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ARITHMETICAL PROPERTIES OF REAL NUMBERS RELATED TO BETA-EXPANSIONS

Hajime Kaneko

Abstract: The main purpose of this paper is to study the arithmetical properties of values $\sum_{m=0}^{\infty}\beta^{-w(m)}$, where β is a fixed Pisot or Salem number and $w(m)$ $(m=0,1,\ldots)$ are distinct sequences of nonnegative integers with $w(m+1)>w(m)$ for any sufficiently large m. We first introduce the algebraic independence results of such values. Our results are applicable to certain sequences $w(m)$ $(m=0,1,\ldots)$ with $\lim_{m\to\infty} w(m+1)/w(m)=1$. For example, we prove that two numbers

$$\sum_{m=1}^{\infty}\beta^{-\lfloor\varphi(m)\rfloor},\quad \sum_{m=3}^{\infty}\beta^{-\lfloor a(m)\rfloor}$$

are algebraically independent, where $\varphi(m)=m^{\log m}$ and $a(m)=m^{\log\log m}$.

Moreover, we also give the linear independence results of real numbers. Our results are applicable to the values $\sum_{m=0}^{\infty}\beta^{-\lfloor m^{\rho}\rfloor}$, where β is a Pisot or Salem number and ρ is a real number greater than 1.

Keywords: algebraic independence, power series, beta expansion, Pisot numbers, Salem numbers.

1. Introduction

Throughout this paper, we denote the set of nonnegative integers (resp. positive integers) by $\mathbb{N}$ (resp. $\mathbb{Z}^{+}$). We write the integral and fractional parts of a real number x by $\lfloor x\rfloor$ and $\{x\}$, respectively. Moreover, $\lceil x\rceil$ is the minimal integer not less than x. We use the Vinogradov symbols $\gg$ and $\ll$, as well as the Landau symbols O, o with their regular meanings. Finally, $f\sim g$ means that the ratio f/g tends to 1

In what follows, we investigate the arithmetical properties of the values of power series $f(X)$ at algebraic points. For simplicity, we first consider the case where $f(X)$ has the form

$$f(X)=\sum_{m=0}^{\infty}X^{w(m)},$$

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where $(w(m))_{m=0}^{\infty}$ is a sequence of nonnegative integers satisfying $w(m) < w(m+1)$ for any sufficiently large m. We call $f(X)$ a gap series if

$$\lim_{m\to\infty} \frac{w(m+1)}{w(m)} = \infty.$$

We say that $f(X)$ is a lacunary series if

$$\liminf_{m\to\infty} \frac{w(m+1)}{w(m)} > 1.$$

Note that if $f(X)$ is a lacunary series, then there exists a positive real number δ such that

$$w(m) > (1+\delta)^m$$

for any sufficiently large m.

In the rest of this secction, suppose that α is an algebraic number with $0 < |\alpha| < 1$. In paper [7], Bugeaud posed a problem on the transcendence of the values of power series $f(X)$ as follows: If $(w(m))_{m=0}^{\infty}$ increases sufficiently rapidly, then $\sum_{m=0}^{\infty} \alpha^{w(m)}$ is transcendental.

Corvaja and Zannier [8] showed that if $f(X) = \sum_{m=0}^{\infty} X^{w(m)}$ is a lacunary series, then $\sum_{m=0}^{\infty} \alpha^{w(m)}$ is transcendental. For instance, let x, y be real numbers with $x > 0$ and $y > 1$. Then two numbers

$$\sum_{m=0}^{\infty} \alpha^{\lfloor x(m!)\rfloor}, \qquad \sum_{m=0}^{\infty} \alpha^{\lfloor y^m\rfloor}$$

are transcendental.

Adamczewski [1] improved the result above in the case of $\alpha = \beta^{-1}$, where β is a Pisot or Salem number. Recall that Pisot numbers are algebraic integers greater than 1 whose conjugates except themselves have absolute values less than 1. Note that any rational integers greater than 1 are Pisot numbers. Salem numbers are algebraic integers greater than 1 such that the conjugates except themselves have moduli less than 1 and that there exists at least one conjugate with modulus 1. Adamczewski [1] showed that if

$$\liminf_{m\to\infty} \frac{w(m+1)}{w(m)} > 1,$$

then $\sum_{m=0}^{\infty} \beta^{-w(m)}$ is transcendental for any Pisot or Salem number β.

We now introduce known results on the algebraic independence of certain lacunary series at fixed algebraic points. First we consider the case where $f(X)$ is a gap series. Durand [10] showed that if α is a real algebraic number with $0 < \alpha < 1$, then the continuum set

$$\left\{ \sum_{m=0}^{\infty} \alpha^{\lfloor x(m!)\rfloor} \,\middle|\, x \in \mathbb{R}, x > 0 \right\} \tag{1.1}$$

is algebraically independent. Moreover, Shiokawa [17] gave a criterion for the algebraic independence of the values of certain gap series. Using his criterion, we deduce for general algebraic number α with $0 < |\alpha| < 1$ that the set (1.1) is algebraically independent.

Next, we consider the case where $f(X)$ is not a gap series. Using Mahler's method for algebraic independence, Nishioka [15] proved that the set

$$\left\{ \sum_{m=0}^{\infty} \alpha^{k^m} \,\middle|\, k = 2, 3, \dots \right\}$$

is algebraically independent. Moreover, Tanaka [18] showed that if positive real numbers $w_1, \dots, w_m$ are linearly independent over $\mathbb{Q}$, then the set

$$\left\{ \sum_{m=0}^{\infty} \alpha^{\lfloor w_i k^m \rfloor} \,\middle|\, i = 1, \dots, m,\ k = 2, 3, \dots \right\}$$

is algebraically independent.

On the other hand, it is generally difficult to study algebraic independence in the case where $f(X)$ is not lacunary. In Section 2 we review known results on the criteria for transcendence of the value $\sum_{m=0}^{\infty} \beta^{-w(m)}$, where β is a Pisot or Salem number and $(w(m))_{m=0}^{\infty}$ is a certain sequence of nonnegative integers with

$$\lim_{m \to \infty} \frac{w(m+1)}{w(m)} = 1.$$

In Section 3 we give the main results on the algebraic independence of real numbers. Our results are applicable to the algebraic independence of the two values

$$\sum_{m=1}^{\infty} \beta^{-\lfloor m^{\log m} \rfloor}, \qquad \sum_{m=3}^{\infty} \beta^{-\lfloor m^{\log \log m} \rfloor}.$$

In the same section we also investigate the linear independence of real numbers applicable to $\sum_{m=0}^{\infty} \beta^{-\lfloor m^{\rho} \rfloor}$ for a real number $\rho > 1$. The main criterion for linear independence, which is used to prove the main results, is denoted in Section 4. For the proof of the algebraic independence and the linear independence, we need no functional equation because our criterion is flexible. We prove the main results in Section 5. Moreover, we show the main criterion in Section 6.

2. Transcendental results related to the numbers of nonzero digits

In this section we review criteria for the transcendence of the value $\sum_{n=0}^{\infty} t_n \beta^{-n}$, where $(t_n)_{n=0}^{\infty}$ is a bounded sequence of nonnegative integers and β is a Pisot or Salem number. First we consider the case where $\beta = b$ is an integer greater than 1. We denote the base-b expansion of a real number η by

$$\eta = \sum_{n=0}^{\infty} s_n^{(b)}(\eta) b^{-n},$$

where $s_0^{(b)}(\eta) = \lfloor \eta \rfloor$ and $s_n^{(b)}(\eta) \in \{0, 1, \ldots, b-1\}$ for any positive integer n. We may assume that $s_n^{(b)}(\eta) \leqslant b-2$ for infinitely many n's. For any positive integer N, put

$$\lambda_b(\eta; N) := \mathrm{Card}\{n \in \mathbb{N} \mid n < N, s_n^{(b)}(\eta) \neq 0\},$$

where Card denotes the cardinality.

Borel [5] conjectured for each integral base $b \geqslant 2$ that any algebraic irrational number is normal in base-b, which is still an open problem. For any real number $\rho > 1$, put

$$\gamma(\rho; X) := \sum_{m=0}^{\infty} X^{\lfloor m^\rho \rfloor}.$$

If Borel's conjecture is true, then $\gamma(\rho; b^{-1})$ is transcendental because $\gamma(\rho; b^{-1})$ is a non-normal irrational number in base-b. However, the transcendence of such values is not known except the case of $\rho = 2$. If $\rho = 2$, then Duverney, Nishioka, Nishioka, Shiokawa [11] and Bertrand [4] independently proved for any algebraic number α with $0 < |\alpha| < 1$ that $\gamma(2; \alpha)$ is transcendental.

Bailey, Borwein, Crandall, and Pomerance [3] gave a criterion for the transcendence of real numbers, using lower bounds for the numbers of nonzero digits in the binary expansions of algebraic irrational numbers. Let η be an algebraic irrational number with degree D. Bailey, Borwein, Crandall, and Pomerance [3] showed that there exist positive constants $C_1(\eta)$ and $C_2(\eta)$, depending only on η, satisfying

$$\lambda_2(\eta; N) \geqslant C_1(\eta) N^{1/D}$$

for any integer N with $N \geqslant C_2(\eta)$. Note that $C_1(\eta)$ is effectively computable but $C_2(\eta)$ is not. For any integral base $b \geqslant 2$, Adamczewski, Faverjon [2] and Bugeaud [6] gave effective versions of lower bounds for $\lambda_b(\eta; N)$ as follows: There exist effectively computable positive constants $C_3(b, \eta)$ and $C_4(b, \eta)$, depending only on b and η, satisfying

$$\lambda_b(\eta; N) \geqslant C_3(b, \eta) N^{1/D} \tag{2.1}$$

for any integer N with $N \geqslant C_4(b, \eta)$. Using (2.1), we obtain for any real number $\rho > 1$ that $\gamma(\rho; b^{-1})$ is not an algebraic number of degree less than ρ. In fact, $\gamma(\rho; b^{-1})$ is an irrational number satisfying

$$\lambda_b\big(\gamma(\rho; b^{-1}); N\big) \sim N^{1/\rho}$$

as N tends to infinity. Thus, (2.1) does not hold if $D < \rho$.

