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COMMENSURABILITY IN MORDELL-WEIL GROUPS OF ABELIAN VARIETIES AND TORI

Grzegorz Banaszak, Dorota Blinkiewicz

Abstract: We investigate local to global properties for commensurability in Mordell-Weil groups of abelian varieties and tori via reduction maps.

Keywords: commensurability, abelian variety, torus, reduction map, Mordell-Weil group.

1. Introduction

Local to global properties for detecting linear relations in Mordell-Weil groups of abelian varieties and tori have been investigated by numerous authors: [Sch](1975), [C-RS](1997), [B1](1998), [BGK1](2003), [La](2003), [We](2003), [Kh-P](2004), [BGK2](2005), [Bar](2006), [BG](2008), [GG](2009), [B2](2009), [JP](2010), [Pe](2010), [BK](2011), [J1](2013), [Rz](2015) and others. Commensurability questions in the Mordell-Weil groups have not yet been investigated in relation to reduction maps. In this paper we establish the relations between local to global detecting properties and local to global commensurability properties. We apply these results to Mordell-Weil groups of abelian varieties and tori. The structure of the paper is as follows. At the end of this introduction we define local to global commensurability properties. We also define notion of strong commensurability in abelian groups with finite torsion. Then we define local to global properties for strong commensurability. In section 2 we investigate relations between local to global commensurability properties and local to global detecting properties. In section 3 we give examples of classes of abelian varieties and tori where the local to global strong commensurability property holds. In both cases we show examples of classes of abelian varieties and tori where the criterion fails. As a corollary we obtain, in each case, four different Deligne 1-motives over a ring of integers, which become all equal to a torsion 1-motive, after base change and

2010 Mathematics Subject Classification: primary: 11G10; secondary: 14Kxx application of reduction map for almost all residue fields. In section 4 we give examples where one can check the strong commensurability in Mordell-Weil groups of abelian varieties and tori by finite number of reductions.

We use the following notation.

(1) F a number field,
(2) $\mathcal{O}_F$ the ring of integers in F,
(3) S finite set of primes in $\mathcal{O}_F$,
(4) $\mathcal{O}_{F,S}$ the ring of S-integers,
(5) v a prime ideal in $\mathcal{O}_F$,
(6) $k_v := \mathcal{O}_F/v$,
(7) $B(F)$ a finitely generated abelian group,
(8) $B_v(k_v)$ a finite group for each v.

We assume in this and the next section that for almost all v there is a homomorphism of groups (the reduction homomorphism):

$$r_v \,:\, B(F) \to B_v(k_v). \tag{1.1}$$

In the following we introduce the notion of strong commensurability. We also introduce the local to global commensurability and strong commensurability properties.

1.1. Commensurability and strong commensurability

Recall that in any group G two subgroups H, H' are called commensurable if:

$$[H \,:\, H \cap H'] < \infty \qquad \text{and} \qquad [H' \,:\, H \cap H'] < \infty. \tag{1.2}$$

Definition 1.1. Let B be an abelian group with finite torsion subgroup B_{tor}. Two subgroups Λ and Λ' are called *strongly commensurable* if:

$$\Lambda \subset \Lambda \cap \Lambda' + B_{\mathrm{tor}} \qquad \text{and} \qquad \Lambda' \subset \Lambda \cap \Lambda' + B_{\mathrm{tor}}. \tag{1.3}$$

Remark 1.2. It is clear that if Λ and Λ' are strongly commensurable then they are commensurable.

1.2. Local to global detecting properties

In the investigation of linear relations in Mordell-Weil groups of abelian varieties and tori the following two properties were studied.

Detecting Property: Let $P \in B(F)$ be any element and $\Lambda \subset B(F)$ be any subgroup. If $r_v(P) \in r_v(\Lambda)$, for almost all v, then $P \in \Lambda + B(F)_{\mathrm{tor}}$.

Weak Detecting Property: Let $P \in B(F)$ be any element and $\Lambda \subset B(F)$ be any subgroup. If $r_v(P) \in r_v(\Lambda)$, for almost all v, then there exists $n \in \mathbb{N}$ such that $nP \in \Lambda$.

