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EXPLICIT EXPRESSION OF A BARBAN & VEHOV THEOREM

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Abstract: We prove that

$$S=\sum_{n\leqslant N}\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2/n\leqslant 166\frac{\log N}{\log z}$$

where $N\geqslant z\geqslant 100$, where the $\lambda_d^{(1)}$ is the weight introduced by Barban & Vehov in 1968, namely

$$\lambda_d^{(1)}=\begin{cases}\mu(d) & \text{when } d\leqslant z,\\ \mu(d)\frac{\log(z^2/d)}{\log z} & \text{when } z<d\leqslant z^2,\\ 0 & \text{when } z^2<d,\end{cases}$$

where μ is the Moebius function.

Keywords: explicit estimates, Möbius function.

1. Introduction and results

This paper is a contribution to the famous optimisation problem of Barban & Vehov. These two authors noticed in [1] that

$$\sum_{n\leqslant N}\Bigg(\sum_{\substack{d\leqslant z\\ d|n}}\mu(d)+\sum_{\substack{z<d\leqslant z2\\ d|n}}\mu(d)\frac{\log(z^2/d)}{2\log z}\Bigg)^2\ll\frac{N}{\log z} \tag{1.1}$$

for every $N\geqslant 1$. Such a result was previously known only for $N\geqslant z^4$, reducing drastically the range of applications. As a matter of fact, they considered a more general sum but gave no details. These were published by Y. Motohashi in [5] and in a much simpler form by R. Graham in [2]. They select two parameters $1\leqslant z_1\leqslant z_2\leqslant N$, and set $\Lambda_i(d)=\mu(d)\max(\log(z_i/d),0)$ for $i=1,2$. They showed that

$$\sum_{n\leqslant N}\Bigg(\sum_{d|n}\Lambda_1(d)\Bigg)\Bigg(\sum_{e|n}\Lambda_2(e)\Bigg)=N\log z_1+O(N). \tag{1.2}$$

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These weights were widely generalized by Y. Motohashi in [3, Lemma 5] (see also his book [4]), We restrict our attention to a specific choice and set, following Y. Motohashi's notation:

$$\lambda_d^{(1)} = \begin{cases} \mu(d) & \text{when } d \leqslant z, \\ \mu(d)\frac{\log(z^2/d)}{\log z} & \text{when } z < d \leqslant z^2, \\ 0 & \text{when } z^2 < d. \end{cases} \tag{1.3}$$

Previous authors considered slightly more general weights with a y instead of the z^2 that we use here. Y. Motohashi noticed in 1978 that it is enough in many proofs to establish the following weaker estimate:

$$\sum_{n\leqslant N} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n} \ll \frac{\log N}{\log z}. \tag{1.4}$$

Our aim is to produce an explicit form of this result.

Theorem 1.1. *We have, when $z \leqslant N$ and $z \geqslant 10^2$,*

$$\sum_{n\leqslant N} \Big(\sum_{d|n} \lambda_d^{(1)}\Big)^2 / n \leqslant 166 \frac{\log N}{\log z}.$$

From now on we call S the sum to be studied (i.e. the left-hand side above). In between, we improve on the main result of [8] in Lemma 2.2 and give a more precise form of [8, Theorem 1] in Lemma 2.1 below.

Notation. The notation $f = O^*(g)$ means that

$$|f| \leqslant g$$

2. Auxiliary lemmas

Lemma 2.1. *For $x \geqslant 1$ and for any positive integer r,*

$$-\frac{11}{15}(1+\varepsilon) \leqslant \sum_{\substack{\ell\leqslant x,\\ (\ell,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \leqslant 1+\varepsilon.$$

Proof. This lower bound is proved in [8, (10)] when $x \geqslant 9$ while the upper bound is [8, Theorem 1] for any positive x. Let us complete the proof of the lower bound for $x \in [1, 9)$. It is enough to consider $x \leqslant 7$. The sum is larger than

$$1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} = -\frac{37}{210} \geqslant -\frac{11}{15}.$$

This concludes the proof. ■

Lemma 2.2. *When $\varepsilon \in [0, 1/10]$, for any real number $x \geqslant 1$ and any integer $r \geqslant 1$, we have*

$$\left|\sum_{\substack{\ell\leqslant x,\\(\ell,r)=1}} \frac{\mu(\ell)\log(x/\ell)}{\ell^{1+\varepsilon}} - (1+\varepsilon)\right| \leqslant \frac{9}{10} + (1+\varepsilon)x^{\varepsilon}\frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)},$$

where $\varphi_{1+\varepsilon}(r)/r^{1+\varepsilon} = \prod_{p|r}(1-p^{-1-\varepsilon})$.