By (2.1), we also deduce a criterion for the transcendence of real numbers as follows: Let η be a positive irrational number. Suppose for any real positive real number ε that

$$\liminf_{N \to \infty} \frac{\lambda_b(\eta; N)}{N^\varepsilon} = 0. \tag{2.2}$$

Then η is a transcendental number. Note that the criterion above was essentially obtained by Bailey, Borwein, Crandall, and Pomerance [3]. Note that if $\sum_{m=0}^{\infty} X^{w(m)}$ is lacunary, then $\eta = \sum_{m=0}^{\infty} b^{-w(m)}$ satisfies (2.2) by

$$\lambda_b(\eta; N) = O(\log N).$$

We give another example of transcendental numbers. For any real numbers $y > 0$ and $R \geqslant 1$, we put

$$\varphi(y; R) := \exp\big((\log R)^{1+y}\big) = R^{(\log R)^y}.$$

Moreover, we set

$$\xi(y; X) := 1 + \sum_{m=1}^{\infty} X^{\lfloor \varphi(y;m) \rfloor}.$$

Note that $\xi(y; X)$ is not lacunary by

$$\lim_{m\to\infty} \frac{\varphi(y; m+1)}{\varphi(y; m)} = 1.$$

We get that $\eta := \xi(y; b^{-1})$ is transcendental for any integer $b \geqslant 2$ because η satisfies (2.2).

In what follows, we consider the case where β is a general Pisot or Salem number. We introduce results in [14] related to the β-expansion of algebraic numbers. For any formal power series $f(X) = \sum_{n=0}^{\infty} t_n X^n$, we put

$$S(f) := \{n \in \mathbb{N} \mid t_n \neq 0\}.$$

Moreover, for any nonempty set $\mathcal{A}$ of nonnegative integers, we set

$$\lambda(\mathcal{A}; N) := \mathrm{Card}(\mathcal{A} \cap [0, N)).$$

We denote the degree of a field extension L/K by $[L : K]$.

Theorem 2.1 ([14]). *Let A be a positive integer and let $f(X) = \sum_{n=0}^{\infty} t_n X^n$ be a power series with integral coefficients. Assume that $0 \leqslant t_n \leqslant A$ for any nonnegative integer n and that there exist infinitely many n's satisfying $t_n \neq 0$. Let β be a Pisot or Salem number. Suppose that $\eta = f(\beta^{-1})$ is an algebraic number with $[\mathbb{Q}(\beta, \eta) : \mathbb{Q}(\beta)] = D$. Then there exist effectively computable positive constants $C_5(A, \beta, \eta)$ and $C_6(A, \beta, \eta)$, depending only on A, β and η satisfying*

$$\lambda\big(S(f); N\big) \geqslant C_5(A, \beta, \eta) \left(\frac{N}{\log N}\right)^{1/D}$$

for any integer N with $N \geqslant C_6(A, \beta, \eta)$.

In the rest of this section, let β be a Pisot or Salem number. Using Theorem 2.1, we obtain for any real number $\rho > 1$ that

$$\big[\mathbb{Q}\big(\gamma(\rho; \beta^{-1}), \beta\big) : \mathbb{Q}(\beta)\big] \geqslant \lceil \rho \rceil$$

by

$$\lambda(S(\gamma(\rho;X));N) \sim N^{1/\rho} \tag{2.3}$$

as N tends to infinity.

Note that Theorem 2.1 is applicable to the study of the nonzero digits in the β-expansions of algebraic numbers. We recall the definition of β-expansion defined by Rényi [16] in 1957. Let $T_\beta : [0,1) \to [0,1)$ be the β-transformation defined by $T_\beta(x) = \{\beta x\}$ for $x \in [0,1)$. Then the β-expansion of a real number $\eta \in [0,1)$ is denoted as

$$\eta = \sum_{n=1}^{\infty} s_n^{(\beta)}(\eta)\beta^{-n},$$

where $s_n^{(\beta)}(\eta) = \lfloor \beta T_\beta^{n-1}(\eta) \rfloor$ for any $n \geqslant 1$. Note that $0 \leqslant s_n^{(\beta)}(\eta) \leqslant \lfloor \beta \rfloor$ for any $n \geqslant 1$. Put

$$\lambda_\beta(\eta;N) := \mathrm{Card}\{n \in \mathbb{Z}^+, n \leqslant N, s_n^{(\beta)}(\eta) \neq 0\}$$

for any positive integer N. Applying Theorem 2.1 with $B = \lfloor \beta \rfloor$, we deduce that if $\eta \in [0,1)$ is an algebraic number with $[\mathbb{Q}(\beta,\eta) : \mathbb{Q}(\beta)] = D$, then

$$\lambda_\beta(\eta;N) \gg \left(\frac{N}{\log N}\right)^{1/D}$$

for any sufficiently large integer N.

Using Theorem 2.1, we also deduce a criterion for the transcendence of real numbers as follows: Let $f(X)$ be a power series whose coefficients are bounded nonnegative integers. Suppose that $f(X)$ is not a polynomial and that

$$\liminf_{m\to\infty} \frac{\lambda_\beta(S(f);N)}{N^\varepsilon} = 0$$

for any positive real number ε. Then $f(\beta^{-1})$ is transcendental. Note that the criterion above was already obtained in [13] and that the criterion is applicable even if the representation $\sum_{n=0}^{\infty} t_n\beta^{-n}$ does not coincide with the β-expansion of $f(\beta^{-1})$. In the same way as the case where $\beta = b \geqslant 2$ is an integer, we obtain for any positive real number y that $\xi(y;\beta^{-1})$ is transcendental.

In the end of this section we introduce a corollary of Theorem 2.1, which we need to prove our criteria for linear independence.

Corollary 2.2. *Let A be a positive integer and $f(X)$ a nonpolynomial power series whose coefficients are bounded nonnegative integers. Assume that there exists a positive real number δ satisfying*

$$\lambda\bigl(S(f);R\bigr) < R^{-\delta+1/A}$$

for infinitely many integer $R \geqslant 0$. Then, for any Pisot or Salem number β, we have

$$\bigl[\mathbb{Q}\bigl(f(\beta^{-1}),\beta\bigr) : \mathbb{Q}(\beta)\bigr] \geqslant A+1.$$

3. Main results

3.1. Results on algebraic independence

We use the same notation as Section 2.

Theorem 3.1. *Let β be a Pisot or Salem number. Then the continuum set*

$$\{\xi(y;\beta^{-1}) \mid y \in \mathbb{R},\ y \geqslant 1\} \tag{3.1}$$

is algebraically independent.

Note that if $\beta = b$ is an integer greater than 1, then the algebraic independence of (3.1) was proved in [12]. Moreover, Theorem 1.4 in [12] implies that if y_1 and y_2 are distinct positive real numbers, then the two values $\xi(y_1, b^{-1})$ and $\xi(y_2, b^{-1})$ are algebraically independent. However, the algebraic independence of the set

$$\{\xi(y;b^{-1}) \mid y \in \mathbb{R},\ y > 0\}$$

is unknown. Next, we generalize Theorem 1.4 in [12] as follows:

Theorem 3.2. *Let y_1 and y_2 be distinct positive real numbers. Then the two values $\xi(y_1;\beta^{-1})$ and $\xi(y_2;\beta^{-1})$ are algebraically independent for any Pisot or Salem number β.*

We now give further results on the algebraic independence of two values.

Theorem 3.3. *For any Pisot or Salem number β the two values*

$$\sum_{m=1}^{\infty} \beta^{-\lfloor m^{\log m}\rfloor}, \qquad \sum_{m=3}^{\infty} \beta^{-\lfloor m^{\log\log m}\rfloor}$$

are algebraically independent.

Many results on the algebraic independence treat the values of power series with same types, for instances, the values of Fredholm series $\sum_{m=0}^{\infty} X^{k^m}$ for $k = 2, 3, \ldots$ (see also Section 1). On the other hand, our results are applicable to the values of two power series of different types. We now introduce a typical example as follows:

Theorem 3.4. *Let y and x be real numbers with $y > 0, x > 1$. Then, the two values $\xi(y;\beta^{-1})$ and $\sum_{m=0}^{\infty} \beta^{-\lfloor x^m\rfloor}$ are algebraically independent for any Pisot or Salem number β.*

3.2. Results on linear independence

Let $\mathcal{F}$ be the set of nonpolynomial power series $g(X)$ with bounded nonnegative integral coefficients satisfying the following two assumptions:

1. For an arbitrary positive real number ε, we have

   $$\lambda(S(g);R) = o(R^{\varepsilon})$$

   as R tends to infinity.

2. There exists a positive constant C such that

$$[R, CR] \cap S(g) \neq \emptyset$$

for any sufficiently large R.

In this subsection, we study arithmetical properties of the values of power series with different types. In particular, we study the linear independence of the values $f(\beta^{-1})^{k_1} g(\beta^{-1})^{k_2}$ ($k_1, k_2 \in \mathbb{N}$), where β is a Pisot or Salem number, $f(X) = \gamma(\rho; X)$ for some $\rho > 1$, and $g(X) \in \mathcal{F}$. In order to state our results, we give a lemma on the zeros of certain polynomials. For any positive integer k, put

$$G_k(X) := (1 - X)^k + (k-1)X - 1.$$

Lemma 3.5. *Suppose that $k \geqslant 3$. Then the following holds:*

1) *There exists a unique zero σ_k of $G_k(X)$ on the interval $(0, 1)$.*
2) *Let x be a real number with $0 < x < 1$. Then $G_k(x) < 0$ (resp. $G_k(x) > 0$) if and only if $x < \sigma_k$ (resp. $x > \sigma_k$).*
3) *$(\sigma_k)_{k=3}^{\infty}$ is strictly decreasing.*

Proof. Observe that $G_k'(X) = -k(1-X)^{k-1} + k - 1$ is monotone increasing on the interval $(0,1)$ and that $G_k'(X)$ has a unique zero $\widetilde{\sigma_k}$ on $(0,1)$. Thus, $G_k(X)$ is monotonically decreasing on $(0, \widetilde{\sigma_k}]$ and monotonically increasing on $(\widetilde{\sigma_k}, 1)$. Hence, the first and second statements of the lemma follow from $G_k(0) = 0$ and $G_k(1) = k - 2 > 0$.

Next, we assume that $k \geqslant 4$. Using

$$G_{k-1}(\sigma_{k-1}) = (1 - \sigma_{k-1})^{k-1} + (k-2)\sigma_{k-1} - 1 = 0,$$

we get

$$G_k(\sigma_{k-1}) = (1 - \sigma_{k-1})^k + (k-1)\sigma_{k-1} - 1 = (k-2)\sigma_{k-1}^2 > 0.$$

Hence, we obtain $\sigma_k < \sigma_{k-1}$ by the second statement of the lemma. ■

Theorem 3.6. *Let A be a positive integer and ρ a real number. Suppose that*

$$\begin{cases} \rho > A & \text{if } A \leqslant 3, \\ \rho > \sigma_A^{-1} & \text{if } A \geqslant 4. \end{cases} \tag{3.2}$$

Then, for any $g(X) \in \mathcal{F}$ and any Pisot or Salem number β, the set

$$\{\gamma(\rho; \beta^{-1})^{k_1} g(\beta^{-1})^{k_2} \mid k_1, k_2 \in \mathbb{N}, k_1 \leqslant A\}$$

is linearly independent over $\mathbb{Q}(\beta)$.

