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PRODUCTS OF CONSECUTIVE VALUES OF SOME QUARTIC POLYNOMIALS

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Abstract: In this paper, we investigate some special quartic polynomials P whose coefficients for $x^4, x^3, \dots, 1$ are $a^2, 2a(a+b), a^2+b^2+3ab+2ac, (a+b)(b+2c), (a+b+c)c$, where $a,b,c \in \mathbb{Z}$, and consider the question whether the product $\prod_{k=1}^m P(k)$ is a perfect square for infinitely many $m \in \mathbb{N}$ or for only finitely many $m \in \mathbb{N}$. The answer depends on the solutions of the Pell type diophantine equation $(a+b+c)(ax^2+bx+c) = y^2$. Our results imply, for example, that the product $\prod_{k=1}^m (4k^4+8k^2+9)$ is a perfect square for infinitely many $m \in \mathbb{N}$, whereas the product $\prod_{k=1}^m (k^4+7k^2+16)$ is a perfect square for $m=3$ only, when it equals $230400 = 480^2$.

Keywords: integer polynomial, Pell's equation, perfect square.

1. Introduction

Let P be a polynomial in $\mathbb{Z}[x]$ with positive leading coefficient. In general, the question of whether there are infinitely many or only finitely many positive integers m (or, more generally, pairs of positive integers $\ell < m$) for which the product $\prod_{k=1}^m P(k)$ (resp. $\prod_{k=\ell}^m P(k)$) is a perfect square or a higher power is completely open. Only in case $P(x) = x + b$, where $b \in \mathbb{Z}$, the theorem of Erdös and Selfridge [8] asserting that the product of two or more consecutive integers is never a power gives a complete answer to this problem. The case of a general linear polynomial $P(x) = ax + b$, where $a \geqslant 2$ and b are integers, has a long history, but it is not yet completely solved. It has been considered, for instance, in [10] and [14], where one can find many references on this problem.

In [1], the problem on whether the product $\prod_{k=1}^m (k^2+1)$ can be a perfect square has been raised. (Of course, this corresponds to the quadratic polynomial $P(x) = x^2+1$.) The negative answer is given in [5]. Similar problems for quadratic polynomials $4x^2+1$, $2x^2-2x+1$ and for polynomials of the form $x^\ell+1$, where $\ell \geqslant 2$, have been considered in [9] and [2], [3], [4], [17], respectively, whereas some

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special cubic polynomials appear in [12], [15]. In [6], some bounds on the density of squares in the sequence $\prod_{k=1}^{m} P(k)$, $m = 1, 2, 3, \dots$, have been obtained for a general irreducible polynomial $P \in \mathbb{Z}[x]$.

2. Main results

This paper is a continuation and in some sense a generalization of two recent results ([11] and [13]) related to some special quartic polynomials. In 2016, by a completely elementary approach, Gürel [13] has shown that the product $\prod_{k=1}^{m}(4k^4+1)$ is a perfect square for infinitely many $m \in \mathbb{N}$, whereas $\prod_{k=1}^{m}(k^4+4)$ is a perfect square only for $m = 2$.

This approach was then developed by Gaitanas [11] who generalized it to some other special quartic polynomials. His idea was to use the identity

$$Q(x+Q(x)) = Q(x)Q(x+1) \tag{1}$$

for the monic quadratic polynomial $Q(x) = x^2+ax+b \in \mathbb{Z}[x]$.

One should say that a more general identity was already used by the author in an entirely different context (see [7]). For a quadratic polynomial

$$Q(x) := ax^2+bx+c \in \mathbb{C}[x], \qquad a \neq 0, \tag{2}$$

and a complex number $t \neq 0$ it was shown that

$$Q\Big(x+\frac{t}{a}Q(x)\Big) = \frac{t^2}{a}Q(x)Q\Big(x+\frac{1}{t}\Big). \tag{3}$$

The proof of (3) given in [7] is a simple exercise. Note that (3) implies (1) for $a = t = 1$.

Inserting $t = 1$ into (3) (but do not assuming that $a = 1$) we find that

$$\begin{aligned} Q(x)Q(x+1) &= aQ\Big(x+\frac{Q(x)}{a}\Big) = aQ\Big(x+x^2+\frac{bx}{a}+\frac{c}{a}\Big) \\ &= P_{a,b,c}(x), \end{aligned} \tag{4}$$

where $P_{a,b,c}(x)$ is a quartic polynomial of the form

$$\begin{aligned} P_{a,b,c}(x) := {}& a^2x^4+2a(a+b)x^3+(a^2+b^2+3ab+2ac)x^2 \\ &+(a+b)(b+2c)x+(a+b+c)c. \end{aligned} \tag{5}$$

With this notation, we have the following:

Theorem 1. *For any integers a, b, c satisfying $a \neq 0$ and $a+b+c \neq 0$, the product $\prod_{k=1}^{m} P_{a,b,c}(k)$ is a perfect square for $m \in \mathbb{N}$ if and only if the equation*

$$(a+b+c)(ax^2+bx+c) = y^2 \tag{6}$$

has a solution (x, y) with $x = m+1$ and $y \in \mathbb{N}$.