Proof. We first treat the case when $x \geqslant 10$. On following the paper [8], we define

$$S_1 = \sum_{\substack{\ell\leqslant x,\\(\ell,r)=1}} \mu(\ell) \sum_{m\leqslant d/\ell} m^{\epsilon}\tau(m). \tag{2.1}$$

Thus,

$$S_1 = \sum_{\substack{\ell\leqslant x,\\(\ell,r)=1}} f_{r,\epsilon}(n)$$

where the multiplicative function $f_{r,\epsilon}$ is defined by

$$f_{r,\epsilon}(n) = \sum_{\substack{\ell\leqslant x,\\(\ell,\delta)=1}} \frac{\mu(\ell)}{\ell^{1+\varepsilon}}\tau(n/\ell).$$

In multiplicative form

$$f_{r,\epsilon}(n) = \prod_{\substack{p^{\nu}||n,\\p|r}} \left(\nu+1-\frac{\nu}{p^{\varepsilon}}\right) \prod_{\substack{p^{\nu}||n,\\p|r}} (\nu+1).$$

Note that $f_{r,\epsilon} \geqslant 1$. As a consequence, we have $S_1 \geqslant x^{1+\varepsilon}/(1+\varepsilon) - x^{\varepsilon}$ for every $x \geqslant 1$. On the other hand, by using [8, Lemma 3.7], we also have

$$S_1 = \frac{x^{1+\varepsilon}}{1+\varepsilon}\sum_{\substack{\ell\leqslant x,\\(\ell,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Big(\log\frac{x}{d} + 2\gamma - \frac{1}{1+\varepsilon}\Big) + O^*(0.961 \times 1.33(1+2\varepsilon)x^{1+\varepsilon})$$

We set $\alpha = 2\gamma - \frac{1}{1+\varepsilon}$ and rewrite the above in the form

$$\begin{aligned} S_1 &= \frac{x^{1+\varepsilon}}{1+\varepsilon}\sum_{\substack{\ell\leqslant x,\\(\ell,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Big(\log\frac{x}{d} + \alpha\Big) + O^*(1.279(1+2\varepsilon)x^{1+\varepsilon}) \\ &= S_1^* + \alpha S_0 + O^*(1.279(1+2\varepsilon)x^{1+\varepsilon}). \end{aligned}$$

Thus, we have

$$\frac{x^{1+\varepsilon}}{1+\varepsilon} - x^{\varepsilon} - 1.279(1+2\varepsilon)x^{1+\varepsilon} \leqslant S_1^* + \alpha S_0 \leqslant 1.279(1+2\varepsilon)x^{1+\varepsilon} + x^{1+2\varepsilon}\frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)}.$$

By Lemma 2.1, we find that

$$\frac{1}{(1+\varepsilon)} - x^{-1} - 1.279(1+2\varepsilon) - \alpha \leqslant x^{-1-\varepsilon}S_1^* \leqslant 1.279(1+2\varepsilon) + x^{\varepsilon}\frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)} + \frac{11}{15}\alpha.$$

Hence

$$\begin{aligned}\frac{-\varepsilon}{(1+\varepsilon)} - x^{-1} - 1.279(1+2\varepsilon) - \alpha &\leqslant x^{-1-\varepsilon}S_1^* - 1 \\ &\leqslant 0.279 + 2\times 1.279\varepsilon + x^{\varepsilon}\frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)} + \frac{11}{15}\alpha.\end{aligned}$$

Notice that

$$(1+\varepsilon)x^{-1-\varepsilon}S_1^* = \sum_{\substack{d\leqslant x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\log\frac{x}{d} = S_2$$

is the quantity we want to approximate. We reduce the above inequality to

$$-M_1 - (1+\varepsilon)x^{\varepsilon}\frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)} \leqslant S_2 - (1+\varepsilon) \leqslant M_2 + (1+\varepsilon)x^{\varepsilon}\frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)},$$

where

$$\begin{aligned}M_1 &= \varepsilon + (1+\varepsilon)x^{-1} + 1.279(1+2\varepsilon)(1+\varepsilon) + 2\gamma(1+\varepsilon) - 1 - (1+\varepsilon)x^{\varepsilon}\frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)} \\ &\leqslant \varepsilon + (1+\varepsilon)x^{-1} + 1.279(1+2\varepsilon)(1+\varepsilon) + 2\gamma(1+\varepsilon) - 1 - (1+\varepsilon)x^{\varepsilon}\end{aligned}$$

since $\frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)} \geqslant 1$ and

$$M_2 = 0.279(1+\varepsilon) + 2\times 1.279\varepsilon(1+\varepsilon) + \frac{22}{15}\gamma(1+\varepsilon) - \frac{11}{15}.$$

We want to find an upper bound for $\max(M_1, M_2)$. The function $x \mapsto (1+\varepsilon)x^{-1} - (1+\varepsilon)x^{\varepsilon}$ is non-increasing in x and is hence not more than $\frac{1}{10}(1+\varepsilon) - (1+\varepsilon)10^{\varepsilon}$. We thus bound M_1 by

$$\begin{aligned}M_1 &\leqslant \varepsilon + (1+\varepsilon)\left(\frac{1}{10} - 10^{\varepsilon}\right) + 1.279(1+2\varepsilon)(1+\varepsilon) + 2\gamma(1+\varepsilon) - 1 \\ &\leqslant 0.1 + (1+\varepsilon)\left(\frac{1}{10} - 10^{\varepsilon} + 1.279\times 1.2 + 2\gamma\right) - 1\end{aligned}$$

since $\varepsilon \in [0, 1/10]$. The derivative of this function of ε is

$$\frac{1}{10} - 10^{\varepsilon} + 1.279\times 1.2 + 2\gamma - (1+\varepsilon)10^{\varepsilon}\log 10.$$

It is non-increasing and negative at $\varepsilon = 0$. This point is thus where the maximum of the initial function is reached and thus

$$M_1 \leqslant 0.1 + \frac{1}{10} - 1 + 1.279 \times 1.2 + 2\gamma - 1 \leqslant 0.89.$$

Concerning M_2, we find that

$$M_2 \leqslant 0.279 \times 1.1 + 2 \times 1.279 \times 0.1 \times 1.1 + \frac{22}{15}\gamma \times 1.1 - \frac{11}{15} \leqslant 0.79.$$

Hence the result in this case.