We give numerical examples of σ_n^{-1} ($n \geqslant 4$) as follows:

$$\sigma_4^{-1} = 5.278\ldots, \qquad \sigma_5^{-1} = 8.942\ldots, \qquad \sigma_6^{-1} = 13.60\ldots.$$

Corollary 3.7. *Let A, ρ be as in Theorem 3.6.*

1) *For any real number $y > 1$ and any Pisot or Salem number β, the set*

$$\left\{ \gamma(\rho;\beta^{-1})^{k_1} \left(\sum_{m=0}^{\infty} \beta^{-\lfloor y^m \rfloor} \right)^{k_2} \;\middle|\; k_1, k_2 \in \mathbb{N}, k_1 \leqslant A \right\}$$

is linearly independent over $\mathbb{Q}(\beta)$.

2) *For any positive real number y and any Pisot or Salem number β, the set*

$$\{\gamma(\rho;\beta^{-1})^{k_1}\xi(y;\beta^{-1})^{k_2} \mid k_1, k_2 \in \mathbb{N}, k_1 \leqslant A\}$$

is linearly independent over $\mathbb{Q}(\beta)$.

Using the asymptotic behavior of the sequence $(\sigma_m)_{m=3}^{\infty}$, we deduce the following:

Corollary 3.8. *Let ε be an arbitrary positive real number. Then there exists an effectively computable positive constant $A_0(\varepsilon)$, depending only on ε satisfying the following: Let A be an integer with $A \geqslant A_0(\varepsilon)$ and ρ a real number with $\rho > (\varepsilon + 1/2)A^2$. Then, for any $g(X) \in \mathcal{F}$ and any Pisot or Salem number β, the set*

$$\{\gamma(\rho;\beta^{-1})^{k_1} g(\beta^{-1})^{k_2} \mid k_1, k_2 \in \mathbb{N}, k_1 \leqslant A\}$$

is linearly independent over $\mathbb{Q}(\beta)$.

4. Main criterion for linear independence

Let k be a nonnegative integer and $f(X) \in \mathbb{Z}[[X]] \backslash \mathbb{Z}[X]$. We denote the Minkowski sum of $S(f)$ by

$$kS(f) := \begin{cases} \{0\} & (k = 0), \\ \{s_1 + \cdots + s_k \mid s_1, \ldots, s_k \in S(f)\} & (k \geqslant 1). \end{cases}$$

Moreover, for any $(k_1, \ldots, k_r) \in \mathbb{N}^r$ and $f_1(X), \ldots, f_r(X) \in \mathbb{Z}[[X]] \backslash \mathbb{Z}[X]$, we set

$$\sum_{h=1}^{r} k_h S(f_h) := \{s_1 + \cdots + s_r \mid s_h \in k_h S(f_h) \text{ for } h = 1, \ldots, r\}.$$

Remark 1. Suppose that $0 \in S(f_i)$ for $i = 1, \ldots, r$. Then, for any $(k_1, \ldots, k_r) \in \mathbb{N}^r$ and $(k'_1, \ldots, k'_r) \in \mathbb{N}^r$ with $k_i \geqslant k'_i$ for any $i = 1, \ldots, r$, we have

$$\sum_{h=1}^{r} k_h S(f_h) \supset \sum_{h=1}^{r} k'_h S(f_h).$$

Let $\mathcal{A}$ be a nonempty set of nonnegative integers and R a real number with $R > \min \mathcal{A}$. Then we put

$$\theta(R; \mathcal{A}) := \max\{n \in \mathcal{A} \mid n < R\}.$$

Theorem 4.1. *Let A, r be integers with $A \geqslant 1$ and $r \geqslant 2$. Let $f_i(X) = \sum_{n=0}^{\infty} t_i(n) X^n (i = 1, \ldots, r)$ be nonpolynomial power series with integral coefficients satisfying*

$$0 \leqslant t_i(n) \leqslant C_7$$

for any $i = 1, \ldots, r$ and $n \geqslant 0$, where C_7 is a positive constant. We assume that $f_1(X), \ldots, f_r(X)$ satisfy the following three assumptions:

1. *There exists a positive real number δ satisfying*

$$\lambda(S(f_1); R) = o\left(R^{-\delta+1/A}\right)$$

as R tends to infinity. Moreover, for any $i = 2, \ldots, r$ and any real number ε, we have

$$\lambda(S(f_i); R) = o\Big(\lambda\big(S(f_{i-1}); R\big)^{\varepsilon}\Big)$$

as R tends to infinity.

2. *There exist positive constants C_8, C_9 such that*

$$[R, C_8 R] \cap S(f_r) \neq \emptyset$$

for any real number R with $R \geqslant C_9$.

3. *Let $k_1, \ldots, k_r$ be nonnegative integers. Suppose that*

$$\begin{cases} k_1 \leqslant A - 1 & \text{if } r = 2, \\ k_1 \leqslant A & \text{if } r \geqslant 3. \end{cases} \tag{4.1}$$

Then we have

$$R - \theta\left(R; \sum_{h=1}^{r-2} k_h S(f_h) + (1 + k_{r-1}) S(f_{r-1})\right) < \frac{R}{\prod_{h=1}^{r} \lambda(S(f_h); R)^{k_h}} \tag{4.2}$$

for any sufficiently large R, depending only on $k_1, \ldots, k_r$.

Then, for any Pisot or Salem number β, the set

$$\{f_1(\beta^{-1})^{k_1} f_2(\beta^{-1})^{k_2} \cdots f_r(\beta^{-1})^{k_r} \mid k_1, k_2, \ldots, k_r \in \mathbb{N}, k_1 \leqslant A\}$$

is linearly independent over $\mathbb{Q}(\beta)$.

Remark 2. Theorem 4.1 is a generalization of Theorem 2.1 in [12] because it is applicable to more general linear independence results. The third assumption of Theorem 2.1 in [12] is essentially as follows: Let $k_1, \ldots, k_{r-1}, k_r$ be nonnegative integers. Suppose that there exists a positive integer $\kappa = \kappa(k_1, \ldots, k_{r-1})$, depending only on $k_1, \ldots, k_{r-1}$, satisfying

$$R - \theta\left(R; \sum_{h=1}^{r-2} k_h S(f_h) + \kappa S(f_{r-1})\right) < \frac{R}{\prod_{h=1}^{r} \lambda(S(f_h); R)^{k_h}}$$

for any sufficiently large R, depending only on $k_1, \ldots, k_{r-1}, k_r$. In Theorem 4.1 we only consider the case of $\kappa = 1 + k_{r-1}$ for simplicity, which is sufficient to prove our main results.

5. Proof of main results

In this section we prove results in Section 3, using Theorem 4.1.

5.1. Proof of results on algebraic independence

Proof of Theorem 3.1. Let $y_1, y_2, \ldots, y_r$ be real numbers with $1 \leqslant y_1 < y_2 < \cdots < y_r$. We show that $f_i(X) := \xi(y_i; X)$ $(i = 1, \ldots, r)$ fulfill the assumptions of Theorem 4.1 for any positive integer A. Recall that we proved Theorem 1.3 in [12], showing for any integer $b \geqslant 2$ that $f_1(b^{-1}), \ldots, f_r(b^{-1})$ satisfy the assumptions of Theorem 2.1 in [12]. In particular, we verified the third assumption with $\kappa = 1 + k_{r-1}$. In the same way, we can check that $f_1(X), \ldots, f(X)$ fulfill the assumptions of Theorem 4.1. ■

Proof of Theorem 3.2. We can show Theorem 3.2 in the same way as the proof of Theorem 1.4 in [12] by checking the assumptions of Theorem 4.1. ■

Proof of Theorem 3.3. Put

$$a(R) := \exp\big((\log R)(\log\log R)\big) = R^{\log\log R}$$

for $R \geqslant 3$ and

$$f_1(X) := 1 + \sum_{m=3}^{\infty} X^{\lfloor a(m) \rfloor}, \qquad f_2(X) := \xi(1; X).$$

Then $f_1(X)$ and $f_2(X)$ satisfy the second assumption of Theorem 4.1. We denote the inverse function of $a(R)$ by $b(R)$. For simplicity, let $\varphi(R) := \varphi(1; R)$. Note that the inverse function of $\varphi(R)$ is

$$\psi(R) := \exp\left((\log R)^{1/2}\right).$$

Then we have

$$(\log b(R))(\log\log b(R)) = \log R = (\log \psi(R))^2.$$

Thus, we see

$$\frac{\log b(R)}{\log R} = \frac{1}{\log\log b(R)} = o(1)$$

as R tends to infinity, and so, for any positive real number ε,

$$\lambda\bigl(S(f_1);R\bigr) \sim b(R) = o\left(R^{\varepsilon}\right). \tag{5.1}$$

Moreover, using

$$\frac{\log \psi(R)}{\log b(R)} = \left(\frac{\log\log b(R)}{\log b(R)}\right)^{1/2} = o(1),$$

we get, for any ε,

$$\lambda\bigl(S(f_2);R\bigr) \sim \psi(R) = o\left(b(R)^{\varepsilon}\right) = o\Bigl(\lambda\bigl(S(f_1);R\bigr)^{\varepsilon}\Bigr). \tag{5.2}$$

Hence, we showed that the first assumption of Theorem 4.1 is satisfied for any positive integer A. In what follows, we verify that the third assumption of Theorem 4.1 is fulfilled. It suffices to show the following: For any positive integer k and any real number ε with $0 < \varepsilon < 1$, we have

$$R - \theta(R; kS(f_1)) \ll Rb(R)^{-k+\varepsilon} \tag{5.3}$$

for any sufficiently large R, depending only on k and ε, where the implied constant in the symbol $\ll$ does not depend on R, but on k and ε. In fact, let k_1 and k_2 be any nonnegative integers. Using (5.3) with $k = 1 + k_1$ and $\varepsilon = 1/2$, we get

$$R - \theta(R; (1+k_1)S(f_1)) \ll Rb(R)^{-k_1-1/2} = o\left(\frac{R}{\prod_{h=1}^{2}\lambda(S(f_h);R)^{k_h}}\right)$$

as R tends to infinity, where for the last inequality we use (5.1) and (5.2).

