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GALLAGHERIAN *PGT* ON $PSL(2,\mathbb{Z})$

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Abstract: Taking the Iwaniec explicit formula as a starting point, we bound the exponent in the error term of the prime geodesic theorem for the modular surface to 2/3, outside a set of finite logarithmic measure.

Keywords: prime geodesic theorem, Selberg zeta function, modular group.

1. Introduction

Let $\Gamma = PSL\,(2,\mathbb{Z})$ be the modular group and Z_Γ its Selberg zeta function defined by

$$Z_\Gamma(s) = \prod_{\{P_0\}}\prod_{k=0}^{\infty}(1-N(P_0)^{-s-k}), \qquad \mathrm{Re}(s)>1,$$

and meromorphicaly continued to the whole complex plane. The product is over hyperbolic conjugacy classes in Γ. The primitive conjugacy classes P_0 correspond to primitive closed geodesics on the modular surface $\Gamma\setminus\mathcal{H}$, where $\mathcal{H}$ is the upper half-plane equipped with the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. The length of the primitive closed geodesic joining two fixed points, necessarily the same for all representatives of a class P_0, equals $\log(N(P_0))$. We are interested in distribution of these geodesics, i.e., in the number $\pi_\Gamma(x)$ of classes P_0 such that $N(P_0)\leqslant x$, for $x>0$.

It is believed that the error term in the prime geodesic theorem

$$\pi_\Gamma\,(x) \approx \int_0^x \frac{dt}{\log t} \qquad (x\to\infty)$$

is $O(x^{\frac{1}{2}+\varepsilon})$ since Z_Γ satisfies the Riemann hypothesis.

Namely, there exists a broad analogy between the prime geodesic theorem and the prime number theorem based on the role played by the distribution of zeros of

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the Selberg zeta function resp. Riemann zeta function in two respective contexts. Furthermore, the non-trivial zeros $\frac{1}{2}+i\gamma=\frac{1}{2}\pm i\sqrt{\lambda-\frac{1}{4}}$ of Z_Γ correspond to the eigenvalues $\lambda \geqslant \frac{1}{4}$ of the essentially self-adjoint Laplace-Beltrami operator $\Delta=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ on $\Gamma\setminus\mathcal{H}$.

However, the fact that the Selberg zeta is a meromorphic function of order 2, as opposed to the Riemann zeta which is of order 1, causes tremendous difficulties in attempts to reach the expected value $\frac{1}{2}+\varepsilon$ in the error term. In the case of cofinite Fuchsian groups $\Gamma\subset PSL(2,\mathbb{R})$, the best estimate of the remainder in the prime geodesic theorem is still Randol's $O\left(\frac{x^{\frac{3}{4}}}{\log x}\right)$. The original Randol's paper [18] is concerned with cocompact Fuchsian groups. The case of noncompact finite volume Riemann surfaces can be dealt with either by means of an explicit formula for the Jorgenson-Lang class and the new integral representation of the logarithmic derivative of the Selberg zeta function (see [4, p. 185], [5, p. 128]) or through Randol's method as in [3, Theorem 2], where the details are given for a Fuchsian group of the first kind and a multiplier system with a weight on it. The analogue of Randol's estimate is also valid in higher dimensions [17], [2]. Indeed, the remainder is $O\left(\frac{x^{\frac{3}{4}(d-1)}}{\log x}\right)$ on d - dimensional hyperbolic manifolds with cusps.

Hejhal's comprehensive treatise [11], [12] raised the question to what extent the best possible exponent in PGT depends on the group Γ (see, e.g., [11, p. 320 top]).

The first breakthrough was due to Iwaniec [13] who obtained $\frac{35}{48}+\varepsilon$, precisely in the setting of $PSL(2,\mathbb{Z})$. Iwaniec's estimate was further improved to $\frac{7}{10}+\varepsilon$ by Luo-Sarnak [16], to $\frac{71}{102}+\varepsilon$ by Cai [7] and to $\frac{25}{36}+\varepsilon$ by Soundararajan-Young [20]. The results by Luo, Rudnick and Sarnak [15] on small eigenvalues of the Laplace-Beltrami operator yielded the validity of $\frac{7}{10}+\varepsilon$ also for congruence subgroups. Koyama next obtained $\frac{7}{10}+\varepsilon$ for cocompact discrete subgroups of $PSL(2,\mathbb{R})$ coming from a quaternion algebra $D=(a,b/\mathbb{Q})$ with $(a,b)=1$ and the prime 2 being unramified (see [14]).

Inspired by Gallagher's approach [10] in the Riemann zeta case, we were able to prove $\frac{7}{10}+\varepsilon$ bound for all cocompact Fuchsian groups, at the cost of excluding a set of finite logarithmic measure [1]. The latter result can be easily transferred to noncompact cofinite Fuchsian groups satisfying the condition

$$\sum_{\delta>0}\frac{x^{\beta-1/2}}{\delta^2}=O\left(\frac{1}{1+(\log x)^2}\right)\qquad(x\to\infty),$$

where $\beta+i\delta$ are poles of the scattering determinant (cf. [12, p. 477]).