Furthermore, for a finite extension K of $\mathbb{Q}$ of degree $d=[K:\mathbb{Q}]$, let $\sigma_1,\dots,\sigma_d$ be the distinct embeddings of K into $\mathbb{C}$. Then, for any algebraic integers $a,b,c\in K$ satisfying $a\neq 0$ and $a+b+c\neq 0$, and any positive integers $\ell\leqslant m$ the product

$$\prod_{k=\ell}^{m}\prod_{j=1}^{d}P_{\sigma_j(a),\sigma_j(b),\sigma_j(c)}(k) \tag{7}$$

is a perfect square if and only if the equation

$$\prod_{j=1}^{d}(\sigma_j(a)\ell^2+\sigma_j(b)\ell+\sigma_j(c))\prod_{j=1}^{d}(\sigma_j(a)x^2+\sigma_j(b)x+\sigma_j(c))=y^2 \tag{8}$$

has a solution (x,y) with $x=m+1$ and $y\in\mathbb{N}$.

By Siegel's theorem [16], equation (8) has only finitely many solutions if the polynomial $\prod_{j=1}^{d}(\sigma_j(a)x^2+\sigma_j(b)x+\sigma_j(c))\in\mathbb{Z}[x]$ has at least three simple roots.

In order to investigate the equation (6) we put

$$d:=a(a+b+c) \tag{9}$$

and

$$D:=b^2-4ac. \tag{10}$$

Evidently, (6) has at most finitely many solutions $(x,y)\in\mathbb{N}^2$ if $d<0$, so it suffices to investigate the case $d>0$. Then, as $a<0$ implies $a+b+c<0$, we can replace the triplet (a,b,c) by $(-a,-b,-c)$, which leaves both (5) and (6) unchaged. For this reason, we only consider the case $a>0$, $a+b+c>0$.

Theorem 2. *Let a,b,c be integers satisfying $a>0$ and $a+b+c>0$. If d defined* (9) *is a perfect square then the equation* (6) *has at most finitely many solutions in positive integers (x,y) when D defined in* (10) *satisfies $D\neq 0$ and infinitely many solutions when $D=0$. If d is not a perfect square and, in addition, either $2a+b\geqslant 0$ or $D<0$, then* (6) *has infinitely many solutions $(x,y)\in\mathbb{N}^2$.*

Note that in case d is not a perfect square D cannot be zero. Indeed, $D=b^2-4ac=0$ implies that b is even. Hence,

$$d=a(a+b+c)=a^2+ab+(b/2)^2=(a+b/2)^2$$

is a perfect square. However, it can happen that $2a+b<0$ and $D>0$. For full description of this case one needs to introduce much more technical conditions, which we will not do in this note.

In the next section, we will prove Theorems 1 and 2. Then, in Section 4 we will give several examples illustrating Theorem 1.

3. Proof of the Theorems 1 and 2

Proof of Theorem 1. By (2) and (4), the product $\prod_{k=1}^m P_{a,b,c}(x)$ is equal to

$$Q(1)Q(m+1)\prod_{k=2}^{m} Q(k)^2,$$

where the product $\prod_{k=2}^m Q(k)^2$ is omitted if $m=1$. This is a perfect square iff

$$Q(1)Q(m+1) = (a+b+c)(a(m+1)^2+b(m+1)+c)$$

is a perfect square. This proves the first part of the theorem.

The proof of the second part is exactly the same, since the product (7) is equal to

$$\prod_{j=1}^{d}(\sigma_j(a)\ell^2+\sigma_j(b)\ell+\sigma_j(c))\prod_{j=1}^{d}(\sigma_j(a)(m+1)^2+\sigma_j(b)(m+1)+\sigma_j(c))$$

multiplied by the product

$$\prod_{k=\ell+1}^{m}\prod_{j=1}^{d}(\sigma_j(a)k^2+\sigma_j(b)k+\sigma_j(c))^2. \tag{11}$$

Clearly, (11) is a perfect square for $m \geqslant \ell+1$ (it is omitted for $m=\ell$), since $\prod_{j=1}^d(\sigma_j(a)k^2+\sigma_j(b)k+\sigma_j(c)) \in \mathbb{Z}$ for each $k \in \mathbb{N}$. ■