Remark 1.3. Obviously Detecting Property implies Weak Detecting Property. On the other hand Weak Detecting Property does not imply Detecting Property. Namely, let A/F be an abelian variety over F such that there is a nontorsion element $P \in A(F)$. Let $n \in \mathbb{N}$, $n > 1$ and let l be a prime number coprime to n. Take $B(F) := A(F)$, $B_v(k_v) := A_v(k_v)_l$. Let $\Lambda = \mathbb{Z}nP$. Obviously $r_v(P) \in r_v(\Lambda)$ and $nP \in \Lambda$ but $P \notin \Lambda + B(F)_{\mathrm{tor}}$.

1.3. Local to global commensurability properties

Local to Global Commensurability Property: Let Λ, $\Lambda' \subset B(F)$ be any two subgroups. The subgroups Λ and Λ' are commensurable if and only if there exists a natural number c such that for almost all v:

$$[r_v(\Lambda) : r_v(\Lambda \cap \Lambda')] \leq c \qquad \text{and} \qquad [r_v(\Lambda') : r_v(\Lambda \cap \Lambda')] \leq c. \tag{1.4}$$

Local to Global Strong Commensurability Property: Let Λ, $\Lambda' \subset B(F)$ be any two subgroups. Subgroups Λ and Λ' are strongly commensurable if and only if for almost all v:

$$r_v(\Lambda) \subset r_v(\Lambda \cap \Lambda') + r_v(B(F)_{\mathrm{tor}}) \quad \text{and} \quad r_v(\Lambda') \subset r_v(\Lambda \cap \Lambda') + r_v(B(F)_{\mathrm{tor}}). \tag{1.5}$$

2. Remarks on commensurability

Proposition 2.1. *The Local to Global Strong Commensurability Property implies the Local to Global Commensurability Property.*

Proof. Assume the Local to Global Strong Commensurability Property. Let Λ, $\Lambda' \subset B(F)$ be any two subgroups. Assume that for Λ, Λ' for almost all v the conditions (1.4) hold for some $c \in \mathbb{N}$. Define two subgroups of $B(F)$

$$H := c!\,\Lambda + \Lambda \cap \Lambda' \qquad \text{and} \qquad H' := c!\,\Lambda' + \Lambda \cap \Lambda'.$$

Observe that $\Lambda \cap \Lambda' \subset H \cap H'$, $H \subset \Lambda$ and $H' \subset \Lambda'$. Hence $H \cap H' = \Lambda \cap \Lambda'$. Then by (1.4):

$$r_v(H) = r_v(c!\,\Lambda) + r_v(\Lambda \cap \Lambda') \subset r_v(\Lambda \cap \Lambda') \subset r_v(H \cap H') + r_v(B(F)_{\mathrm{tor}}),$$
$$r_v(H') = r_v(c!\,\Lambda') + r_v(\Lambda \cap \Lambda') \subset r_v(\Lambda \cap \Lambda') \subset r_v(H \cap H') + r_v(B(F)_{\mathrm{tor}}).$$

Hence the conditions (1.5) hold for H and H'. By the Local to Global Strong Commensurability Property:

$$c!\,\Lambda + \Lambda \cap \Lambda' = H \subset H \cap H' + B(F)_{\mathrm{tor}} = \Lambda \cap \Lambda' + B(F)_{\mathrm{tor}},$$
$$c!\,\Lambda' + \Lambda \cap \Lambda' = H' \subset H \cap H' + B(F)_{\mathrm{tor}} = \Lambda \cap \Lambda' + B(F)_{\mathrm{tor}}.$$

Hence $c!\,\Lambda \subset \Lambda \cap \Lambda' + B(F)_{\mathrm{tor}}$ and $c!\,\Lambda' \subset \Lambda \cap \Lambda' + B(F)_{\mathrm{tor}}$. It obviously implies that $[\Lambda :\, \Lambda \cap \Lambda'] < \infty$ and $[\Lambda' :\, \Lambda \cap \Lambda'] < \infty$. ■

Proposition 2.2. *The Weak Detecting Property is equivalent to the Local to Global Commensurability Property.*

Proof. Assume the Weak Detecting Property. Let $\Lambda,\, \Lambda' \subset B(F)$ be any two subgroups.