We now consider the case when x is strictly below 10. We define

$$D_r(x,\varepsilon) = \sum_{\substack{d\leqslant x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log\frac{x}{d} - (1+\varepsilon). \tag{2.2}$$

When x is in $[1,2)$, we have directly

$$|D_r(x,\varepsilon)| = |\log x - (1+\varepsilon)| \leqslant \frac{9}{10} + (1+\varepsilon)x^\varepsilon \frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)}$$

since $x^\varepsilon r^{1+\varepsilon}/\varphi_{1+\varepsilon}(r) \geqslant 1$. When x is in $[2,3)$, if r is even then

$$|D_r(x,\varepsilon)| \leqslant |\log 3 - (1+\varepsilon)| \leqslant \frac{9}{10} + (1+\varepsilon)x^\varepsilon \frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)},$$

while, if r is odd, we have

$$D_r(x,\varepsilon) = \log x - \frac{1}{2^{1+\varepsilon}} \log\frac{x}{2} - (1+\varepsilon)$$

which is increasing in x. It is enough to consider $x = 2$ and $x = 3$ and the result follows readily.

In general, when $x \in [n, n+1)$ and n is some positive integer, we discuss according to the gcd of r with $P(n) = \prod_{p\leqslant n} p$. We get in this manner a function of the shape $a\log x + b$, from which we deduce that, ε being fixed, is either non-increasing or non-decreasing. In either case, it is enough to handle the case of the two endpoints n and $n+1$.

When $x < 10$ and either $(r,6) > 1$ or $x < 6$, the $1/d^{1+\varepsilon}$ term that appears is with a negative coefficient. Thus,

$$D_r(x,\varepsilon) \leqslant \log x - (1+\varepsilon) \leqslant \log(10) - 1 \leqslant \frac{9}{10} + 1 + \varepsilon. \tag{2.3}$$

When $(r,6) > 1$, this inequality still holds since

$$-\frac{1}{2^{1+\varepsilon}} \log\frac{x}{2} + \frac{1}{6^{1+\varepsilon}} \log\frac{x}{6} \leqslant 0.$$

We now consider the lower bound. The worst case is $\varepsilon = 0$ for the $d^{1+\varepsilon}$ term, i.e. we have, when $7 \leqslant x < 10$

$$D_r(x,\varepsilon) \geqslant \log x - \frac{1}{2}\log\frac{x}{2} - \frac{1}{3}\log\frac{x}{3} - \frac{1}{5}\log\frac{x}{5} - \frac{1}{7}\log\frac{x}{7} - \left(1 + \frac{1}{10}\right)$$

which is readily shown to be $\geqslant -0.2$. When $x \in [5,7)$, a similar proof (erase the contribution of 7 above) shows that $D_r(x,\varepsilon) \geqslant -0.14$. Same when $x \in [3,5)$, though this time the relevant function is increasing and this proves that $D_r(x,\varepsilon) \geqslant -0.21$. When $x \in [2,3)$, we get $D_r(x,\varepsilon) \geqslant -0.41$. ■

Lemma 2.3. *For $y \geqslant 1$ we have*

$$\sum_{\delta \leqslant y} \frac{\mu^2(\delta)}{\varphi(\delta)} \leqslant \log y + 1.332582 + \frac{3.95}{\sqrt{y}}.$$

This lemma follows from [6, Theorem 1.2].

Lemma 2.4. *For a real number $y \geqslant 1$ we have*

$$\frac{6}{\pi^2}\log y + 0.578 \leqslant \sum_{\delta \leqslant y} \frac{\mu^2(\delta)}{\delta} \leqslant \frac{6}{\pi^2}\log y + 1.166.$$

This lemma follows from [9, Lemma 3.4].

Lemma 2.5. *We have*

$$\sum_{\delta \leqslant y} \frac{\mu^2(\delta)\varphi(\delta)}{\delta^2} = a\log y + b + O^*(0.174)$$

with

$$a = \prod_{p \geqslant 2}(p^3 - 2p + 1)/p^3 = 0.4282 + O^*(10^{-4})$$

and

$$b/a = \gamma + \sum_{p \geqslant 2} \frac{3p-2}{p^3 - 2p + 1}\log p = 2.046 + O^*(10^{-4}).$$

This lemma follows from [9, Lemma 3.4].

Lemma 2.6. *When s is a real number satisfying $|s-1| \leqslant \frac{1}{2}$ we have*

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + O^*(20|s-1|^2)$$

where $\gamma = 0.57721...$ and $\gamma_1 = 0.07281...$ are the Laurent-Stieltjes constants.

This lemma follows from [9, Lemma 5.3]. Let the von Mangoldt function be defined by

$$\Lambda(n) = \begin{cases} \log p & \textit{if } n = p^k \textit{ for a prime number } p \textit{ and an integer } k \geqslant 1\ , \\ 0 & \textit{otherwise.} \end{cases}.$$

Lemma 2.7. *For $x > 0$ we have*

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n) < 1.03883\, x.$$

This lemma follows from [10].