Proof of Theorem 2. Suppose first that $d=a(a+b+c)=v^2$ for some positive integer v. Then, the equation (6) is equivalent to

$$(vx)^2+ux+w=y^2, \tag{12}$$

where $u=b(a+b+c)$ and $w=c(a+b+c)$. Here, the left hand side is between $(vx-\max(|u|,|w|))^2$ and $(vx+\max(|u|,|w|))^2$ for x large enough. Thus, (12) has infinitely many solutions in $(x,y)\in\mathbb{N}^2$ if and only if for some $q\in\mathbb{Z}$ satisfying $|q|\leqslant\max(|u|,|w|)$ one has

$$(vx)^2+ux+w=(vx+q)^2 \tag{13}$$

for infinitely many $x\in\mathbb{N}$. This happens only when (13) is the identity. Consequently, the discriminant of v^2x^2+ux+w is zero, that is, $u^2=4v^2w$, or, equivalently, $D=b^2-4ac=0$. Otherwise, if $D\neq 0$ then (12) has at most finitely many solutions in $(x,y)\in\mathbb{N}^2$.

In all what follows we will prove that if d is not a perfect square and either $2a+b\geqslant 0$ or $D<0$ then (6) has infinitely many solutions in $(x,y)\in\mathbb{N}^2$.

Setting $X=2ax+b$ and $Y=y/(a+b+c)$ and using the identity $(2ax+b)^2-(b^2-4ac)=4a(ax^2+bx+c)$, we can rewrite (6) in the following form:

$$X^2-4dY^2=D. \tag{14}$$

Note that $(X_0,Y_0)=(2a+b,1)\in\mathbb{Z}^2$ is a solution of (14).

We also consider the equation

$$X^2 - 4dY^2 = 1. \tag{15}$$

Since $4d > 0$ is not a perfect square, this is a Pell equation, so that its solutions in positive integers are $(X_n, Y_n) \in \mathbb{N}^2$, where $(X_1, Y_1) \in \mathbb{N}^2$ is a fundamental solution, and $X_n + 2\sqrt{d}Y_n = (X_1 + 2\sqrt{d}Y_1)^n$ for $n = 1, 2, \dots$. It follows that the pairs

$$((2a+b)X_n + 4dY_n, (2a+b)Y_n + X_n), \qquad n = 1, 2, \dots, \tag{16}$$

obtained from the products $(X_0+2\sqrt{d}Y_0)(X_n+2\sqrt{d}Y_n)$ are some solutions of (14).

Suppose first that $2a+b \geqslant 0$. Then each pair in (16) belongs to $\mathbb{N}^2$. Furthermore, by (9) and (15), we see that X_1 modulo $2a$ is either 1 or -1. In both cases, $X_2 = X_1^2 + 4dY_1^2$ modulo $2a$ is 1. Consequently, for each $n \in \mathbb{N}$ the number $U_n := (2a+b)X_{2n} + 4dY_{2n}$ is a positive integer, which is b modulo $2a$, and $V_n := (2a+b)Y_{2n} + X_{2n}$ is a positive integer too. Choosing

$$x = \frac{U_n - b}{2a} \qquad \text{and} \qquad y = (a+b+c)V_n \tag{17}$$

we get a positive solution of (6). This, by choosing different n's, gives infinitely many solutions of (6) in $(x, y) \in \mathbb{N}^2$.

Suppose now that $D = b^2 - 4ac < 0$. Then, the argument is the same as above, but, since $2a+b$ can be negative, we need to show that both U_n and V_n tend to $+\infty$ as $n \to \infty$. (Then, we can take the solutions as in (17) but with n large enough.)

To show that $U_n = (2a+b)X_{2n} + 4dY_{2n} \to \infty$ as $n \to \infty$, we first observe that $X_{2n} \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} 4dY_{2n}/X_{2n} = 2\sqrt{d}$, by (15). So, it remains to verify the inequality

$$2a + b + 2\sqrt{d} > 0. \tag{18}$$

The inequality (18) clearly holds for $2a+b \geqslant 0$, whereas for $2a+b < 0$ it is equivalent to $4d > (2a+b)^2$. This inequality indeed holds, because

$$4d - (2a+b)^2 = 4a^2 + 4ab + 4ac - 4a^2 - 4ab - b^2 = 4ac - b^2 = -D > 0.$$

Similarly, to show that $V_n = (2a+b)Y_{2n} + X_{2n} \to \infty$ as $n \to \infty$, we observe that $Y_{2n} \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} X_{2n}/Y_{2n} = 2\sqrt{d}$, by (15). Hence, we arrive to the same inequality (18), which is already verified. ∎

4. Examples

Example 1. Selecting $(a, b, c) = (2, -2, 1)$ in (5), we find that $P_{2,-2,1}(x) = 4x^4+1$. With this choice, $d = a(a+b+c) = 2$ is not a perfect square and $D = b^2 - 4ac = -4 < 0$. Theorem 2 implies that (6) has infinitely many solutions in $(x, y) \in \mathbb{N}^2$. Therefore, by Theorem 1,

$$\prod_{k=1}^{m} (4k^4 + 1)$$

is a perfect square for infinitely many $m \in \mathbb{N}$. This reproduces the result of Gürel [13]. A similar choice $(a,b,c) = (2,-2,3)$ shows that

$$\prod_{k=1}^{m}(4k^4+8k^2+9)$$

is a perfect square for infinitely many $m \in \mathbb{N}$.