($\Rightarrow$) Let $\Lambda,\, \Lambda'$ be commensurable. The condition (1.4) obviously holds taking $c := \max\{[\Lambda : \Lambda \cap \Lambda'],\, [\Lambda' : \Lambda \cap \Lambda']\}$.

($\Leftarrow$) Assume that (1.4) holds for some $c \in \mathbb{N}$. Then:

$$r_v(c!\,\Lambda) \subset r_v(\Lambda \cap \Lambda'), \tag{2.1}$$

$$r_v(c!\,\Lambda') \subset r_v(\Lambda \cap \Lambda') \tag{2.2}$$

for almost all v. Let $P_1, \ldots, P_m \in c!\,\Lambda$ be a set of generators. By (2.1) $r_v(P_i) \in r_v(\Lambda \cap \Lambda')$ for almost all v for all $i = 1, \ldots, m$. By the Weak Detecting Property assumption there exists $k \in \mathbb{N}$ such that $kP_i \in \Lambda \cap \Lambda'$ for all $i = 1, \ldots, m$. Hence $kc!\,\Lambda \subset \Lambda \cap \Lambda' \subset \Lambda$. Since $B(F)$ is finitely generated then $\Lambda / kc!\,\Lambda$ is a finite abelian group hence $\Lambda / \Lambda \cap \Lambda'$ is a finite abelian group. In the same way we prove that $\Lambda' / \Lambda \cap \Lambda'$ is a finite abelian group.

Assume the Local to Global Commensurability Property. Let $P \in B(F)$ and let $\Lambda \subset B(F)$ satisfy $r_v(P) \in r_v(\Lambda)$, for almost all v. Define $\Lambda' := \mathbb{Z}P + \Lambda \subset B(F)$. Observe that $\Lambda \cap \Lambda' = \Lambda$. Moreover, $r_v(\Lambda) = r_v(\Lambda')$ for almost all v. Hence (1.4) holds with $c = 1$. Therefore, $[\Lambda' : \Lambda] < \infty$ because we assumed the Local to Global Commensurability Property. Hence there is $n \in \mathbb{N}$ such that $nP \in \Lambda$. ■

Proposition 2.3. *The Detecting Property is equivalent to the Local to Global Strong Commensurability Property.*

Proof. The proof is left as an exercise for the reader. ■

Remark 2.4. The Proposition 2.1 also follows by Propositions 2.2, 2.3 and Remark 1.3.

3. Commensurability in Mordell-Weil groups

In the following we present examples where Propositions 2.1, 2.2 and 2.3 can be applied.

Let A be an abelian variety over F and T be a torus over F. In this section we will assume that $B := A$ or T. Let F'/F be a finite extension such that

$$B \otimes_F F' = \prod_{i=1}^{t} B_i^{e_i}, \qquad \text{where} \quad \begin{cases} B_i = A_i, & \text{for} \quad B = A, \\ t = 1,\ B_1 = \mathbb{G}_m/F', & \text{for} \quad B = T, \end{cases}$$

with A_i/F' simple, pairwise nonisogenous abelian varieties. We will also assume that $\mathrm{End}_{F'}(B_i) = \mathrm{End}_{\bar{F}}(B_i)$. Let $\mathrm{End}_{F'}(B_i)^0 := \mathrm{End}_{F'}(B_i) \otimes \mathbb{Q}$. Let $B(F) := A(F)$ or $T(F)$, $B_v(k_v) := A_v(k_v)$ or $T_v(k_v)$ and $r_v : B(F) \to B_v(k_v)$ be the reduction map for each prime of good reduction.

Let $\mathcal{T}$ be a model of T for a finite set S of primes in $\mathcal{O}_F$ containing the primes of bad reduction. Then $T = \mathcal{T} \times_{\operatorname{spec} \mathcal{O}_{F,S}} \operatorname{spec} F$. For $B = T$, by $r_v: B(F) \to B_v(k_v)$ we understand the reduction map $r_v : \mathcal{T}(\mathcal{O}_{F,S}) \to T_v(k_v)$. In addition, in this case "a subgroup of $B(F) := T(F)$" means in this paper "a subgroup of $\mathcal{T}(\mathcal{O}_{F,S})$".

