^{18}w^9 + 290304v^{15}w^{12} + 163840v^{18}w^{15} + 331776v^{15}w^{15} - 896v^{18}w^6\\
&+ 143360v^{18}w^{12} - 4536v^{15}w^6 + 648v^3w^{15} + 648v^{15}w^3 - 4536v^6w^{15}\\
&+ 224v^{18}w^3 - 28w^{15} + w^{18} + v^{18} + 55w^{11} + 35w^{14} - 5w^{17} - 21w^{13} - 72w^{10}\\
&+ 15w^{16} - 1557v^3w^{11} - 927v^3w^{14} + 36v^3w^{17} + 459v^3w^{13} + 1539v^3w^{10}\\
&- 270v^3w^{16} + 23508v^6w^{11} + 22716v^6w^{14} + 288v^6w^{17} - 3618v^6w^{13} - 810v^6w^{10}\\
&- 2160v^6w^{16} + v^9w - 137v^9w^2 + 4121v^9w^4 + 24664v^9w^8 - 49v^9w^5\\
&- 26888v^9w^7 - 107552v^9w^{11} - 131872v^9w^{14} + 256v^9w^{17} - 784v^9w^{13}\\
&- 49328v^9w^{10} + 17536v^9w^{16} - 9v^{12}w - 135v^{12}w^2 + 5679v^{12}w^4 - 3240v^{12}w^8\\
&+ 1809v^{12}w^5 - 47016v^{12}w^7 - 19008v^{12}w^{11} - 13824v^{12}w^{14} + 150912v^{12}w^{13}\\
&- 3168v^{12}w^{10} + 101376v^{12}w^{16} + 9v^{15}w + 135v^{15}w^2 + 1854v^{15}w^4 - 49248v^{15}w^8\\
&+ 1836v^{15}w^5 - 24912v^{15}w^7 + 158976v^{15}w^{11} + 387072v^{15}w^{14} + 377856v^{15}w^{13}\\
&+ 26496v^{15}w^{10} + 147456v^{15}w^{16} + 10v^{18}w + 60v^{18}w^2 + 560v^{18}w^4 - 18432v^{18}w^8\\
&+ 672v^{18}w^5 - 7040v^{18}w^7 + 43008v^{18}w^{11} + 245760v^{18}w^{14} + 229376v^{18}w^{13}\\
&- 14336v^{18}w^{10} + 65536v^{18}w^{16} - 1179v^6w^5) = 0.
\end{aligned} \tag{5.3}$$

From the definitions of v and w, we have $v = o(q^{1/3})$ and $w = o(q^3)$ as $q \to 0$, it can be seen that the first factor of the equation (5.3) vanishes for q sufficiently small; whereas the second factor does not vanish. Thus by the Identity Theorem, first factor vanishes identically. This completes the proof. ■

Theorem 5.2. *If $v = V(q)$ and $w = V(q^{17})$, then*

$$\begin{aligned}
&w^{18} - wv\left[(2vw)^{16}+1\right] + v^{18} + 17\left\{\left(1-5v^3+5v^6+9v^9-v^{12}\right)v^4w\right.\\
&+\left(16384v^{15}+36864v^{12}+12288v^9-7488v^6+712v^3-1\right)vw^{16}\\
&+8\left(79-1059v^3+2952v^6+1024v^9-3072v^{12}\right)v^3w^{15}+(178\\
&-8192v^{15}-75776v^{12}-50560v^9+15024v^6-2002v^3\big)v^2w^{14}\\
&+\left(36864v^{15}+33792v^{12}-13120v^9-9560v^6+1001v^3+9\right)vw^{13}\\
&+\left(8192v^{12}+38016v^9+17496v^6-10177v^3+1059\right)v^3w^{12}\\
&+2\left(117-1195v^3+3016v^6-19328v^9-25280v^{12}-2560v^{15}\right)v^2w^{11}\\
&+\left(12288v^{15}-13120v^{12}-26784v^9-3016v^6+939v^3+5\right)vw^{10}\\
&+\left(369-2187v^3-11536v^6+17496v^9+23616v^{12}\right)v^3w^9+2\big(205v^3\\
&-320v^{15}+7512v^{12}+3016v^9-3348v^6+24\big)v^2w^8-\big(7488v^{15}+9560v^{12}\\
&+3016v^9+2416v^6-395v^3+5\big)vw^7-\big(8472v^{12}+10177v^9+2187v^6+16\\
&-594v^3\big)v^3w^6+2\left(72v^{15}-1001v^{12}-1195v^9+205v^6+66v^3-9\right)v^2w^5\\
&+\left(712v^{15}+1001v^{12}+939v^9+395v^6-74v^3+1\right)vw^4+\big(632v^{12}+1059v^9\\
&+369v^6-16v^3-6\big)v^3w^3+2\left(9(2v)^3+1-5(2v)^6-(2v)^{12}-5(2v)^9\right)v^2w^{17}\\
&\left.+\left(2v^{15}+178v^{12}+234v^9+48v^6-18v^3+1\right)v^2w^2\right\}=0.
\end{aligned}\tag{5.4}$$