The purpose of the present note is to demonstrate what further reduction of the exponent is possible through Gallagher's approach, if the underlying group is $PSL(2,\mathbb{Z})$. The answer is given by the following theorem.

Theorem 1. *Let* $\Gamma = PSL(2,\mathbb{Z})$ *be the modular group and* $\varepsilon > 0$ *arbitrarily small. There exists a set* A *of finite logarithmic measure such that*

$$\pi_\Gamma(x) = \int_0^x \frac{dt}{\log t} + O\left(x^{\frac{2}{3}} (\log\log x)^{\frac{1}{3}+\varepsilon}\right) \qquad (x \to \infty,\ x \notin A).$$

2. Preliminaries: the Iwaniec explicit formula and Gallagher's lemma

The projective special linear group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$ is the group of Möbius transformations $z \to \frac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{R}$, $ad-bc=1$, that acts on the upper half-plane $\mathcal{H} = \{z = x+iy : y > 0\}$ by homeomorphisms which preserve the hyperbolic distance. Discrete subgroups of $PSL(2,\mathbb{R})$ are called Fuchsian groups. If all non-identity elements in a Fuchsian group Γ have a trace larger than 2 (i.e., $|a+d| > 2$), then Γ is strictly hyperbolic. It is also said to be cocompact, since the quotient space $\Gamma \setminus \mathcal{H}$ can be identified in this case with a compact Riemann surface of a genus $g \geqslant 2$.

For every hyperbolic conjugacy class P there exists the primitive hyperbolic conjugacy class P_0 such that $P = {P_0}^k$ for some $k \in \mathbb{N}_0$. The norm $N(P)$ is the square root of the larger of two real eigenvalues of transformations in P. Obviously, $N(P) = N(P_0)^k$ (thus illustrating one of the reasons why $N(P_0)$ are called pseudo-primes).

In analogy to the prime number theorem situation, where the distribution of primes $\pi(x) = \sum_{p \leqslant x} 1$ is more conveniently studied through the behavior of the Chebyshev function $\psi(x) = \sum_{p^k \leqslant x} \log p$, one usually considers the function

$$\psi_\Gamma(x) = \sum_{N(P) \leqslant x} \log N(P_0) = \sum_{N(P_0)^k \leqslant x} \log N(P_0).$$

(or its integrated versions) while dealing with the prime geodesic theorem.

A prominent place among cocompact Fuchsian groups is taken by the modular group $PSL(2,\mathbb{Z})$. The modular group and the modular surface $PSL(2,\mathbb{Z})\setminus\mathcal{H}$ have important applications both in arithmetic and physics (see, e.g., [19] on a research side and [6] on an advanced textbook level).

For our purpose, we shall make use of two lemmas that played important role in [13] and [10]. The first one is the Iwaniec explicit formula [13] with an error term for the Chebyshev function ψ_Γ on $\Gamma = PSL(2,\mathbb{Z})$.

Lemma A. *For* $1 \leqslant T \leqslant \frac{x^{\frac{1}{2}}}{(\log x)^2}$, *one has*

$$\psi_\Gamma(x) = x + \sum_{|\gamma| \leqslant T} \frac{x^\rho}{\rho} + O\left(\frac{x}{T} (\log x)^2\right),$$

where $\rho = \frac{1}{2} + i\gamma$ *denote zeros of* Z_Γ.

The second one is Gallagher's lemma [9] that enabled him to reduce the error term in the prime number theorem under the Riemann hypothesis from $\psi(x) = x + O\left(x^{\frac{1}{2}}(\log x)^2\right)$ to $\psi(x) = x + O\left(x^{\frac{1}{2}}(\log\log x)^2\right)$ outside a set of finite logarithmic measure.

Lemma B. *Let A be a discrete subset of $\mathbb{R}$ and $\theta \in (0,1)$. For any sequence $c(\nu) \in \mathbb{C}$, $\nu \in A$, let the series*

$$S(u) = \sum_{\nu \in A} c(\nu)\, e^{2\pi i \nu u}$$

be absolutely convergent. Then

$$\int_{-U}^{U} |S(u)|^2\, du \leqslant \left(\frac{\pi\theta}{\sin \pi\theta}\right)^2 \int_{-\infty}^{+\infty} \left| \frac{U}{\theta} \sum_{t \leqslant \nu \leqslant t + \frac{\theta}{U}} c(\nu) \right|^2 dt.$$

3. Proof of Theorem 1

The goal is to find a proper bound for $\sum\limits_{|\gamma| \leqslant T} \frac{x^\rho}{\rho}$ in Lemma A. For $x \in \left[e^n, e^{n+1}\right)$, we have

$$\begin{aligned} \int_{e^n}^{e^{n+1}} \left| \sum_{|\gamma| \leqslant T} \frac{x^\rho}{\rho} \right|^2 dx &= \int_{e^n}^{e^{n+1}} x^2 \left| \sum_{|\gamma| \leqslant T} \frac{x^{i\gamma}}{\rho} \right|^2 \frac{dx}{x} \\ &\ll e^{2n} \int_{e^n}^{e^{n+1}} \left| \sum_{|\gamma| \leqslant T} \frac{x^{i\gamma}}{\rho} \right|^2 \frac{dx}{x}. \end{aligned} \tag{1}$$