We first consider the case of $k = 1$. Note that

$$\bigl(\log a(x)\bigr)' = \frac{a(x)'}{a(x)} = \frac{1+\log\log x}{x}.$$

Using the mean value theorem for $\log a(x)$, we see for $m \geqslant 3$ that

$$a(m) \leqslant a(m+1) \ll a(m).$$

Using the mean value theorem for $a(x)$, we get that there exists a real number ρ with $0 < \rho < 1$ satisfying

$$\begin{aligned} a(m+1) - a(m) &= a'(m+\rho) \\ &= \frac{1+\log\log(m+\rho)}{m+\rho} a(m+\rho) \ll \frac{a(m)}{(m+1)^{1-\varepsilon}}. \end{aligned}$$

For any sufficiently large R, there exists an integer $m \geqslant 3$ such that

$$\lfloor a(m) \rfloor < R \leqslant \lfloor a(m+1) \rfloor.$$

Thus, we obtain

$$\begin{aligned} R - \theta\big(R; S(f_1)\big) &= R - \lfloor a(m) \rfloor \leqslant a(m+1) - a(m) + 1 \\ &\ll \frac{a(m)}{(m+1)^{1-\varepsilon}} \ll \frac{R}{b(a(m+1))^{1-\varepsilon}} \leqslant \frac{R}{b(R)^{1-\varepsilon}}, \end{aligned}$$

which implies (5.3) in the case of $k = 1$.

Next we assume that (5.3) holds for a fixed positive integer k and an arbitrary positive real number ε. In what follows, we verify (5.3) for $k+1$ with fixed $\varepsilon < 1$. Put

$$R_0 := R - \theta\big(R; kS(f_1)\big).$$

It suffices to consider the case of

$$R_0 \geqslant \frac{R}{b(R)^{k+1}}. \tag{5.4}$$

In fact, suppose that (5.4) does not hold. Since $0 \in S(f_1)$ by the definition of $f_1(X)$, we have

$$\theta\big(R; kS(f_1)\big) \in kS(f_1) \subset (k+1)S(f_1)$$

by Remark 1. Thus, we get

$$R - \theta\big(R; (k+1)S(f_1)\big) \leqslant R_0 < \frac{R}{b(R)^{k+1}},$$

which implies (5.3).

In what follows, we assume that (5.4) is satisfied. In particular, applying (5.1) to (5.4), we see

$$R_0 \geqslant R^{1-\varepsilon/4} \tag{5.5}$$

for any sufficiently large R. Moreover, the inductive hypothesis implies that

$$R_0 \ll \frac{R}{b(R)^{k-\varepsilon/2}}. \tag{5.6}$$

Let

$$\eta := \theta\big(R; kS(f_1)\big) + \theta\big(R_0; S(f_1)\big) \in (k+1)S(f_1).$$

Observing that

$$R - \eta = R_0 - \theta\big(R_0; S(f_1)\big) > 0,$$

we see

$$\theta\big(R; (k+1)S(f_1)\big) \geqslant \eta.$$

Thus, we get

$$\begin{aligned} R-\theta\big(R;(k+1)S(f_1)\big) &\leqslant R-\eta \\ &= R_0-\theta\big(R_0;S(f_1)\big) \ll \frac{R_0}{b(R_0)^{1-\varepsilon/4}}, \end{aligned}$$

where for the last inequality we apply (5.3) with $k=1$ because R_0 is sufficiently large by (5.5). By (5.5) and (5.6), we obtain

$$R-\theta\big(R;(k+1)S(f_1)\big) \ll \frac{R}{b(R)^{k-\varepsilon/2}b(R^{1-\varepsilon/4})^{1-\varepsilon/4}}. \tag{5.7}$$

Using the fact that $(\log a(x))/(\log x)$ is ultimately increasing with

$$x=b(R)>x'=b(R^{1-\varepsilon/4}),$$

we get

$$\begin{aligned} \frac{\log R}{\log b(R)} = \frac{\log a(x)}{\log x} &\geqslant \frac{\log a(x')}{\log x'} \\ &= \left(1-\frac{\varepsilon}{4}\right)\frac{\log R}{\log b(R^{1-\varepsilon/4})}. \end{aligned}$$

Consequently,

$$b\left(R^{1-\varepsilon/4}\right) \geqslant b(R)^{1-\varepsilon/4},$$

and so

$$\frac{1}{b(R^{1-\varepsilon/4})^{1-\varepsilon/4}} \leqslant \frac{1}{b(R)^{(1-\varepsilon/4)^2}} \leqslant \frac{1}{b(R)^{1-\varepsilon/2}} \tag{5.8}$$

by $(1-\varepsilon/4)^2 \geqslant 1-\varepsilon/2$. Combining (5.7) and (5.8), we deduce that

$$(0<)R-\theta\big(R;(k+1)S(f_1)\big) \ll \frac{R}{b(R)^{k+1-\varepsilon}},$$

which implies (5.3). ■

Proof of Theorem 3.4. Let

$$f_1(X)=\xi(y;X), \qquad f_2(X)=\sum_{m=0}^{\infty} X^{\lfloor x^m \rfloor}, \qquad \widetilde{f_2}(X):=\xi(2y;X).$$

Then $f_1(X)$ and $f_2(X)$ satisfy the second assumption of Theorem 4.1. We denote the inverse function of $\varphi(2y;R)$ by

$$\psi(2y;R)=\exp\left((\log R)^{1/(1+2y)}\right).$$

Then we have

$$\lambda\big(S(f_2);R\big)=o\left(\psi(2y;R)^{\varepsilon}\right)=o\left(\lambda\big(S(\widetilde{f_2});R\big)^{\varepsilon}\right)$$

as R tends to infinity. Thus, $f_1(X)$ and $f_2(X)$ satisfy the first and third assumptions of Theorem 4.1 because we checked in the proof of Theorem 3.2 that $f_1(X)$ and $\widetilde{f_2}(X)$ fulfill the same assumptions. ■

5.2. Proof of results on linear independence

Proof of Theorem 3.6. We show that the assumptions of Theorem 4.1 are satisfied, where A is defined as in Theorem 3.6, $r = 2$, $f_1(X) := \gamma(\rho; X)$, and $f_2(X) := g(X)$. The second assumption follows from the second assumption of $\mathcal{F}$.

In order to check the first assumption, it suffices to show that

$$\frac{1}{\rho} < \frac{1}{A} \tag{5.9}$$

by (2.3) and the first assumption of $\mathcal{F}$. We may assume that $A \geqslant 4$ by (3.2). Using

$$\begin{aligned}\log\left(1-\frac{1}{A}\right)^A &= -A\sum_{n=1}^{\infty}\frac{1}{n}A^{-n}\\ &> -A\sum_{n=1}^{\infty}A^{-n} = -1-\frac{1}{A-1},\end{aligned}$$

we get by $A \geqslant 4$ that

$$\begin{aligned}\left(1-\frac{1}{A}\right)^A &> \exp\left(-1-\frac{1}{A-1}\right)\\ &\geqslant \exp\left(-\frac{4}{3}\right) > \frac{1}{4} \geqslant \frac{1}{A}.\end{aligned}$$

Hence, we obtain

$$G_A\left(\frac{1}{A}\right) = \left(1-\frac{1}{A}\right)^A - \frac{1}{A} > 0,$$

which implies (5.9) by (3.2) and the second statement of Lemma 3.5. In what follows, we check the third assumption of Theorem 4.1. The following lemma was inspired by the results of Daniel [9].

Lemma 5.1. *Let k be a positive integer. Then*

$$R - \theta\big(R; kS(f_1)\big) = O\left(R^{(1-1/\rho)^k}\right) \tag{5.10}$$

for any $R \geqslant 1$, where the implied constant in the symbol O does not depend on R, but on k.

Proof. First we consider the case of $k = 1$. Using the mean value theorem, we see that

$$\begin{aligned}\lfloor (m+1)^\rho \rfloor - \lfloor m^\rho \rfloor &= (m+1)^\rho - m^\rho + O(1)\\ &= O\left(m^{\rho-1}\right) = O\left(\lfloor m^\rho \rfloor^{1-1/\rho}\right)\end{aligned} \tag{5.11}$$

for any positive integer m. For any sufficiently large R, take a positive integer m with

$$\lfloor m^\rho \rfloor < R \leqslant \lfloor (m+1)^\rho \rfloor$$

Then we get

$$R - \theta\big(R; S(f_1)\big) \leqslant \lfloor (m+1)^\rho \rfloor - \lfloor m^\rho \rfloor = O\left(R^{1-1/\rho}\right)$$

by (5.11).

Next, we assume that (5.10) holds for a positive integer k. Let

$$R_0 := R - \theta\big(R; kS(f_1)\big) \in \mathbb{Z}^+.$$

The inductive hypothesis implies that

$$R_0 = O\left(R^{(1-1/\rho)^k}\right). \tag{5.12}$$

In the same way as the proof of (5.3), putting

$$\eta := \theta\big(R; kS(f_1)\big) + \theta\big(R_0; S(f_1)\big),$$

we see

$$R - \theta\big(R; (k+1)S(f_1)\big) \leqslant R - \eta = R_0 - \theta\big(R_0; S(f_1)\big).$$

Consequently, using (5.10) with $k = 1$ and $R = R_0$, we obtain

$$0 < R - \theta\big(R; (k+1)S(f_1)\big) = O\left(R_0^{1-1/\rho}\right) = O\left(R^{(1-1/\rho)^{k+1}}\right)$$

by (5.12). ∎

Put $\log_R x := (\log x)/(\log R)$. Using Lemma 5.1 with $k = 1 + k_1$, we get

$$\log_R F_1(R) := \log_R\Big(R - \theta\big(R; (1+k_1)S(f_1)\big)\Big) \leqslant \left(1 - \frac{1}{\rho}\right)^{1+k_1} + o(1)$$

as R tends to infinity. Moreover, using (2.3) and the first assumption of $\mathcal{F}$, we see

$$\log_R F_2(R) := \log_R\left(\frac{R}{\prod_{i=1}^2 \lambda(S(f_i); R)^{k_i}}\right) = 1 - \frac{k_1}{\rho} + o(1).$$