The proof of the equation (5.4) is similar to the proof of the equation (5.1), except in the place of the result (3.1), the result (3.6) is used.

Remark 5. A different proof of the equation (5.4) can be found in [8].

Theorem 5.3. *If $v = V(q)$ and $w = V(q^{19})$, then*

$$\begin{aligned}
&v^{20}+w^{20}-wv\left\{(2wv)^{18}+1\right\}+19\left\{4\left(-8192w^{15}-6144w^{12}-1280w^9+22w^3-1\right.\right.\\
&\left.+144w^6\right)wv^{19}+2w^2v^{18}\left[53-2^4\left(4096w^{15}+8192w^{12}+3072w^9-944w^6+77w^3\right)\right]\\
&-4w^3v^{17}\left(243-2492w^3-8608w^6+94464w^9+133120w^{12}+32768w^{15}\right)+wv^{16}\\
&\times\left(-11+2124w^3-13096w^6+57728w^9-463872w^{12}-569344w^{15}-32768w^{18}\right)\\
&-4w^2v^{15}\left(133120w^{15}+152832w^{12}+12496w^9-6266w^6+1077w^3-77\right)\\
&-2w^3v^{14}\left(131072w^{15}+305664w^{12}+229568w^9+40136w^6-15237w^3+623\right)\\
&+wv^{13}\left(9+1637w^3-852w^6-64704w^9-505600w^{12}-463872w^{15}-24576w^{18}\right)\\
&+w^2v^{12}\left(472-3133w^3+17721w^6-66584w^9-459136w^{12}-377856w^{15}\right)\\
&+2w^3v^{11}\left(269+5017w^3-57124w^6-33292w^9-24992w^{12}-49152w^{15}\right)\\
&+2wv^{10}\left(5+451w^3+4044w^6-55316w^9-32352w^{12}+28864w^{15}-2560w^{18}\right)\\
&+w^2v^9\left(192-781w^3+8323w^6-114248w^9-80272w^{12}+34432w^{15}\right)\\
&+w^3v^8\left(30208w^{15}+25064w^{12}+17721w^9+8323w^6-7174w^3+738\right)
\end{aligned}$$

$$
\begin{aligned}
&+ wv^7 \left(576w^{18} - 13096w^{15} - 852w^{12} + 8088w^9 - 7900w^6 + 906w^3 - 6\right)\\
&+ w^2v^6 \left(9968w^{15} + 30474w^{12} + 10034w^9 - 7174w^6 + 1194w^3 - 64\right)\\
&- w^3v^5 \left(2464w^{15} + 4308w^{12} + 3133w^9 + 781w^6 - 1194w^3 + 130\right)\\
&+ wv^4 \left(88w^{18} + 2124w^{15} + 1637w^{12} + 902w^9 + 906w^6 - 139w^3 + 1\right) \qquad (5.5)\\
&+ w^2v^3 \left(4 - 130w^3 + 738w^6 + 538w^9 - 1246w^{12} - 972w^{15}\right)\\
&+ w^3v^2 \left(4 - 64w^3 + 192w^6 + 472w^9 + 308w^{12} + 106w^{15}\right)\\
&+ w^4v \left(9w^9 - 11w^{12} + 1 - 4w^{15} - 6w^3 + 10w^6\right)\big\} = 0.
\end{aligned}
$$

The proof of the equation (5.5) is similar to the proof of the equation (5.1), except in the place of the result (3.1), the result (3.9) is used.

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