3. Proof of Theorem 1.1 for $N \geqslant z^2$

To simplify the typographic work, we let

$$S = \sum_{n \leqslant N} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n}. \tag{3.1}$$

3.1. Reduction to an infinite sum

Let $\epsilon > 0$ be a real parameter which we will choose later. We use Rankin's trick to write

$$S \leqslant \sum_{n \geqslant 1} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n} \left(\frac{N}{n}\right)^{\epsilon} = N^{\epsilon} \sum_{n \geqslant 1} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n^{\epsilon+1}}.$$

We set

$$\omega = \varepsilon + 1. \tag{3.2}$$

3.2. Reduction to two sums

Let

$$L(y,d) = \begin{cases} \mu(d) \log(y/d) & \text{if } d \leqslant y, \\ 0 & \text{otherwise.} \end{cases}$$

We notice that

$$\lambda_d^{(1)} = \frac{L(z^2,d) - L(z,d)}{\log z}.$$

By using the classical inequality $|x+y|^2 \leqslant 2(|x|^2 + |y|^2)$, we find that

$$(\log z)^2 \sum_{n \geqslant 1} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n^{\omega}} \leqslant 2 \sum_{n \geqslant 1} \frac{\left(\sum_{d|n} L(z^2,d)\right)^2}{n^{\omega}} + 2 \sum_{n \geqslant 1} \frac{\left(\sum_{d|n} L(z,d)\right)^2}{n^{\omega}}.$$

3.3. Individual upper bound

We are looking for the upper bound of

$$\sum_{n\geqslant 1}\frac{\Big(\sum_{d|n}L(y,d)\Big)^2}{n^\omega}$$

for $y=z$ and $y=z^2$. We expand the square and notice that

$$\sum_{n\geqslant 1}\frac{\Big(\sum_{d|n}L(y,d)\Big)^2}{n^\omega}=\sum_{d_1,d_2}\frac{L(y,d_1)L(y,d_2)}{[d_1,d_2]^\omega}\zeta(\omega).$$

We are looking for the upper bound of the sum

$$S_1(y)=\sum_{d_1,d_2}\frac{L(y,d_1)L(y,d_2)}{[d_1,d_2]^\omega}=\sum_{d_1,d_2}\frac{L(y,d_1)L(y,d_2)(d_1,d_2)^\omega}{d_1^\epsilon d_2^\omega}. \tag{3.3}$$

We use the process of diagonalisation of Selberg. To do so, we have

$$\varphi_\omega(d)=\prod_{p|d}(p^\omega-1). \tag{3.4}$$

When $d\geqslant 1$ is squarefree, this function satisfies the following

$$\sum_{p|d}\varphi_\omega(d)=\big(\varphi_\omega\star\mathbb{1}\big)(d)=d^\omega.$$

From this we infer that

$$S_1(y)=\sum_{\delta\geqslant 1}\varphi_\omega(\delta)\sum_{d_1,d_2\ \delta|d_1\ \delta|d_2}\frac{L(y,d_1)L(y,d_2)}{d_1^\omega d_2^\omega}=\sum_{\delta\leqslant y}\varphi_\omega(\delta)\left(\sum_{\substack{d\\ \delta|d}}\frac{L(y,d)}{d^\omega}\right)^2.$$

We are now looking for an estimate of $\sum_{d/\delta|d}L(y,d)/d^\omega$. According to the definition of L we find that

$$\sum_{\substack{d\\ \delta|d}}\frac{L(y,d)}{d^\omega}=\sum_{\substack{d\leqslant y\\ \delta|d}}\frac{\mu(d)\log(\frac{y}{d})}{d^\omega}=\sum_{\substack{\ell\leqslant y/\delta\\ d=\delta\ell}}\frac{\mu(\delta\ell)\log(\frac{y}{\delta\ell})}{(\delta\ell)^\omega}.$$

We can assume that $(\delta,\ell)=1$ since otherwise $\mu(\delta\ell)=0$. Since μ is multiplicative, we then have $\mu(\delta\ell)=\mu(\delta)\mu(\ell)$. Thus, we get

$$\sum_{\substack{d\\ \delta|d}}\frac{L(y,d)}{d^\omega}=\frac{\mu(\delta)}{\delta^\omega}\sum_{\substack{\ell\leqslant y/\delta\\ (\delta,\ell)=1}}\frac{\mu(\ell)\log(\frac{y}{\delta\ell})}{\ell^\omega}.$$

By Lemma 2.2, we have the following upper bound

$$\left|\sum_{\substack{d \\ \delta|d}} \frac{L(y,d)}{d^\omega} - \frac{\mu(\delta)}{\delta^\omega}(1+\varepsilon)\right| \leqslant \frac{1}{\delta^{1+\varepsilon}}\left(\frac{9}{10} + (\varepsilon+1)(\frac{y}{\delta})^\varepsilon \frac{\delta^{1+\varepsilon}}{\varphi_\omega(\delta)}\right),$$

i.e. with $\omega = 1+\varepsilon$:

$$\left|\sum_{\substack{d \\ \delta|d}} \frac{L(y,d)}{d^\omega} - \frac{\mu(\delta)}{\delta^\omega}(1+\varepsilon)\right| \leqslant \frac{1}{\delta^{\omega}}\left(\frac{9}{10} + \omega(\frac{y}{\delta})^{\omega-1} \frac{\delta^{\omega}}{\varphi_\omega(\delta)}\right).$$