Thus, we obtain

$$\log_R F_1(R) - \log_R F_2(R) \leqslant G_{1+k_1}\left(\frac{1}{\rho}\right) + o(1)$$

as R tends to infinity. For the proof of (4.2), it suffices to show that

$$G_{1+k_1}\left(\frac{1}{\rho}\right) < 0. \tag{5.13}$$

If $k_1 = 0$ or $k_1 = 1$, then (5.13) is clear by $G_1(X) = -X$ and $G_2(X) = -X(1-X)$. If $k_1 = 2$, then we have $G_3(X) = -X(1 - 3X + X^2)$ and $\sigma_3 = (3 - \sqrt{5})/2$. By (5.9) and (4.1), we get

$$\frac{1}{\rho} < \frac{1}{A} \leqslant \frac{1}{1 + k_1} = \frac{1}{3} < \sigma_3,$$

which implies (5.13) by the second statement of Lemma 3.5. Finally, suppose that $k_1 \geqslant 3$. Using (3.2), (4.1), and the third statement of Lemma 3.5, we obtain

$$\frac{1}{\rho} < \sigma_A \leqslant \sigma_{1+k_1},$$

which means (5.13). Therefore, we proved Theorem 3.6. ■

Proof of Corollary 3.7. The first statement of Corollary 3.7 follows from Theorem 3.6 by

$$\sum_{m=0}^{\infty} X^{\lfloor y^m \rfloor} \in \mathcal{F}.$$

The second statement of the corollary is similarly verified by $\xi(y; X) \in \mathcal{F}$. In fact, the first assumption of $\mathcal{F}$ follows from the fact that, for any real number M,

$$\lim_{R\to\infty} \frac{\varphi(y; R)}{R^M} = \infty.$$

Moreover, using the mean value theorem for $\log \varphi(y; R)$, we can show that

$$\lim_{R\to\infty} \frac{\varphi(y; R+1)}{\varphi(y; R)} = 1.$$

■

Proof of Corollary 3.8. By Theorem 3.6 and the second statement of Lemma 3.5, it suffices to show that $(\varepsilon + 1/2)A^2 > \sigma_A^{-1}$, namely,

$$0 > G_A\left(\left(\frac{1}{2} + \varepsilon\right)^{-1} A^{-2}\right)$$

for any sufficiently large A, depending only on $\varepsilon > 0$. We now fix an arbitrary positive real number ε. In the proof of Corollary 3.8, the implied constant in the symbol O does not depend on A, but on ε. Observe that

$$\begin{aligned}\log\left(1 - \left(\frac{1}{2} + \varepsilon\right)^{-1} A^{-2}\right)^{A} &= A\left(-\left(\frac{1}{2} + \varepsilon\right)^{-1} A^{-2} + O\left(A^{-4}\right)\right) \\ &= -\left(\frac{1}{2} + \varepsilon\right)^{-1} A^{-1} + O\left(A^{-3}\right)\end{aligned}$$

and that

$$\left(1-\left(\frac{1}{2}+\varepsilon\right)^{-1}A^{-2}\right)^{A} = \exp\left(-\left(\frac{1}{2}+\varepsilon\right)^{-1}A^{-1}+O\left(A^{-3}\right)\right)$$
$$= 1-\left(\frac{1}{2}+\varepsilon\right)^{-1}A^{-1}+\frac{1}{2}\left(\frac{1}{2}+\varepsilon\right)^{-2}A^{-2}+O\left(A^{-3}\right).$$

Thus, we get

$$G_A\left(\left(\frac{1}{2}+\varepsilon\right)^{-1}A^{-2}\right) = \left(1-\left(\frac{1}{2}+\varepsilon\right)^{-1}A^{-2}\right)^{A}$$
$$-1+\left(\frac{1}{2}+\varepsilon\right)^{-1}A^{-1}-\left(\frac{1}{2}+\varepsilon\right)^{-1}A^{-2}$$
$$= -\varepsilon\left(\frac{1}{2}+\varepsilon\right)^{-2}A^{-2}+O\left(A^{-3}\right)<0$$

for any sufficiently large A, depending only on ε. ■

6. Proof of Theorem 4.1

Put

$$\overline{f_i}(X) := \begin{cases} f_i(X) & \text{if } f_i(0)\neq 0, \\ 1+f_i(X) & \text{if } f_i(0)=0. \end{cases}$$

Then $\overline{f_1}(X),\ldots,\overline{f_r}(X)$ satisfy the assumptions of Theorem 4.1. The second assumption is easily checked. Moreover, the first and third assumptions are also seen by

$$\theta\left(R;\sum_{h=1}^{r-2}k_hS(\overline{f_h})+(1+k_{r-1})S(\overline{f_{r-1}})\right)$$
$$\geqslant \theta\left(R;\sum_{h=1}^{r-2}k_hS(f_h)+(1+k_{r-1})S(f_{r-1})\right)$$

and, for $h=1,\ldots,r$,

$$\lambda\left(S\left(\overline{f_h}\right);R\right)\sim\lambda(S(f_h);R)$$

as R tends to infinity. For the proof of Theorem 4.1, it suffices to show that

$$\left\{\overline{f_1}(\beta^{-1})^{k_1}\overline{f_2}(\beta^{-1})^{k_2}\cdots\overline{f_r}(\beta^{-1})^{k_r}\;\middle|\;k_1,k_2,\ldots,k_r\in\mathbb{N},k_1\leqslant A\right\}$$

is linearly independent over $\mathbb{Q}(\beta)$. In particular, rewriting $\overline{f_i}(X)$ by $f_i(X)$ for $i=1,\ldots,r$, we may assume that $f_i(0)\neq 0$ for any $i=1,\ldots,r$.

For simplicity, put, for $i = 1, \ldots, r$,

$$\xi_i := f_i(\beta^{-1}), \qquad S_i := S(f_i), \qquad \lambda_i(R) := \lambda\bigl(S(f_i); R\bigr).$$

Using Corollary 2.2 and the first assumption of Theorem 4.1, we see that

$$[\mathbb{Q}(\xi_1, \beta) : \mathbb{Q}(\beta)] \geqslant A + 1$$

and that $\xi_2, \ldots, \xi_r$ are transcendental.

We introduce notation for the proof of Theorem 4.1. For any nonempty subset $\mathcal{A}$ of $\mathbb{N}$ and any positive integer k, let $\mathcal{A}^k$ denote the n-fold Cartesian product. For convenience, set

$$\mathcal{A}^0 := \{0\}.$$

Let $k \in \mathbb{N}$ and $\boldsymbol{p} = (p_1, \ldots, p_k) \in \mathbb{N}^k$. We put

$$|\boldsymbol{p}| := \begin{cases} 0 & (k = 0), \\ p_1 + \cdots + p_k & (k \geqslant 1) \end{cases}$$

and, for $i = 1, \ldots, r$,

$$t_i(\boldsymbol{p}) := \begin{cases} 1 & (k = 0), \\ t_i(p_1) \cdots t_i(p_k) & (k \geqslant 1). \end{cases}$$

Moreover, for any $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$, let

$$\underline{X}^{\boldsymbol{k}} = \prod_{i=1}^{r} X_i^{k_i}, \qquad \underline{\xi}^{\boldsymbol{k}} := \prod_{i=1}^{r} \xi_i^{k_i}, \qquad \underline{\lambda}(N)^{\boldsymbol{k}} := \prod_{i=1}^{r} \lambda_i(N)^{k_i}.$$

We calculate $\underline{\xi}^{\boldsymbol{k}}$ in the same way as the proof of Theorem 2.1 in [12]. The method was inspired by the proof of Theorem 7.1 in [3]. Let $\boldsymbol{k} \in \mathbb{N}^r \backslash \{(0, \ldots, 0)\}$. Then we have

$$\begin{aligned} \underline{\xi}^{\boldsymbol{k}} &= \prod_{i=1}^{r} \left(\sum_{m_i \in S_i} t_i(m_i) \beta^{-m_i} \right)^{k_i} \\ &= \prod_{i=1}^{r} \sum_{\boldsymbol{m}_i \in S_i^{k_i}} t_i(\boldsymbol{m}_i) \beta^{-|\boldsymbol{m}_i|} =: \sum_{m=0}^{\infty} \beta^{-m} \rho(\boldsymbol{k}; m), \end{aligned} \tag{6.1}$$

where

$$\rho(\boldsymbol{k}; m) = \sum_{\substack{\boldsymbol{m}_1 \in S_1^{k_1}, \ldots, \boldsymbol{m}_r \in S_r^{k_r} \\ |\boldsymbol{m}_1| + \cdots + |\boldsymbol{m}_r| = m}} t_1(\boldsymbol{m}_1) \cdots t_r(\boldsymbol{m}_r) \in \mathbb{N}.$$

Note that $\rho(\boldsymbol{k}; m)$ is positive if and only if

$$m \in \sum_{h=1}^{r} k_h S_h.$$

We see that

$$\rho(\boldsymbol{k};m) \leqslant \sum_{\substack{\boldsymbol{m}_1\in S_1^{k_1},\ldots,\boldsymbol{m}_r\in S_r^{k_r} \\ |\boldsymbol{m}_1|+\cdots+|\boldsymbol{m}_r|=m}} C_7^{|\boldsymbol{k}|} \leqslant C_7^{|\boldsymbol{k}|}(1+m)^{|\boldsymbol{k}|}. \tag{6.2}$$

We give an analogue of Lemma 4.1 in [12].

