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A HYPERCYCLICITY CRITERION FOR NON-METRIZABLE TOPOLOGICAL VECTOR SPACES

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Dedicated to the memory of
Professor Paweł Domański

Abstract: We provide a sufficient condition for an operator T on a non-metrizable and sequentially separable topological vector space X to be sequentially hypercyclic. This condition is applied to some particular examples, namely, a composition operator on the space of real analytic functions on $]0,1[$, which solves two problems of Bonet and Domański [3], and the "snake shift" constructed in [5] on direct sums of sequence spaces. The two examples have in common that they do not admit a densely embedded F-space Y for which the operator restricted to Y is continuous and hypercyclic, i.e., the hypercyclicity of these operators cannot be a consequence of the comparison principle with hypercyclic operators on F-spaces.

Keywords: hypercyclic operators.

The study of the dynamics of linear operators has experienced a great development in recent years, with two monographs [1] and [8], and many research papers. Usually the interest is in the dynamics of (continuous and linear) operators $T \in L(X)$ defined on separable Fréchet spaces X. Metrizability and completeness of the space offers the possibility to apply Baire category arguments, which are very useful in this context. A few articles concentrate on the dynamics of operators on non-metrizable topological vector spaces (see, e.g., [3, 5, 9]).

We recall that an operator $T \in L(X)$ on a topological vector space X is *hypercyclic* if there are $x \in X$ whose orbit $\mathrm{Orb}(x,T) := \{x, Tx, T^2x, \dots\}$ is dense in X. We will say that T is *sequentially hypercyclic* if there is $x \in X$ such that, for each $y \in X$, there exists an increasing sequence of integers $(n_k)_k$ such that $\lim_k T^{n_k}x = y$. Also, to avoid confusion with more general concepts, we say that X is *sequentially separable* if there exists a countable set $A \subset X$ such that any $z \in X$ is the limit of a sequence in A.

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In many cases one obtains (sequential) hypercyclicity of an operator $T \in L(X)$ for a non-metrizable X by finding a Fréchet space Y, a hypercyclic operator S on Y, and a continuous map $\Psi : Y \to X$ with dense range such that $T \circ \Psi = \Psi \circ S$. This is the so-called *comparison principle.* Exceptions to this procedure are the hypercyclic operators on non-metrizable topological vector spaces obtained in [5] and [9], where the hypercyclic vectors are constructed directly. We will obtain criteria under which operators on general topological vector spaces are sequentially hypercyclic.

1. Criteria for sequential hypercyclicity

In this section we will provide useful sufficient conditions for sequential hypercyclicity of operators on (non-metrizable) topological vector spaces.

Proposition 1. *Let X be a sequentially separable topological vector space and $T \in L(X)$ such that there exist a sequentially dense set $X_0 := \{x_n \ ; \ n \in \mathbb{N}\} \subset X$, a sequence of maps $S_n : X_0 \to X$, $n \in \mathbb{N}$, a subspace $Y \subset X$ with a finer topology τ such that (Y, τ) is an F-space for which we fix a countable basis of balanced 0-neighbourhoods $(V_n)_n$ with $V_n + V_n \subset V_{n-1}$, $n > 1$,and an increasing sequence $(n_k)_k$ of natural numbers ($n_0 := 0$) satisfying:*

(i) $T^{n_k} S_{n_j} x_j \in V_{2k}$, $k > 1$, $j = 1, \dots, k-1$,
(ii) $T^{n_k} S_{n_j} x_j \in V_j$, $k \geqslant 0$, $j > k$,
(iii) $x_k - T^{n_k} S_{n_k} x_k \in V_k$, $k \in \mathbb{N}$.

Then T is sequentially hypercyclic.

Proof. Let

$$x := \sum_{j=1}^{\infty} S_{n_j} x_j,$$

which belongs to Y by (ii) for $k = 0$, since Y is an F-space. Conditions (i), (ii) and (iii) yield that

$$\begin{aligned} x_k - T^{n_k} x = &-\left(T^{n_k} \left(\sum_{j=1}^{k-1} S_{n_j} x_j \right) \right) + (x_k - T^{n_k} S_{n_k} x_k) \\ &-\left(T^{n_k} \left(\sum_{j=k+1}^{\infty} S_{n_j} x_j \right) \right) \in V_{k-2}, \end{aligned}$$

for all $k \geqslant 2$, and we conclude that T is sequentially hypercyclic. ∎

Actually, to apply this criterion in some particular examples, we will use other conditions which are stronger, but easy to verify.

Definition 2. We say that a sequence $(x_j)_j$ is *eventually contained* in a set A (denoted by $(x_j)_j \subset_{ec} A$) if there is an integer j_0 such that $x_j \in A$ for $j \geqslant j_0$.