This implies that

$$\sum_{\delta\leqslant y}\varphi_\omega(\delta)\left|\sum_{\substack{d \\ \delta|d}} \frac{L(y,d)}{d^\omega} - \frac{\mu(\delta)}{\delta^\omega}(1+\varepsilon)\right|^2 \leqslant \sum_{\delta\leqslant y}\varphi_\omega(\delta)\frac{\mu^2(\delta)}{\delta^{2\omega}}\left(\frac{9}{10} + \omega(\frac{y}{\delta})^{\omega-1} \frac{\delta^{\omega}}{\varphi_\omega(\delta)}\right)^2.$$

We use the identity $a^2 = (a-e)^2 + 2ae - e^2$ to infer from the above that

$$\begin{aligned} S_1(y) \leqslant{}& \sum_{\delta\leqslant y}\varphi_\omega(\delta)\frac{\mu^2(\delta)}{\delta^{2\omega}}\left(\frac{9}{10} + \omega(\frac{y}{\delta})^{\omega-1} \frac{\delta^{\omega}}{\varphi_\omega(\delta)}\right)^2 \\ &+ 2(1+\varepsilon)\sum_{\delta\leqslant y}\frac{\mu(\delta)\varphi_\omega(\delta)}{\delta^\omega}\sum_{\substack{d \\ \delta|d}}\frac{L(y,d)}{d^\omega} - \sum_{\delta\leqslant y}\varphi_\omega(\delta)\frac{\mu^2(\delta)}{\delta^{2\omega}}(1+\varepsilon)^2. \end{aligned}$$

Let us investigate the factor of $2(1+\varepsilon)$. It is

$$\begin{aligned} \sum_{\delta|d\leqslant y}\frac{\mu(\delta)\varphi_\omega(\delta)\mu(d)\log(y/d)}{d(\delta)^\omega} &= \sum_{d\leqslant y}\frac{\mu(d)\log(y/d)}{d^\omega}\sum_{\delta|d}\mu(\delta)\frac{\varphi_\omega(\delta)}{\delta^\omega} \\ &= \sum_{d\leqslant y}\frac{\mu(d)\log(y/d)}{d^\omega}\prod_{p|d}\left(1-\frac{p^\omega-1}{p^\omega}\right) \\ &= \sum_{d\leqslant y}\frac{\mu(d)\log(y/d)}{d^{2\omega}}. \end{aligned}$$

Then we have

$$\begin{aligned} \left|\sum_{d\leqslant y}\frac{\mu(d)\log(y/d)}{d^{2\omega}}\right| &\leqslant \sum_{d\leqslant y}\frac{\mu^2(d)\log(y/d)}{d^2} = \sum_{d\leqslant y}\frac{\mu^2(d)}{d^2}\int_d^y\frac{dt}{t} \\ &= \int_1^y\sum_{d\leqslant t}\frac{\mu^2(d)}{d^2}\frac{dt}{t} \leqslant \frac{\zeta(2)}{\zeta(4)}\log y. \end{aligned}$$

The above upper bound reduces to:

$$S_1(y) \leqslant \sum_{\delta\leqslant y}\left(\left(\frac{9}{10}\right)^2 - \omega^2 + 2\omega\frac{9}{10}\frac{\delta^\omega}{\varphi_\omega(\delta)}\left(\frac{y}{\delta}\right)^{\omega-1} + \omega^2\frac{\delta^{2\omega}}{\varphi_\omega^2(\delta)}\left(\frac{y}{\delta}\right)^{2(\omega-1)}\right)\frac{\varphi_\omega(\delta)\mu^2(\delta)}{\delta^{2\omega}} + 2\omega\frac{\zeta(2)}{\zeta(4)}\log y$$

Then we have

$$S_1(y) \leqslant \left(\left(\frac{9}{10}\right)^2 - \omega^2\right)\sum_{\delta\leqslant y}\frac{\varphi_\omega(\delta)\mu^2(\delta)}{\delta^{2\omega}} + \frac{18}{10}\omega y^{\omega-1}\sum_{\delta\leqslant y}\frac{\mu^2(\delta)}{\delta^{2\omega-1}} + \omega^2 y^{2\omega-2}\sum_{\delta\leqslant y}\frac{\mu^2(\delta)}{\varphi_\omega(\delta)\delta^{2(\omega-1)}} + 3.04\omega\log y.$$

On using the two inequalities

$$\varphi_\omega(\delta) \leqslant \delta^\omega \qquad \text{and} \qquad \frac{1}{\varphi_\omega(\delta)} \leqslant \frac{1}{\varphi(\delta)},$$

we obtain

$$S_1(y) \leqslant \left(\left(\frac{9}{10}\right)^2 - \omega^2\right)\sum_{\delta\leqslant y}\frac{\varphi(\delta)\mu^2(\delta)}{\delta^{2\omega}} + 1.8\omega y^{\omega-1}\sum_{\delta\leqslant y}\frac{\mu^2(\delta)}{\delta^{2\omega-1}} + \omega^2 y^{2\omega-2}\sum_{\delta\leqslant y}\frac{\mu^2(\delta)}{\varphi(\delta)\delta^{2(\omega-1)}} + 3.04\omega\log y.$$

We use $\frac{1}{\delta^{2\omega}} \geqslant \frac{e^{-2c}}{\delta^2}$, $\frac{1}{\delta^{2\omega-1}} \leqslant \frac{1}{\delta}$, and $\frac{1}{\delta^{2\omega-2}} \leqslant 1$.