Through the substitution $x = e^n \cdot e^{2\pi\left(u + \frac{1}{4\pi}\right)}$, the last integral is transformed into

$$2\pi \int_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}} \left| \sum_{|\gamma| \leqslant T} \frac{e^{\left(n+\frac{1}{2}\right)i\gamma}}{\rho} e^{2\pi i \gamma u} \right|^2 du.$$

Lemma B, with $\theta = U = \frac{1}{4\pi}$ and $c_\gamma = \frac{e^{\left(n+\frac{1}{2}\right)i\gamma}}{\rho}$ for $|\gamma| \leqslant T$, $c_\gamma = 0$ otherwise, implies

$$\int_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}} \left| \sum_{|\gamma| \leqslant T} \frac{e^{\left(n+\frac{1}{2}\right)i\gamma}}{\rho} e^{2\pi i \gamma u} \right|^2 du \leqslant \left(\frac{\frac{1}{4}}{\sin\frac{1}{4}}\right)^2 \int_{-\infty}^{+\infty} \left(\sum_{\substack{t < \gamma \leqslant t+1 \\ |\gamma| \leqslant T}} \frac{1}{|\rho|} \right)^2 dt. \tag{2}$$

Now, obviously $|c_{-\gamma}| = |c_\gamma|$. According to the Weyl law, $\sharp\{\gamma : t < |\gamma| \leqslant t+1\} = O(t)$. Thus, $\sum\limits_{t<|\gamma|\leqslant t+1} \frac{1}{|\rho|} = O(1)$, yielding

$$\int\limits_{-\infty}^{+\infty} \left(\sum_{\substack{t<\gamma\leqslant t+1 \\ |\gamma|\leqslant T}} \frac{1}{|\rho|} \right)^2 dt = O\left(\int\limits_0^T dt \right) = O(T). \tag{3}$$

Combining (1), (2) and (3), we get

$$\int\limits_{e^n}^{e^{n+1}} \left| \sum_{|\gamma|\leqslant T} \frac{x^\rho}{\rho} \right|^2 dx = O\left(e^{2n}T\right). \tag{4}$$

Let $A_n = \left\{ x \in \left[e^n, e^{n+1}\right) : \left| \sum\limits_{|\gamma|\leqslant T} \frac{x^\rho}{\rho} \right| > x^{\frac12} T^{\frac12} (\log x)^{\frac12} (\log\log x)^{\frac12+\frac32\varepsilon} \right\}$. Its logarithmic measure is controlled by

$$\begin{aligned} \mu^* A_n &= \int\limits_{A_n} \frac{dx}{x} = \int\limits_{A_n} xT(\log x)(\log\log x)^{1+3\varepsilon} \frac{dx}{x^2 T(\log x)(\log\log x)^{1+3\varepsilon}} \\ &\leqslant \frac{1}{e^{2n}Tn(\log n)^{1+3\varepsilon}} \int\limits_{e^n}^{e^{n+1}} \left| \sum_{|\gamma|\leqslant T} \frac{x^\rho}{\rho} \right|^2 dx \\ &\ll \frac{e^{2n}T}{e^{2n}Tn(\log n)^{1+3\varepsilon}} = \frac{1}{n(\log n)^{1+3\varepsilon}}. \end{aligned}$$

Thus,

$$\left| \sum_{|\gamma|\leqslant T} \frac{x^\rho}{\rho} \right| \leqslant x^{\frac12} T^{\frac12} (\log x)^{\frac12} (\log\log x)^{\frac12+\frac32\varepsilon} \tag{5}$$

outside a set $A = \cup A_n$ of finite logarithmic measure.

The optimal choice is $T \approx \frac{x^{\frac13}\log x}{(\log\log x)^{\frac13+\varepsilon}}$. Then, Lemma A and the relation (5) yield

$$\psi_\Gamma(x) = x + O\left(x^{\frac23} \log x (\log\log x)^{\frac13+\varepsilon}\right) \qquad (x\to\infty,\ x\notin A). \tag{6}$$

From (6), we obtain the assertion of Theorem 1 in a standard way, making use of the expressions

$$\pi_\Gamma(x) = \int_2^x \frac{1}{\log t} d\theta_\Gamma(t) \qquad \text{and} \qquad \psi_\Gamma(x) = \sum_{n=1}^{\infty} \theta_\Gamma\left(x^{\frac1n}\right),$$

where $\theta_\Gamma(x) = \sum\limits_{N(P_0)\leqslant x} \log N(P_0)$.

Remark. One should mention that G. Cherubini and J. Guerreiro have recently posted on arxiv.org a preprint [8], where Theorem 1.4 possibly gives the exponent $\frac{5}{8}+\varepsilon$ in the error term of the prime geodesic theorem for the modular group, outside a set of finite logarithmic measure. Their proof is more evolved and completely different from the one presented here.

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