Lemma 6.1. *Let* $\boldsymbol{k} \in \mathbb{N}^r \backslash \{(0,\ldots,0)\}$ *and let* $N \in \mathbb{Z}^+$. *Then we have*

$$\sum_{m=0}^{N-1} \rho(\boldsymbol{k};m) \leqslant C_7^{|\boldsymbol{k}|} \underline{\lambda}(N)^{\boldsymbol{k}} \tag{6.3}$$

and

$$\mathit{Card}\{m \in \mathbb{N} \mid m < N, \rho(\boldsymbol{k};m) > 0\} \leqslant C_7^{|\boldsymbol{k}|} \underline{\lambda}(N)^{\boldsymbol{k}}. \tag{6.4}$$

Proof. We see that (6.4) follows from (6.3) because $\rho(\boldsymbol{k};m) \in \mathbb{N}$ for any m. Put $S(i;N) := S_i \cap [0,N)$ for $i = 1,\ldots,r$. Then we get

$$\begin{aligned}\sum_{m=0}^{N-1} \rho(\boldsymbol{k};m) &= \sum_{\substack{\boldsymbol{m}_1\in S_1^{k_1},\ldots,\boldsymbol{m}_r\in S_r^{k_r} \\ |\boldsymbol{m}_1|+\cdots+|\boldsymbol{m}_r|<N}} t_1(\boldsymbol{m}_1)\cdots t_r(\boldsymbol{m}_r) \\ &\leqslant C_7^{|\boldsymbol{k}|} \sum_{\boldsymbol{m}_1\in S(1;N)^{k_1}} \sum_{\boldsymbol{m}_2\in S(2;N)^{k_2}} \cdots \sum_{\boldsymbol{m}_r\in S(r;N)^{k_r}} 1 \\ &= C_7^{|\boldsymbol{k}|} \underline{\lambda}(N)^{\boldsymbol{k}},\end{aligned}$$

which implies (6.3). ∎

Assume that the set $\{\underline{\xi}^{\boldsymbol{k}} \mid \boldsymbol{k} = (k_1,\ldots,k_r) \in \mathbb{N}^r, k_1 \leqslant A\}$ is linearly independent over $\mathbb{Q}(\beta)$. Then there exists $P(X_1,\ldots,X_r) \in \mathbb{Z}[\beta][X_1,\ldots,X_r]\backslash\mathbb{Z}[\beta]$ such that the degree of $P(X_1,\ldots,X_r)$ in X_1 is at most A and that

$$P(\xi_1,\ldots,\xi_r) = 0. \tag{6.5}$$

Let D be the total degree of $P(X_1,\ldots,X_r)$. Without loss of generality, we may assume that $X_r(-1+X_r)$ divides $P(X_1,\ldots,X_r)$ and that if $r \geqslant 3$, then X_{r-1} divides $P(X_1,\ldots,X_r)$. Put

$$P(X_1,\ldots,X_r) =: \sum_{\boldsymbol{k}\in\Lambda} A_{\boldsymbol{k}} \underline{X}^{\boldsymbol{k}}, \tag{6.6}$$

where Λ is a nonempty finite subset of $\mathbb{N}^r$ and $A_{\boldsymbol{k}} \in \mathbb{Z}[\beta]\backslash\{0\}$ for any $\boldsymbol{k} \in \Lambda$. For any $\boldsymbol{k} = (k_1,\ldots,k_r) \in \Lambda$, we have $k_r \geqslant 1$ because X_r divides $P(X_1,\ldots,X_r)$. Moreover, if $r \geqslant 3$, then

$$k_{r-1} \geqslant 1 \tag{6.7}$$

because X_{r-1} divides $P(X_1,\ldots,X_r)$.

The lexicographic order $\succ$ on $\mathbb{N}^r$ is defined as follows: Let $\boldsymbol{k} = (k_1, \ldots, k_r)$ and $\boldsymbol{k}' = (k'_1, \ldots, k'_r)$ be distinct elements of $\mathbb{N}^r$. Put $l := \min\{i \mid 1 \leqslant i \leqslant r, k_i \neq k'_i\}$. Then $\boldsymbol{k} \succ \boldsymbol{k}'$ if and only if $k_l > k'_l$. The first assumption of Theorem 4.1 implies that if $\boldsymbol{k} \succ \boldsymbol{k}'$, then

$$\underline{\lambda}(N)^{\boldsymbol{k}'} = o\left(\underline{\lambda}(N)^{\boldsymbol{k}}\right) \tag{6.8}$$

as N tends to infinity.

Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ be the greatest element of Λ with respect to $\succ$. Without loss of generality, we may assume that

$$A_{\boldsymbol{g}} \geqslant 1. \tag{6.9}$$

We see that

$$g_{r-1} \geqslant 1. \tag{6.10}$$

In fact, (6.10) follows from (6.7) if $r \geqslant 3$. Suppose that $r = 2$. Then g_1 is the degree of $P(X_1, X_2)$ in X_1. Thus, g_1 is positive because ξ_2 is transcendental.

Putting

$$\Lambda_1 := \{\boldsymbol{k} = (k_1, \ldots, k_{r-1}, k_r) \mid k_1 = g_1, \ldots, k_{r-1} = g_{r-1}, k_r < g_r\}$$

and

$$\Lambda_2 := \{\boldsymbol{k} = (k_1, \ldots, k_{r-1}, k_r) \mid k_i < g_i \text{ for some } i \leqslant r-1\},$$

we see $\Lambda = \{\boldsymbol{g}\} \cup \Lambda_1 \cup \Lambda_2$. Using the fact that ξ_r is transcendental and that $-1 + X_r$ divides $P(X_1, \ldots, X_r)$, we obtain the following lemma, applying the same method as the proof of Lemma 4.3 in [12] with $F(X_{r-1}, X_r) = 1$:

Lemma 6.2. Λ_1 *and* Λ_2 *are not empty.*

Set

$$\boldsymbol{e} = (g_1, \ldots, g_{r-2}, -1 + g_{r-1}, 1 + D).$$

Recall that the degree g_1 of $P(X_1, \ldots, X_r)$ in X_1 is at most A. Thus, we can apply the third assumption of Theorem 4.1 with $\boldsymbol{k} = (k_1, \ldots, k_r) = \boldsymbol{e}$. In fact, we see

$$k_1 = \begin{cases} -1 + g_1 & \text{if } r = 2, \\ g_1 & \text{if } r \geqslant 3. \end{cases}$$

Hence, there exits a positive constant C_{12} satisfying the following: For any integer R with $R \geqslant C_{12}$, we have

$$\lambda_r(R) \geqslant 5 \tag{6.11}$$

and

$$R - \theta\left(\sum_{h=1}^{r-1} g_h S_h; R\right) < \frac{R}{\underline{\lambda}(R)^{\boldsymbol{e}}}. \tag{6.12}$$

In what follows, we set
$$\theta(R) := \theta\left(\sum_{h=1}^{r-1} g_h S_h; R\right)$$
for simplicity. Using (6.11) and (6.12), we obtain the following lemma in the same way as the proof of Lemma 4.4 in [12].

Lemma 6.3. *Let M, E be real numbers with*
$$M \geqslant C_{12}, \qquad E \geqslant \frac{4M}{\underline{\lambda}(M)^{\boldsymbol{e}}}.$$
Then
$$M + \frac{1}{2}E < \theta(M+E).$$

Using $k_1 \leqslant A$ and the first assumption of Theorem 4.1, we get
$$\lim_{R\to\infty} \frac{R}{\underline{\lambda}(R)^{\boldsymbol{e}}} = \infty.$$
Thus, the set
$$\Xi := \left\{ N \in \mathbb{N} \,\middle|\, \frac{N}{\underline{\lambda}(N)^{\boldsymbol{e}}} \geqslant \frac{n}{\underline{\lambda}(n)^{\boldsymbol{e}}} \text{ for any } n \leqslant N \right\}$$
is infinite. We now verify for any $\boldsymbol{k} = (k_1, \ldots, k_r) \in \Lambda_2$ that
$$\underline{\lambda}(N)^{\boldsymbol{k}} = o\left(\underline{\lambda}(N)^{\boldsymbol{e}}\right) \tag{6.13}$$
as N tends to infinity. For the proof of (6.13), it suffices to check
$$\boldsymbol{e} \succ \boldsymbol{k} \tag{6.14}$$
by (6.8). If $g_i > k_i$ for some $i \leqslant r-2$, then (6.14) holds. Suppose that $g_i = k_i$ for any $i \leqslant r-2$. Then we get $-1 + g_{r-1} \geqslant k_{r-1}$ and $1 + D > k_r$ by $\boldsymbol{k} \in \Lambda_2$, which implies (6.14).

Combining (6.5), (6.6), and (6.1), we get
$$0 = \sum_{\boldsymbol{k}\in\Lambda} A_{\boldsymbol{k}} \underline{\xi}^{\boldsymbol{k}} = \sum_{\boldsymbol{k}\in\Lambda} A_{\boldsymbol{k}} \sum_{m=0}^{\infty} \rho(\boldsymbol{k}; m)\beta^{-m}.$$
For an arbitrary nonnegative integer R, multiplying β^R to the both-hand sides of the equality above, we obtain
$$0 = \sum_{\boldsymbol{k}\in\Lambda} A_{\boldsymbol{k}} \sum_{m=-R}^{\infty} \rho(\boldsymbol{k}; m+R)\beta^{-m}.$$

Putting

$$Y_R := \sum_{\boldsymbol{k}\in\Lambda} A_{\boldsymbol{k}} \sum_{m=1}^{\infty} \rho(\boldsymbol{k}; m+R)\beta^{-m}$$
$$= -\sum_{\boldsymbol{k}\in\Lambda} A_{\boldsymbol{k}} \sum_{m=-R}^{0} \rho(\boldsymbol{k}; m+R)\beta^{-m}, \tag{6.15}$$

we see that Y_R is an algebraic integer because β is a Pisot or Salem number.

Lemma 6.4. *There exist positive integers C_{13} and C_{14} satisfying the following: For any integer R with $R \geqslant C_{14}$, we have*

$$Y_R = 0 \qquad \text{or} \qquad |Y_R| \geqslant R^{-C_{13}}.$$

Proof. Let d be the degree of β and let $\sigma_1, \sigma_2, \ldots, \sigma_d$ be the conjugate embeddings of $\mathbb{Q}(\beta)$ into $\mathbb{C}$ such that $\sigma_1(\gamma) = \gamma$ for any $\gamma \in \mathbb{Q}(\beta)$. Set

$$C_{15} := \max\{|\sigma_i(A_{\boldsymbol{k}})| \mid i = 1, \ldots, d, \boldsymbol{k} \in \Lambda\}.$$

Let $2 \leqslant i \leqslant d$. Using (6.15) and (6.2), and $|\beta_i| \leqslant 1$, we get

$$|\sigma(Y_R)| = \left|\sum_{\boldsymbol{k}\in\Lambda} \sigma_i(A_{\boldsymbol{k}}) \sum_{n=0}^{R} \rho(\boldsymbol{k}; -n+R)\sigma_i(\beta)^n\right|$$
$$\leqslant \sum_{\boldsymbol{k}\in\Lambda} C_{15} \sum_{n=0}^{R} C_7^D (1+R)^D \ll (R+1)^{D+1}.$$

In particular, if $R \gg 1$, then

$$|\sigma(Y_R)| \leqslant R^{D+2}.$$

Hence, if $Y_R \neq 0$, then we obtain

$$1 \leqslant |Y_R| \prod_{i=2}^{d} |\sigma(Y_R)| \leqslant |Y_R| R^{(D+2)(d-1)}. \qquad \blacksquare$$

In the case of $\beta = 2$ and $r = 1$, Bailey, Borwein, Crandall, and Pomerance estimated the numbers $\widetilde{y_N}$ of positive Y_R with $R < N$ in order to give lower bounds for the nonzero digits in binary expansions (Theorem 7.1 in [3]). Moreover, if $\beta = b > 1$ is a rational integer and $r \geqslant 2$, then $\widetilde{y_N}$ is applied to prove a criterion for algebraic independence (Theorem 2.1 in [12]).