We would like to mention, that working with the reduction maps r_v, in the proofs of our results concerning abelian varieties and tori, we can assume without loss of generality that $F = F'$ cf. [BK, pp. 317-318]. In this case $\mathcal{T} := (\mathbb{G}_m/\operatorname{spec} \mathcal{O}_{F,S})^{e_1}$, $\mathcal{T}(\mathcal{O}_{F,S}) = (\mathcal{O}_{F,S}^{\times})^{e_1}$ and $r_v : (\mathcal{O}_{F,S}^{\times})^{e_1} \to (k_v^{\times})^{e_1}$.

Corollary 3.1. *Let $B := A$ or T as above. Assume that e_i is such that*

$$\dim_{\operatorname{End}_{F'}(B_i)^0} H_1(B_i(\mathbb{C}), \mathbb{Q}) \geq e_i \tag{3.1}$$

for each $1 \leq i \leq t$. Then the Local to Global Strong Commensurability Property holds in $B(F)$ with respect to reduction maps.

Proof. The corollary follows by Proposition 2.3 because, under assumption (3.1), the Detecting Property holds for abelian varieties by [BK, Theorem 4.1] and for tori by the result of A. Schinzel [Sch, Theorem 2, p. 398]. ■

The assumption (3.1) in Corollary 3.1 is important as shown in the following counterexamples below based on [BK, pp. 330-332] and [Sch, p. 419].

3.1. Counterexample for tori

Consider the following group scheme $\mathbb{G}_m^2 := \mathbb{G}_m \times_{\operatorname{spec} \mathcal{O}_{F,S}} \mathbb{G}_m$. We observe that $\dim_{\mathbb{Q}} H_1(\mathbb{G}_m(\mathbb{C}), \mathbb{Q}) = 1$. Hence the inequality (3.1) does not hold for $\mathbb{G}_m^2$. In this case we might expect counterexamples to strong commensurability. Indeed, take $F = \mathbb{Q}$ and $S = \{(2), (3), (5)\}$. Hence $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]$. For each $p \neq 2, 3, 5$ consider the reduction map $r_p \times r_p : \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]^{\times} \times \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]^{\times} \to \mathbb{F}_p^{\times} \times \mathbb{F}_p^{\times}$. By abuse of notation we will write r_p instead of $r_p \times r_p$. Consider the following elements which are linearly independent in $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]^{\times} \times \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]^{\times}$:

$$\lambda := \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \lambda_1 := \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda_2 := \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \lambda_3 := \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad \lambda_4 := \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad \lambda' := \begin{bmatrix} 25 \\ 1 \end{bmatrix}.$$

Take the following two subgroups of $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]^{\times} \times \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]^{\times}$:

$$\Lambda := \lambda^{\mathbb{Z}} \cdot \lambda_1^{\mathbb{Z}} \cdot \lambda_2^{\mathbb{Z}} \cdot \lambda_3^{\mathbb{Z}} \cdot \lambda_4^{\mathbb{Z}},$$

$$\Lambda' := \lambda_1^{\mathbb{Z}} \cdot \lambda_2^{\mathbb{Z}} \cdot \lambda_3^{\mathbb{Z}} \cdot \lambda_4^{\mathbb{Z}} \cdot \lambda'^{\mathbb{Z}}.$$

Then

$$\Lambda \cap \Lambda' = \lambda_1^{\mathbb{Z}} \cdot \lambda_2^{\mathbb{Z}} \cdot \lambda_3^{\mathbb{Z}} \cdot \lambda_4^{\mathbb{Z}}.$$