By choosing $\omega = 1 + \frac{c}{\log N} \leqslant 1 + \frac{c}{\log(z^2)}$, where now $c > 0$ is a real parameter that we can choose, we get

$$S_1(y) \leqslant \left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c}\sum_{\delta\leqslant y}\frac{\varphi(\delta)\mu^2(\delta)}{\delta^2} + 1.8\omega y^{\omega-1}\sum_{\delta\leqslant y}\frac{\mu^2(\delta)}{\delta} + \omega^2 y^{2\omega-2}\sum_{\delta\leqslant y}\frac{\mu^2(\delta)}{\varphi(\delta)} + 3.04\omega\log y.$$

On using Lemma 2.3, 2.4 and 2.5 we obtain

$$\begin{aligned} S_1(y) \leqslant{}& \left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c}a\log y + \left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c}b \\ &- 0.174\left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c} + 1.8\omega y^{\omega-1}\left(\frac{6}{\pi^2}\log y + 1.166\right) \\ &+ w^2 y^{2(w-1)}\left(\log y + 1.3325 + \frac{3.95}{\sqrt{y}}\right) + 3.04\omega\log y. \end{aligned} \tag{3.5}$$

3.4. Conclusion

Using the upper bound given by (3.5) for $y = z$ and for $y = z^2$ we obtain the following upper bound of S:

$$S \leqslant \frac{2N^{\omega-1}\zeta(\omega)}{(\log z)^2}\Bigg(\Big(2\frac{a}{e^{2c}}\log z + \frac{b}{e^{2c}} - \frac{0,174}{e^{2c}}\Big)\Big(\Big(\frac{9}{10}\Big)^2 - \omega^2\Big)$$
$$+ 1.8\omega e^{c/2}\Big(\frac{12}{\pi^2}\log z + 1.166\Big) + w^2e^c\Big(2\log z + 1.3325 + \frac{3.95}{z}\Big) + 6.08\omega\log z\Bigg)$$
$$+ \frac{2N^{\omega-1}\zeta(\omega)}{(\log z)^2}\Bigg(\Big(\frac{a}{e^{2c}}\log z + \frac{b}{e^{2c}} - \frac{0,174}{e^{2c}}\Big)\Big(\Big(\frac{9}{10}\Big)^2 - \omega^2\Big)$$
$$+ 1.8\omega e^c\Big(\frac{6}{\pi^2}\log z + 1.166\Big) + w^2e^{2c}\Big(\log z + 1.3325 + \frac{3.95}{\sqrt{z}}\Big)$$
$$+ 3.04\beta(c)\log z.\Bigg)$$

Then we have

$$S \leqslant \frac{2N^{\omega-1}\zeta(\omega)}{(\log z)^2}\Bigg(\Big(1.1e^{-2c} + (2 + e^c - 1.285e^{-3c})\omega^2e^c + \frac{10.8}{\pi^2}\omega e^c\Big)\log z$$
$$+ \Big(\frac{21.6}{\pi^2}\omega e^{c/2} + 9.12\omega\Big)\log z$$
$$+ w^2e^c\Big(1.3325 + \frac{3.95}{10^2} + 1.3325e^c + \frac{3.95e^c}{\sqrt{10^2}} - 1.6e^{-3c}\Big)$$
$$+ 2.1\omega e^c + 2.1\omega e^{c/2} + 1.42e^{-2c}\Bigg),$$

provided that $z^2 \leqslant N$ and $z \geqslant 10^2$. We have

$$\omega = 1 + \frac{c}{\log N} \leqslant 1 + \frac{c}{\log 10^4} = \beta(c)$$

where

$$\beta(c) = 1 + \frac{c}{\log(10^4)}. \tag{3.6}$$

Otherwise, by using Lemma 2.6 with $s = 1 + \epsilon$, we find that

$$\zeta(\omega) = \zeta(1+\epsilon) = \frac{\log N}{c} + \gamma + 20\Big(\frac{c}{\log N}\Big)^2 \leqslant g(c)\log N$$

where

$$g(c) = \frac{1}{c} + \frac{\gamma + 20\big(\frac{c}{\log 10^4}\big)^2}{\log 10^4}. \tag{3.7}$$

We now find that

$$S \leqslant \frac{2N^{\omega-1}g(c)\log N}{(\log z)^2}\Bigg(\bigg(1.1e^{-2c} + (2+e^c-1.285e^{-3c})\beta^2(c)e^c + \frac{10.8}{\pi^2}\beta(c)e^c\bigg)\log z$$

$$+ \left(\frac{21.6}{\pi^2}\beta(c)e^{c/2} + 9.12\beta(c)\right)\log z$$

$$+ \beta^2(c)e^c\bigg(1.3325 + \frac{3.95}{10^2} + 1.3325e^c + \frac{3.95e^c}{\sqrt{10^2}} - 1.6e^{-3c}\bigg)$$

$$+ 2.1\beta(c)e^c + 2.1\beta(c)e^{c/2} + 1.42e^{-2c}\Bigg).$$

We next define $A(c)$ by

$$A(c) = 2e^c g(c)\left(1.1e^{-2c} + (2+e^c-1.285e^{-3c})\beta^2(c)e^c\right)$$
$$+ 2e^c g(c)\left(\frac{10.8}{\pi^2}\beta(c)e^c + \frac{21.6}{\pi^2}\beta(c)e^{c/2} + 9.12\beta(c)\right),$$

and $B(c)$ by

$$B(c) = 2e^c g(c)\bigg(\beta^2(c)e^c\bigg(1.3325 + \frac{3.95}{10^2} + 1.3325e^c + \frac{3.95e^c}{\sqrt{10^2}} - 1.6e^{-3c}\bigg)$$
$$+ 2.1\beta(c)e^c + 2.1\beta(c)e^{c/2} + 1.42e^{-2c}\bigg).$$