Now, we put, for $N \in \mathbb{Z}^+$,

$$y_N := \mathrm{Card}\left\{R \in \mathbb{N} \;\middle|\; R < N, Y_R \geqslant \frac{1}{\beta}\right\}.$$

In the case where β is a Pisot or Salem number and $r = 1$, then y_N is estimated to give lower bounds for the numbers of nonzero digits in β-expansions (Theorem 2.2 in [14]). In what follows, we calculate upper and lower bounds for y_N, which gives contradiction. First, we estimate upper bounds for y_N in Lemma 6.5. Next, we give lower bounds for y_N in Lemma 6.10, estimating upper bounds for $R - \theta(R; \Omega)$ in Lemma 6.9, where

$$\Omega = \left\{ R \in \mathbb{N} \,\middle|\, Y_R \geqslant \frac{1}{\beta} \right\}. \tag{6.16}$$

In what follows, we assume that N is a sufficiently large integer satisfying

$$\left(1 + \frac{1}{N}\right)^D < \frac{\beta + 1}{2}. \tag{6.17}$$

Lemma 6.5. *We have*

$$y_N = o\left(N^{1-\delta/2}\right)$$

as N tends to infinity.

Proof. Put

$$K := \lceil (1 + D) \log_\beta N \rceil,$$

where $\log_\beta N = (\log N)/(\log \beta)$. Then we see

$$y_N \leqslant K + y_{N-K} = K + \sum_{\substack{0 \leqslant R < N-K \\ Y_R \geqslant 1/\beta}} 1 \leqslant K + \beta \sum_{R=0}^{N-K-1} |Y_R|$$

and

$$\begin{aligned} \sum_{R=0}^{N-K-1} |Y_R| &\leqslant \sum_{R=0}^{N-K-1} \sum_{\boldsymbol{k} \in \Lambda} \sum_{m=1}^{\infty} |A_{\boldsymbol{k}}| \beta^{-m} \rho(\boldsymbol{k}; m + R) \\ &= \sum_{\boldsymbol{k} \in \Lambda} |A_{\boldsymbol{k}}| Y(\boldsymbol{k}; N), \end{aligned}$$

where

$$Y(\boldsymbol{k}; N) = \sum_{R=0}^{N-K-1} \sum_{m=1}^{\infty} \beta^{-m} \rho(\boldsymbol{k}; m + R)$$

for $\boldsymbol{k} \in \Lambda$. For the proof of Lemma 6.5, it suffices to show for any $\boldsymbol{k} = (k_1, k_2, \ldots, k_r) \in \Lambda$ that

$$Y(\boldsymbol{k}; N) = o\left(N^{1-\delta/2}\right) \tag{6.18}$$

as N tends to infinity. Observe that

$$0 \leqslant Y(\boldsymbol{k}; N) = \sum_{m=1}^{K} \sum_{R=0}^{N-K-1} \beta^{-m} \rho(\boldsymbol{k}; m+R) + \sum_{m=K+1}^{\infty} \sum_{R=0}^{N-K-1} \beta^{-m} \rho(\boldsymbol{k}; m+R) =: S^{(1)}(\boldsymbol{k}; N) + S^{(2)}(\boldsymbol{k}; N). \tag{6.19}$$

Using (6.3), we get

$$S^{(1)}(\boldsymbol{k}; N) \leqslant \sum_{m=1}^{K} \beta^{-m} \sum_{R=0}^{N-1} \rho(\boldsymbol{k}; R) \leqslant \sum_{m=1}^{\infty} \beta^{-m} \sum_{R=0}^{N-1} \rho(\boldsymbol{k}; R) \leqslant \sum_{m=1}^{\infty} \beta^{-m} C_7^D \underline{\lambda}(N)^{\boldsymbol{k}} \ll \underline{\lambda}(N)^{\boldsymbol{k}}.$$

Thus, the first assumption of Theorem 4.1 with $\varepsilon = \delta/(2D)$ implies that

$$S^{(1)}(\boldsymbol{k}; N) \ll \lambda_1(N)^A \prod_{i=2}^{r} \lambda_i(N)^{k_i} = o\left(N^{1-\delta/2}\right). \tag{6.20}$$

Using (6.2), we see

$$S^{(2)}(\boldsymbol{k}; N) \leqslant \sum_{m=K+1}^{\infty} \beta^{-m} \sum_{R=0}^{N-K-1} C_7^D (m+R+1)^D \ll \sum_{m=K+1}^{\infty} \beta^{-m} N (m+N)^D.$$

Note for any $m \in \mathbb{N}$ that

$$\left(\frac{m+1+N}{m+N}\right)^D \leqslant \left(1 + \frac{1}{N}\right)^D < \frac{\beta+1}{2}$$

by (6.17). Hence, we obtain

$$S^{(2)}(\boldsymbol{k}; N) \ll \beta^{-K-1} N (K+1+N)^D \sum_{m=0}^{\infty} \beta^{-m} \left(\frac{\beta+1}{2}\right)^m \ll \beta^{-K-1} N^{D+1} \leqslant 1. \tag{6.21}$$

Hence, combining (6.19), (6.20), and (6.21), we deduce (6.18). ■

In what follows, we estimate lower bounds for y_N in the case where $N \in \Xi$ is sufficiently large. Recall that Λ_2 is not empty by Lemma 6.2 and that $0 \in S_i$ for $i = 1, \ldots, r$. In particular, for any $\boldsymbol{k} \in \Lambda$, we have $\rho(\boldsymbol{k}; 0) > 0$. Put

$$\{T \in \mathbb{N} \mid T < N, \rho(\boldsymbol{k}; T) > 0 \text{ for some } \boldsymbol{k} \in \Lambda_2\} =: \{0 = T_1 < T_2 < \cdots < T_\tau\}.$$

If N is sufficiently large, then (6.4) and (6.13) imply that

$$\tau \leqslant \sum_{\boldsymbol{k}\in\Lambda_2} C_7^{|\boldsymbol{k}|}\underline{\lambda}(N)^{\boldsymbol{k}} \leqslant \frac{1}{32}\underline{\lambda}(N)^{\boldsymbol{e}}.$$

For convenience, put $T_{1+\tau} := N$. Set

$$\mathcal{J} := \{J = J(j) \mid 1 \leqslant j \leqslant \tau\},$$

where $J(j)$ is an interval of $\mathbb{R}$ defined by $J(j) = [T_j, T_{1+j})$ for $1 \leqslant j \leqslant \tau$.

In what follows, we denote the length of a bounded interval I of $\mathbb{R}$ by $|I|$. Then we have

$$\sum_{J\in\mathcal{J}} |J| = N.$$

Let

$$\mathcal{J}_1 := \left\{ J \in \mathcal{J} \;\middle|\; |J| \geqslant \frac{16N}{\underline{\lambda}(N)^{\boldsymbol{e}}} \right\},$$
$$\mathcal{J}_2 := \{J \in \mathcal{J}_1 \mid J \subset [C_{12}, N)\}.$$

In the same way as the proof of Lemma 4.7 in [12], we obtain the following:

Lemma 6.6. *If $N \in \Xi$ is sufficiently large, then we have*

$$\sum_{J\in\mathcal{J}_1} |J| \geqslant \frac{N}{2}, \qquad \sum_{J\in\mathcal{J}_2} |J| \geqslant \frac{N}{3}.$$

Recall that Λ_1 is not empty by Lemma 6.2. Let $\boldsymbol{k}_1$ be the maximal element of Λ_1 with respect to $\succ$. Set

$$\{R \in \mathbb{N} \mid R < N, \rho(\boldsymbol{k}; R) > 0 \text{ for some } \boldsymbol{k} \in \Lambda_1\} =: \{0 = R_1 < R_2 < \cdots < R_\mu\}$$

and $R_{1+\mu} := N$. Then (6.4) implies that

$$\mu \leqslant \sum_{\boldsymbol{k}\in\Lambda_1} C_7^{|\boldsymbol{k}|}\underline{\lambda}(N)^{\boldsymbol{k}} \leqslant C_{16}\underline{\lambda}(N)^{\boldsymbol{k}_1},$$

where C_{16} is a positive constant.

Let

$$\mathcal{I} := \{I = I(i) \mid 1 \leqslant i \leqslant \mu\},$$

where $I(i)$ is an interval of $\mathbb{R}$ defined by $I(i) = [R_i, R_{i+1})$ for $1 \leqslant i \leqslant \mu$. Set

$$y_N(i) := \operatorname{Card}\left\{ R \in I(i) \;\middle|\; Y_R \geqslant \frac{1}{\beta} \right\}$$

for $i = 1, \ldots, \mu$. Observe that

$$\sum_{I\in\mathcal{I}} |I| = N$$

and that

$$\sum_{i=1}^{\mu} y_N(i) = y_N. \tag{6.22}$$

Set

$$\mathcal{I}_1 := \{I \in \mathcal{I} \mid I \subset J \text{ for some } J \in \mathcal{J}\},$$
$$\mathcal{I}_2 := \left\{ I \in \mathcal{I}_1 \,\middle|\, |I| \geqslant \frac{1}{12C_{16}} \frac{N}{\underline{\lambda}(N)^{\boldsymbol{k}_1}} \right\}.$$

In the same way as the proof of Lemma 4.8 in [12], we obtain the following, using (6.12) and Lemmas 6.3 and 6.6.

Lemma 6.7. *For any sufficiently large $N \in \Xi$, we have*

$$\sum_{I \in \mathcal{I}_1} |I| \geqslant \frac{N}{6}, \qquad \sum_{I \in \mathcal{I}_2} |I| \geqslant \frac{N}{12}. \tag{6.23}$$

In what follows, we assume that $N \in \Xi$ satisfies

$$N^{\delta/2} \geqslant (1 + C_8)C_9. \tag{6.24}$$

Let $1 \leqslant i \leqslant \mu$ with $I(i) \in \mathcal{I}_2$ and let $R \in (R_i, R_{i+1})$. We now show that

$$\rho(\boldsymbol{k}; R) = 0 \tag{6.25}$$

for any $\boldsymbol{k} \in \Lambda_1 \cup \Lambda_2 = \Lambda \backslash \{\boldsymbol{g}\}$. In fact, if $\boldsymbol{k} \in \Lambda_1$, then (6.25) follows from the definition of $R_1, \ldots, R_{\mu+1}$. Suppose that $\boldsymbol{k} \in \Lambda_2$. By the definition of $\mathcal{I}_2$, we have $I(i) \subset J(j)$ for some j with $1 \leqslant j \leqslant \tau$, and so $R \in (T_j, T_{1+j})$. Thus, we get (6.25).