Corollary 3. *Let X be a sequentially separable topological vector space and $T \in L(X)$ such that there exist a sequentially dense set $X_0 := \{x_n : n \in \mathbb{N}\} \subset X$, a sequence of maps $S_n : X_0 \to X$, $n \in \mathbb{N}$, a subspace $Y \subset X$ with a finer topology τ such that (Y, τ) is an F-space, and an increasing sequence $(n_k)_k$ of natural numbers ($n_0 := 0$) satisfying:*

(i)' $(T^{n_k} S_{n_j} x)_k \subset_{ec} Y$ *and converges to* 0 *in* (Y, τ) *for each* $x \in X_0$, *and for all* $j \in \mathbb{N}$,

(ii)' $(T^{n_j} S_{n_k} x)_k \subset_{ec} Y$ *and converges to* 0 *in* (Y, τ) *for each* $x \in X_0$, *and for all* $j \geqslant 0$,

(iii)' $(x - T^{n_k} S_{n_k} x)_k \subset_{ec} Y$ *and converges to* 0 *in* (Y, τ) *for each* $x \in X_0$.

Then T is sequentially hypercyclic.

2. Composition operators on the space of real analytic functions and shifts on direct sums

In this section we will apply the previous criterion to a composition operator on the space of real analytic functions $\mathscr{A}(]0,1[)$, solving two questions of Bonet and Domański in [3], and to the "snake shift" constructed in [5] on countable direct sums of sequence spaces.

We first recall some basic definitions on the spaces of real analytic functions and composition operators between them. Given an open subset $\Omega \subset \mathbb{R}^d$, we denote by $\mathscr{A}(\Omega)$ the space of real analytic functions defined on Ω. We recall that every $f \in \mathscr{A}(\Omega)$ can be extended holomorphically to a complex neighbourhood $U \subset \mathbb{C}^d$ of Ω, i.e., we can consider $f \in \mathcal{H}(U)$ for some $\Omega \subset U \subset \mathbb{C}^d$ open set. The space $\mathcal{H}(U)$ is endowed with its natural (Fréchet) topology of uniform convergence on compact subsets. Given a compact set $K \subset \mathbb{C}^d$, the space $\mathcal{H}(K)$ of holomorphic germs on K with its natural locally convex topology is

$$\mathcal{H}(K) = \operatorname{ind}_{n \in \mathbb{N}} \mathcal{H}(U_n),$$

where $(U_n)_n$ is a basis of $\mathbb{C}^d$-neighbourhoods of K. Thus, the space $\mathscr{A}(\Omega)$ has a description as a countable projective limit

$$\mathscr{A}(\Omega) = \operatorname{proj}_{j \in \mathbb{N}} \mathcal{H}(K_j),$$

where $(K_j)_j$ is a fundamental sequence of compact subsets of Ω.

Several basic facts about spaces of real analytic functions were studied by Domański and Vogt [7], including the surprising result that this natural space has no basis.

Given a real analytic map $\varphi : \Omega \to \Omega$, the composition operator $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$, $f \mapsto f \circ \varphi$, is continuous. The dynamics of composition operators on spaces of real analytic functions was thoroughly studied in [3]. The dynamics of other natural operators, namely weighted backward shifts, on spaces of real analytic functions was recently studied in [6].

Bonet and Domański [3] asked whether the composition operator C_φ, $\varphi(z) := z^2$, is (sequentially) hypercyclic on $\mathscr{A}(]0,1[)$. They also asked if every sequentially hypercyclic operator $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is so that there exists a complex neighbourhood U of Ω such that φ extends holomorphically to U, $\varphi(U) \subset U$ and $C_\varphi : \mathcal{H}(U) \to \mathcal{H}(U)$ is hypercyclic. The following example provides a positive answer to the first question, and a negative answer to the second one.

Example 4. Let $\varphi(z) := z^2$, $X := \mathscr{A}(]0,1[)$, and $T := C_\varphi$. Let $p_n(z)$ be a dense sequence of polynomials. We set $x_n(z) = z(1-z)p_n(z)$, $n \in \mathbb{N}$, which forms a sequentially dense set in X. Let $X_0 := \{x_n \ ; \ n \in \mathbb{N}\}$ and $Y := \mathcal{H}(U)$, for U the open disk centered at $1/2$ of radius $1/2$. Let $\log z$ be a branch of the logarithm defined on $\mathbb{C}\backslash]-\infty, 0]$. We set $S_n = C_{\gamma_n}$, where $\gamma_n(z) = \exp(\frac{1}{2^n}\log z)$, $n \in \mathbb{N}$. It is clear that the conditions of Corollary 3 are satisfied for the sequence of all natural numbers, and T is sequentially hypercyclic. Indeed, $T^k S_k f = f$ on $]0,1[$ for all $f \in \mathcal{H}(U)$ (considered as a subspace of $\mathscr{A}(]0,1[)$). Since $f \in \mathcal{H}(U)$, we even have $T^k S_k f = f \in \mathcal{H}(U)$ for each $k \in \mathbb{N}$, so that (iii)' is trivially satisfied. Given a compact set $K \subset U$, $\varphi^n \to 0$ and $\gamma_n \to 1$ uniformly on K. Therefore,

$$\lim_{n\to\infty} (T^n S_j x_m)(z) = \lim_{n\to\infty} z^{2^{n-j}}(1 - z^{2^{n-j}})p_m(z^{2^{n-j}}) = 0, \qquad \forall j, m \in \mathbb{N},$$

and

$$\begin{aligned}\lim_{n\to\infty} &(T^j S_n x_m)(z) \\ &= \lim_{n\to\infty} \gamma_{n-j}(z)(1-\gamma_{n-j}(z))p_m(\gamma_{n-j}(z)) = 0, \qquad \forall j \geqslant 0, \forall m \in \mathbb{N},\end{aligned}$$

uniformly on K. That is, (i)' and (ii)' are also satisfied.