Example 2. For $(a,b,c) = (4,1,-4)$ we obtain $P_{4,1,-4}(x) = 16x^4+40x^3-3x^2-35x-4$. Now, $d = a(a+b+c) = 4$ is a perfect square and $D = b^2-4ac = 65 \neq 0$. Theorem 2 implies that (6) has only finitely many solutions in $(x,y) \in \mathbb{N}^2$. In fact, (6) is $4x^2+x-4 = y^2$. This equation has two solutions in positive integers $(x,y) = (1,1)$ and $(4,8)$. Indeed, for $x = 2,3$ the expression $4x^2+x-4$ is not a perfect square. It is also not a perfect square for $x \geqslant 5$, since then $(2x)^2 < 4x^2+x-4 < (2x+1)^2$. Therefore, by Theorem 1,

$$\prod_{k=1}^{m}(16k^4+40k^3-3k^2-35k-4)$$

is a perfect square for $m = 3$ only, when it equals $15366400 = 3920^2$.

Example 3. For $(a,b,c) = (1,-1,t^2)$, where $t \in \mathbb{N}$, we have

$$P_{1,-1,t}(x) = x^4+(2t^2-1)x^2+t^4.$$

With this choice, $d = a(a+b+c) = t^2$ is a perfect square and $D = b^2-4ac = 1-4t^2 \neq 0$. Since (6) is $t^2(x^2-x+t^2) = y^2$, we must have $t|y$, which leads to $x^2-x+t^2 = z^2$, where $z \in \mathbb{N}$. Clearly, it has no solutions in $(x,z) \in \mathbb{N}^2$ with $x \geqslant 2$ when $t = 1$ and has a unique such solution $(x,z) = (4,4)$ when $t = 2$. Thus,

$$\prod_{k=1}^{m}(k^4+k^2+1)$$

is never a perfect square (this is misstated in [11]), whereas

$$\prod_{k=1}^{m}(k^4+7k^2+16)$$

is a perfect square for $m = 3$ only, when it equals $230400 = 480^2$.

Example 4. Take $K = \mathbb{Q}(i)$ and $(a,b,c) = (1,-1,1+i)$. The two embeddings of $\mathbb{Q}(i)$ into $\mathbb{C}$ are the identity $u+iv \mapsto u+iv$ and $u+iv \mapsto u-iv$ (here $u,v \in \mathbb{Q}$). Hence, by (5),

$$P_{1,-1,1+i}(k)P_{1,-1,1-i}(k) = (k^4+4)(k^2+1)^2.$$

Note that equation (8) becomes $2((x^2-x+1)^2+1)=y^2$, so $y=2z$ with $z\in\mathbb{N}$. This gives the equation

$$(x^2-x+1)^2+1=2z^2. \tag{19}$$

Evidently, $\prod_{k=1}^m (k^2+1)^2$ is always a perfect square. Hence, by the second part of Theorem 1, the product

$$\prod_{k=1}^m (k^4+4)$$

is perfect square iff $(x,z)=(m+1,z)$ is a solution of (19) in positive integers $x\geqslant 2$, z. By [13], the above product is a square for $m=2$ only. This corresponds to the solution $(x,z)=(3,5)$ of (19).

Example 5. Let $K=\mathbb{Q}(\sqrt{5})$ and $(a,b,c)=(1,-1,(3+\sqrt{5})/2)$. The two embeddings K into $\mathbb{C}$ are the identity $u+\sqrt{5}v\mapsto u+\sqrt{5}v$ and $u+\sqrt{5}v\mapsto u-\sqrt{5}v$ (here $u,v\in\mathbb{Q}$). Hence, by (5),

$$P_{1,-1,(3+\sqrt{5})/2}(k)P_{1,-1,(3-\sqrt{5})/2}(k)=k^8+4k^6+6k^4-k^2+1.$$

Note that for $\ell=1$ equation (8) becomes $(x^2-x)^2+3(x^2-x)+1=y^2$, which is equivalent to $(2x^2-2x+3)^2-5=(2y)^2$. It has integer solutions only when $2x^2-2x+3=\pm 3$, that is, $x=0$ and $x=1$. Hence, by the second part of Theorem 1, the product

$$\prod_{k=1}^m (k^8+4k^6+6k^4-k^2+1)$$

is never a perfect square.

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