With these definitions, our upper bound of S becomes

$$S \leqslant \frac{\log N}{(\log z)^2}(A(c)\log z + B(c)).$$

We optimize the choice of the parameter c by using PARI/GP [7]: we find that by choosing $c = 0.535$, we obtain that, for $z^2 \leqslant N$ and $z \geqslant 100$

$$S \leqslant \frac{\log N}{(\log z)^2}\left(144.55\log z + 97.9\right) \leqslant 166\frac{\log N}{\log z}.$$

4. Proof of Theorem 1.1 for $z \leqslant N \leqslant z^2$

We begin slightly differently. We notice that

$$(\log z)^2 S \leqslant 2\sum_{n\leqslant N}\frac{\left(\sum_{d|n}L(z^2,d)\right)^2}{n} + 2\sum_{n\leqslant N}\frac{\left(\sum_{d|n}L(z,d)\right)^2}{n}$$

By using Rankin's trick on the second sum, we obtain that

$$(\log z)^2 S \leqslant 2\sum_{n\leqslant N}\frac{\left(\sum_{d|n}L(z^2,d)\right)^2}{n} + 2N^{\epsilon}\sum_{n\geqslant 1}\frac{\left(\sum_{d|n}L(z,d)\right)^2}{n^{\omega}}$$

with $\omega = \epsilon + 1$. We are still asking $\omega = 1 + \frac{c}{\log N}$ for a certain parameter c. On using (3.3) for $y = z$, we have

$$S \leqslant \frac{2}{(\log z)^2} \sum_{n \leqslant N} \frac{\left(\sum_{d|n} L(z^2, d)\right)^2}{n} + \frac{2N^{\epsilon}\zeta(\omega)}{(\log z)^2} S_1(z)$$
$$\leqslant \frac{2}{(\log z)^2} \sum_{n \leqslant N} \frac{\left(\sum_{d|n} L(z^2, d)\right)^2}{n} + \frac{2e^c g(c) \log N}{(\log z)^2} S_1(z).$$

Let

$$S_2 = \frac{2}{(\log z)^2} \sum_{n \leqslant N} \frac{\left(\sum_{d|n} L(z^2, d)\right)^2}{n} \tag{4.1}$$

and

$$S_3 = \frac{2e^c g(c) \log N}{(\log z)^2} S_1(z). \tag{4.2}$$

4.1. Upper bound of S_2

Recall that $L(z^2, d) = \mu(d) \log(z^2/d)$,we get that, for $n \leqslant N \leqslant z^2$,

$$\sum_{d|n} L(z^2, d) = \sum_{d|n} \mu(d)(2 \log z - \log d) = 2 \log z\, \delta_{n=1} - \sum_{d|n} \mu(d) \log d$$
$$= 2 \log z\, \delta_{n=1} + \sum_{d|n} \mu(d)(\log(\frac{n}{d}) - \log n)$$
$$= 2 \log z\, \delta_{n=1} + \sum_{d|n} \mu(d) \log(\frac{n}{d}).$$

since $(\mu \star \log)(n) = \Lambda(n)$ where Λ is the von Mangoldt function. Hence, when $n \leqslant z^2$,

$$\sum_{d|n} L(z^2, d) = 2 \log z\, \delta_{n=1} + \Lambda(n).$$

Therefore,we get

$$\sum_{n \leqslant N} \frac{\left(\sum_{d|n} L(z^2, d)\right)^2}{n} = 4(\log z)^2 + \sum_{n \leqslant N} \frac{\Lambda(n)^2}{n} \leqslant 4(\log z)^2 + \sum_{n \leqslant N} \frac{\Lambda(n) \log n}{n}$$
$$\leqslant 4(\log z)^2 + \sum_{n \leqslant N} \Lambda(n) \left(\frac{\log N}{N} + \int_2^N \frac{\log t - 1}{t^2} dt \right)$$
$$\leqslant 4(\log z)^2 + \sum_{n \leqslant N} \Lambda(n) \frac{\log N}{N} + \int_2^N \sum_{n \leqslant t} \Lambda(n) \frac{\log t - 1}{t^2} dt,$$

so,

$$\begin{aligned}\sum_{n\leqslant N}\frac{\left(\sum_{d|n}L(z^2,d)\right)^2}{n} &\leqslant 4(\log z)^2+\sum_{n\leqslant N}\Lambda(n)\frac{\log N}{N}+\int_2^N\sum_{n\leqslant t}\Lambda(n)\frac{\log t-1}{t^2}dt\\ &\leqslant 4(\log z)^2+\sum_{n\leqslant N}\Lambda(n)\frac{\log N}{N}+\int_2^e\sum_{n\leqslant t}\Lambda(n)\frac{\log t-1}{t^2}dt\\ &\quad+\int_e^N\sum_{n\leqslant t}\Lambda(n)\frac{\log t-1}{t^2}dt.\end{aligned}$$