The idea of the previous example can be extended to certain composition operators $C_\varphi : \mathscr{A}(I) \to \mathscr{A}(I)$ for bounded open intervals I in $\mathbb{R}$. These results will appear elsewhere. We should also note the following alternative argument provided by José Bonet: A classical result of Belitskii and Lyubich [2] (see also [4]) shows that any real analytic diffeomorphism without fixed points $\varphi : \mathbb{R} \to \mathbb{R}$ is real analytic conjugate to the shift $x \mapsto x+1$. As a consequence, for any real analytic diffeomorphism without fixed points $\varphi : I \to I$ on an open interval $I \subset \mathbb{R}$, the composition operator $C_\varphi : \mathscr{A}(I) \to \mathscr{A}(I)$ is sequentially hypercyclic.

Example 5. The snake shift T constructed in [5] was defined on the countable direct sum $X := \oplus_{i\in\mathbb{N}} Y$ of a Fréchet sequence space Y, for the cases $Y = \ell^p$, $1 \leqslant p < \infty$, $Y = c_0$, $Y = s$, the space of rapidly decreasing sequences

$$s := \{x = (x_n)_n \in \mathbb{C}^{\mathbb{N}} \colon \|x\|_k := \sum_{n\in\mathbb{N}} |x_n| n^k < \infty \text{ for all } k \in \mathbb{N}\}.$$

To fix notation, $(e_{i,j})_j$ represents the canonical unit vectors on the i-th summand, $Te_{1,1} = 0$, $Te_{i,j} = \lambda e_{f(i,j)}$, $(i,j) \neq (1,1)$, where the constant $\lambda > 1$ and $f : \mathbb{N} \times \mathbb{N} \setminus \{(1,1)\} \to \mathbb{N} \times \mathbb{N}$ is a suitable bijection. Once a certain sequence of vectors with finite support $(x_j)_j$ in X was fixed so that it is sequentially dense in X, the constructed hypercyclic vector had the form

$$x = \sum_{k\in\mathbb{N}} \sum_{j=m_k}^{n_k} \frac{1}{\lambda^{l_k}} \alpha_j e_{1,j} \in Y,$$

where $T^{l_k}(\sum_{j=m_k}^{n_k} \frac{1}{\lambda^{l_k}} \alpha_j e_{1,j}) = x_k$ and $|\alpha_j| \leqslant k$, $m_k \leqslant j \leqslant n_k$, $k \in \mathbb{N}$, for suitable increasing sequences $(m_k)_k$, $(n_k)_k$ and $(l_k)_k$. In the case $Y = s$, the sequence $(n_k)_k$ was required to be polynomially bounded (actually, $n_k \leqslant 3k^2$, $k \in \mathbb{N}$).

Defining $X_0 = \{x_k : k \in \mathbb{N}\}$ and $Se_{i,j} = \lambda^{-1} e_{f^{-1}(i,j)}$, $(i,j) \in \mathbb{N}\times\mathbb{N}$, $S_n = S^n$, $n \in \mathbb{N}$, one has

$$S_{l_k} x_k = \sum_{j=m_k}^{n_k} \frac{1}{\lambda^{l_k}} \alpha_j e_{1,j}, \quad k \in \mathbb{N}, \text{ that yields condition (iii)' in Corollary 3,}$$

$$T^{l_k} S_{l_j} x_i = 0, \qquad \text{if } \ k > j + i,$$

that is, condition (i)' in Corollary 3, and finally

$$T^{l_j} S_{l_k} x_i \in Y, \qquad \text{if } \ k > j + i,$$

and

$$\lim_k T^{l_j} S_{l_k} x_i = \lim_k \frac{1}{\lambda^{l_k - l_j}} \sum_{r=m_k-l_j}^{m_k-l_j+n_i-m_i} \beta_r e_{1,r} = 0 \qquad \text{in } \ Y,$$

for certain β_r with $|\beta_r| \leqslant i$, for $m_k - l_j \leqslant r \leqslant m_k - l_j + n_i - m_i$, $k > i + j$, which gives condition (ii)' in Corollary 3.

We want to point out that Shkarin constructed in [9] hypercyclic operators on locally convex direct sums of sequences $(X_n)_n$ of separable Fréchet spaces for which infinitely many of them are infinite dimensional, and he characterized inductive limits of sequences of separable Banach spaces which support a hypercyclic operator.

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