Applying the first assumption of Theorem 4.1 with $\varepsilon = \delta/(2D)$, we see by $g_1 \leqslant A$ that

$$\underline{\lambda}(N)^{\boldsymbol{k}_1} = o\left(N^{-\delta/2+1}\right)$$

as $N \in \Xi$ tends to infinity. Thus, we obtain for any sufficiently large $N \in \Xi$ that

$$|I(i)| \geqslant \frac{1}{12C_{16}} \frac{N}{\underline{\lambda}(N)^{\boldsymbol{k}_1}} \geqslant N^{\delta/2}. \tag{6.26}$$

We can apply the second assumption of Theorem 4.1 with

$$R = \frac{|I(i)|}{1 + C_8} \geqslant \frac{N^{\delta/2}}{1 + C_8} \geqslant C_9$$

by (6.26) and (6.24). Thus, we get that there exists $V(N, i) \in S_r$ with

$$\frac{|I(i)|}{1 + C_8} \leqslant V(N, i) \leqslant \frac{C_8 |I(i)|}{1 + C_8}.$$

Put $M = M(N,i) := R_i + V(N,i)$. Then we have

$$R_i + \frac{|I(i)|}{1+C_8} \leqslant M \leqslant R_i + \frac{C_8|I(i)|}{1+C_8}. \tag{6.27}$$

By the definition of R_i, there exists $k_r \leqslant -1 + g_r$ such that

$$R_i \in \sum_{h=1}^{r-1} g_h S_h + k_r S_r.$$

Using Remark 1, we see

$$R_i \in \sum_{h=1}^{r-1} g_h S_h + (-1+g_r) S_r.$$

Thus, we get

$$M \in \sum_{h=1}^{r} g_h S_h \tag{6.28}$$

by $V(N,i) \in S_r$.

Lemma 6.8. *Let $N \in \Xi$ be sufficiently large and let $1 \leqslant i \leqslant \mu$ with $I(i) \in \mathcal{I}_2$. Then $Y_R > 0$ for any R with $R_i \leqslant R < M$.*

Proof. We prove Lemma 6.8 by induction on R. First we show that $Y_{M-1} > 0$. We see

$$\begin{aligned} Y_{M-1} &= A_{\boldsymbol{g}} \sum_{m=1}^{\infty} \beta^{-m} \rho(\boldsymbol{g}; m + M - 1) \\ &\quad + \sum_{\boldsymbol{k} \in \Lambda \setminus \{\boldsymbol{g}\}} A_{\boldsymbol{k}} \sum_{m=1}^{\infty} \beta^{-m} \rho(\boldsymbol{k}; m + M - 1) \\ &=: S^{(3)} + S^{(4)}. \end{aligned} \tag{6.29}$$

By (6.28)

$$S^{(3)} \geqslant \frac{A_{\boldsymbol{g}}}{\beta} \rho(\boldsymbol{g}; M) \geqslant \frac{1}{\beta}. \tag{6.30}$$

We now estimate upper bounds for $|S_4|$. Let m be an integer with

$$1 \leqslant m \leqslant -1 + \lceil 2D \log_\beta N \rceil. \tag{6.31}$$

Using (6.27) and (6.26), we get

$$R_{i+1} - M \geqslant R_{i+1} - R_i - \frac{C_8|I(i)|}{1+C_8} = \frac{|I(i)|}{1+C_8} > m$$

for sufficiently large $N \in \Xi$ and

$$R_{i+1} > m + M - 1 > R_i.$$

Thus, applying (6.25) with $R = m + M - 1$ for any m with (6.31), we obtain by (6.2) that

$$\begin{aligned}
|S^{(4)}| &\leqslant \sum_{\boldsymbol{k}\in\Lambda\setminus\{\boldsymbol{g}\}} |A_{\boldsymbol{k}}| \sum_{m=\lceil 2D\log_\beta N\rceil}^{\infty} \beta^{-m}\rho(\boldsymbol{k}; m + M - 1) \\
&\leqslant \sum_{\boldsymbol{k}\in\Lambda\setminus\{\boldsymbol{g}\}} |A_{\boldsymbol{k}}| \sum_{m=\lceil 2D\log_\beta N\rceil}^{\infty} \beta^{-m} C_7^D (m + N)^D \\
&\ll \sum_{m=\lceil 2D\log_\beta N\rceil}^{\infty} \beta^{-m}(m + N)^D.
\end{aligned}$$

Therefore, (6.17) implies that

$$|S^{(4)}| \ll N^{-2D}\left(\lceil 2D\log_\beta N\rceil + N\right)^D \sum_{m=0}^{\infty} \beta^m \left(\frac{1+\beta}{2}\right)^m = o(1)$$

as N tends to infinity. In particular, if $N \in \Xi$ is sufficiently large, then

$$|S^{(4)}| < \frac{1}{2\beta}. \tag{6.32}$$

Combining (6.29), (6.30), and (6.32), we deduce that if $N \in \Xi$ is sufficiently large, then $Y_{M-1} > 0$.

Next, we assume that $Y_R > 0$ for some R with $R_i < R < M$. Using (6.25), we see

$$\begin{aligned}
Y_{R-1} &= \sum_{\boldsymbol{k}\in\Lambda} A_{\boldsymbol{k}} \frac{1}{\beta}\rho(\boldsymbol{k}; R) + \sum_{\boldsymbol{k}\in\Lambda} A_{\boldsymbol{k}} \sum_{m=2}^{\infty} \beta^{-m}\rho(\boldsymbol{k}; m + R - 1) \\
&= \frac{A_{\boldsymbol{g}}}{\beta}\rho(\boldsymbol{g}; R) + \frac{1}{\beta}\sum_{\boldsymbol{k}\in\Lambda} A_{\boldsymbol{k}} \sum_{m=1}^{\infty} \beta^{-m}\rho(\boldsymbol{k}; m + R) \\
&= \frac{A_{\boldsymbol{g}}}{\beta}\rho(\boldsymbol{g}; R) + \frac{1}{\beta} Y_R.
\end{aligned} \tag{6.33}$$

By the inductive hypothesis

$$Y_{R-1} > \frac{A_{\boldsymbol{g}}}{\beta}\rho(\boldsymbol{g}; R) \geqslant 0.$$

Therefore, we proved Lemma 6.8. ∎

Recall that Ω is defined in (6.16).

Lemma 6.9. *Let $N \in \Xi$ be sufficiently large and let $1 \leqslant i \leqslant \mu$ with $I(i) \in \mathcal{I}_2$. Let R be an integer with*

$$R_i + 4C_{13} \log_\beta N \leqslant R < M.$$

Then we have

$$R - \theta(R;\Omega) \leqslant 2C_{13} \log_\beta N. \tag{6.34}$$

Proof. Put $R_1 := \theta(R;\Omega)$. In the same way as the proof of (6.33), we see for any integer n with $R_i < n < R_{i+1}$ that

$$Y_{n-1} = \frac{A_{\boldsymbol{g}}}{\beta}\rho(\boldsymbol{g};n) + \frac{1}{\beta}Y_n. \tag{6.35}$$

First, we consider the case of $Y_R \geqslant 1$. Then (6.35) implies that

$$Y_{R-1} \geqslant \frac{1}{\beta}$$

and that $R - R_1 = 1$, which implies (6.34).

In what follows, we may assume that $0 < Y_R < 1$ by Lemma 6.8. Let $S := \lceil C_{13} \log_\beta N \rceil$. Suppose for any integer m with $0 \leqslant m \leqslant S$ that

$$\rho(\boldsymbol{g}; R - m) = 0.$$

Noting $M > R > R - 1 > \cdots > R - S > R_i$, we get by (6.35) that

$$1 > Y_R = \beta Y_{R-1} = \cdots = \beta^S Y_{R-S} = \beta^{1+S} Y_{R-S-1} > 0,$$

where we use Lemma 6.8 for the last inequality by $R_i < R - S - 1 < M$. So we get

$$\beta^{S+1} < Y_{R-S-1}^{-1} = |Y_{R-S-1}|^{-1}.$$

Since

$$R - S - 1 \geqslant 2C_{13} \log_\beta N > C_{14}$$

for any sufficiently large N, we apply Lemma 6.4 as follows:

$$\beta^{S+1} < |Y_{R-S-1}|^{-1} \leqslant (R - S - 1)^{C_{13}} < N^{C_{13}}.$$

Thus, we obtain

$$\lceil C_{13} \log_\beta N \rceil + 1 = S + 1 < C_{13} \log_\beta N,$$

a contradiction.

Hence, there exists an integer m' with $0 \leqslant m' \leqslant S$ satisfying $\rho(\boldsymbol{g}; R - m') \geqslant 1$. Applying (6.35) with $n = R - m'$, we get by $Y_{R-m'} > 0$ that

$$Y_{R-m'-1} \geqslant \frac{A_{\boldsymbol{g}}}{\beta}\rho(\boldsymbol{g}; R - m') \geqslant \frac{1}{\beta},$$

where for the last inequality we use (6.9). Hence, we deduce that

$$R - R_1 \leqslant m' + 1 \leqslant 2C_{13} \log_\beta N. \qquad \blacksquare$$

Lemma 6.10.

$$\limsup_{N\to\infty} y_N \cdot \frac{\log N}{N} > 0.$$

Proof. Let $N \in \Xi$ be sufficiently large and let $1 \leqslant i \leqslant \mu$ with $I(i) \in \mathcal{I}_2$. Note that

$$\lim_{N\to\infty} \frac{|I(i)|}{\log_\beta N} = \infty \tag{6.36}$$

by (6.26). Combining (6.27), (6.36), and Lemma 6.9, we see that there exists a constant C_{17} such that

$$y_N(i) \geqslant C_{17} \frac{|I(i)|}{\log N}.$$

Therefore, using (6.22) and (6.23), we obtain

$$y_N \geqslant \sum_{\substack{1\leqslant i\leqslant \mu\\ I(i)\in\mathcal{I}_2}} y_N(i) \geqslant \sum_{I\in\mathcal{I}_2} C_{17}\frac{|I|}{\log N} \gg \frac{N}{\log N}. \qquad \blacksquare$$

Finally, we deduce a contradiction from Lemma 6.5 and 6.10, which proves Theorem 4.1.

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Address: Hajime Kaneko: Institute of Mathematics, and Research Core for Mathematical Sciences, University of Tsukuba, 1-1-1, Tennodai, Tsukuba, Ibaraki, 305-8571, Japan.

E-mail: kanekoha@math.tsukuba.ac.jp

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