According to Lemma 2.7, we have

$$\begin{aligned}\sum_{n\leqslant N}\frac{\left(\sum_{d|n}L(z^2,d)\right)^2}{n} &\leqslant 4(\log z)^2+\psi(N)\frac{\log N}{N}+\int_2^e\log 2\frac{\log t-1}{t^2}dt\\ &\quad+\int_e^N\psi(t)\frac{\log t-1}{t^2}dt\\ &\leqslant 4(\log z)^2+1.03883\log N+\log 2\int_2^e\frac{\log t-1}{t^2}dt\\ &\quad+1.03883\int_e^N\frac{\log t-1}{t}dt.\end{aligned}$$

By using integration by parts, we find that

$$\begin{aligned}\sum_{n\leqslant N}\frac{\left(\sum_{d|n}L(z^2,d)\right)^2}{n} &\leqslant 4(\log z)^2+1.03883\log N-0.0147\\ &\quad+1.03883((\log N)^2-\log N)\\ &\leqslant 4(\log z)^2-0.0147+1.03883(\log N)^2.\end{aligned}$$

Hence

$$\begin{aligned}S_2&=\frac{2}{(\log z)^2}\sum_{n\leqslant N}\frac{\left(\sum_{d|n}L(z^2,d)\right)^2}{n}\\ &\leqslant\frac{2}{(\log z)^2}\left(4(\log z)^2-0.0147+1.03883(\log N)^2\right)\\ &\leqslant\frac{\log N}{(\log z)^2}\left(8\log z+2\frac{-0.0147}{2\log 10}+2.07766\log N\right)\\ &\leqslant\frac{\log N}{(\log z)^2}\left(8\log z-0.0025+4.3166\log z\right).\end{aligned}$$

Then we have

$$S_2\leqslant\frac{\log N}{(\log z)^2}(12.32\log z-0.0025).$$

4.2. Upper bound of S_3

Recall that:

$$S_3 = \frac{2e^c g(c)\log N}{(\log z)^2} S_1(z).$$

From (3.5) for $y = z$,we infer that

$$\begin{aligned} S_1(z) \leqslant & \left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c}a\log z + \left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c}b \\ & - 0.174\left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c} + 1.8\omega e^c\left(\frac{6}{\pi^2}\log z + 1.166\right) \\ & + w^2e^{2c}\left(\log z + 1.3325 + \frac{3.95}{\sqrt{z}}\right) + 3.04\beta(c)\log z. \end{aligned} \tag{4.3}$$

Then we have

$$\begin{aligned} S_1(z) \leqslant & \left(\left(\left(\frac{9}{10}\right)^2 - \omega^2\right)\frac{a}{e^{2c}} + \frac{10.8}{\pi^2}\omega e^c + w^2e^{2c} + 3.04\beta(c)\right)\log z \\ & + \left(\left(\frac{9}{10}\right)^2 - \omega^2\right)\frac{b}{e^{2c}} - 0.174\left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c} \\ & + 2.1\omega e^c + w^2e^{2c}\left(1.3325 + \frac{3.95}{\sqrt{10^2}}\right). \end{aligned} \tag{4.4}$$

Hence

$$S_3 \leqslant \frac{\log N}{(\log z)^2}(U(c)\log z + V(c))$$

with

$$U(c) = 2e^c g(c)\left(\left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c}a + 1.8\frac{6}{\pi^2}\omega e^c + w^2e^{2c} + 3.04\beta(c)\right)$$

and

$$\begin{aligned} V(c) = & 2e^c g(c)\left(\left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c}b - 0.174\left(\left(\frac{9}{10}\right)^2 - \omega^2\right)e^{-2c}\right) \\ & + 2e^c g(c)\left(2.1\omega e^c + w^2e^{2c}\left(1.3325 + \frac{3.95}{\sqrt{10^2}}\right)\right). \end{aligned}$$

Letting $c = 0.4$, we obtain

$$S_3 \leqslant \frac{\log N}{(\log z)^2}(44.25\log z + 65.29).$$

4.3. Conclusion

Putting together the upper bounds of S_2 and S_3, we get

$$\begin{aligned} S &\leqslant \frac{\log N}{(\log z)^2}(12.32\log z - 0.0025) + \frac{\log N}{(\log z)^2}(44.25\log z + 65.29) \\ &\leqslant \frac{\log N}{(\log z)^2}(56.57\log z + 52) \leqslant 68\frac{\log N}{\log z}, \end{aligned}$$

and the theorem is proved.

5. Further results

We find that by choosing $c = 0.555$, we obtain for $z^2 \leqslant N$ and $z \geqslant 10^3$ the inequality

$$S = \sum_{n\leqslant N}\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2/n \leqslant 152\frac{\log N}{\log z}.$$

By choosing $c = 0.421$, we obtain for $z \leqslant N \leqslant z^2$ and $z \geqslant 10^3$

$$S = \sum_{n\leqslant N}\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2/n \leqslant 63\frac{\log N}{\log z}.$$

We find that by choosing $c = 0.565$, we obtain for $z^2 \leqslant N$ and $z \geqslant 10^4$

$$S = \sum_{n\leqslant N}\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2/n \leqslant 146\frac{\log N}{\log z}.$$

Lastly, by choosing $c = 0.44$, we obtain for $z \leqslant N \leqslant z^2$ and $z \geqslant 10^4$ the inequality

$$S = \sum_{n\leqslant N}\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2/n \leqslant 59\frac{\log N}{\log z}.$$

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