Extending the argument of A. Schinzel [Sch, pp. 419-420] we obtain that

$$r_p(\Lambda) = r_p(\Lambda \cap \Lambda') = r_p(\Lambda') \tag{3.2}$$

for $p \neq 2,\ 3,\ 5$. Indeed, let $\overline{\lambda} := r_p(\lambda)$, $\overline{\lambda'} := r_p(\lambda')$ and $\overline{\lambda_i} := r_p(\lambda_i)$ for $i = 1, 2, 3, 4$. The equation

$$\overline{\lambda} = \overline{\lambda_1}^{R_1}\, \overline{\lambda_2}^{R_2}\, \overline{\lambda_3}^{R_3}\, \overline{\lambda_4}^{R_4}$$

is equivalent to the system of equations:

$$\begin{aligned} 1 &\equiv 2^{R_1}\, 3^{R_2}\, 5^{R_3} \mod p,\\ 2^2 &\equiv 2^{R_2}\, 3^{R_3}\, 5^{R_4} \mod p. \end{aligned} \tag{3.3}$$

Let g be a generator of $(\mathbb{Z}/p)^\times$ i.e. $(\mathbb{Z}/p)^\times = < g > \cong \mathbb{Z}/(p-1)$. For $p > 5$ the system of equations (3.3) is equivalent to the system:

$$\begin{aligned} 0 &\equiv c_1 R_1 + c_2 R_2 + c_3 R_3 \mod (p-1),\\ 2c_1 &\equiv c_1 R_2 + c_2 R_3 + c_3 R_4 \mod (p-1), \end{aligned} \tag{3.4}$$

where $2 = g^{c_1}$, $3 = g^{c_2}$, $5 = g^{c_3}$ for some c_1, c_2, c_3.

If $c_1 \equiv 0 \mod (p-1)$ or $c_2 \equiv 0 \mod (p-1)$ then the system of congruences (3.4) has the solution $(R_1,\ R_2,\ R_3,\ R_4) = (0,\ 0,\ 0,\ 0)$, $(0,\ 2,\ 0,\ 0)$, respectively. If $c_3 \equiv 0 \mod (p-1)$, then we obtain the following congruences:

$$\begin{aligned} 0 &\equiv c_1 R_1 + c_2 R_2 \mod (p-1)\\ 2c_1 &\equiv c_1 R_2 + c_2 R_3 \mod (p-1). \end{aligned} \tag{3.5}$$

Observe that (3.5) and [Sch, (70), p. 419] are identical, so we obtain solutions R_1, R_2, R_3 in the same way as A. Schinzel and we can take arbitrary $R_4 \in \mathbb{Z}$.

So assume that $c_1 \not\equiv 0 \mod (p-1)$, $c_2 \not\equiv 0 \mod (p-1)$ and $c_3 \not\equiv 0 \mod (p-1)$. Let $D := \gcd(c_1,\ c_2,\ c_3)$. Observe that

$$\gcd\left(\frac{c_1^2}{D},\ \frac{c_1 c_2}{D},\ c_3\right) \mid c_1.$$

Thus the equation $\frac{c_1^2}{D} R + \frac{c_1 c_2}{D} R' + c_3 R_4 = 2c_1$ has an integer solution R, R', R_4. Putting

$$R_1 := -\frac{c_2}{D} R - \frac{c_3}{D} R', \qquad R_2 := \frac{c_1}{D} R, \qquad R_3 := \frac{c_1}{D} R'$$

we find that R_1, R_2, R_3, R_4 satisfy the system of congruences (3.4). So we have $\overline{\lambda} \in r_p(\Lambda \cap \Lambda')$. Now we must show that $\overline{\lambda'} \in r_p(\Lambda \cap \Lambda')$. The equation

$$\overline{\lambda'} = \overline{\lambda_1}^{R_4}\, \overline{\lambda_2}^{R_3}\, \overline{\lambda_3}^{R_2}\, \overline{\lambda_4}^{R_1}$$

is equivalent to the system of equations:

$$\begin{aligned} 1 &\equiv 5^{R_1}\, 3^{R_2}\, 2^{R_3} \mod p,\\ 5^2 &\equiv 5^{R_2}\, 3^{R_3}\, 2^{R_4} \mod p. \end{aligned} \tag{3.6}$$

For $p > 5$ it is equivalent to the system:

$$\begin{aligned} 0 &\equiv c_1 R_1 + c_2 R_2 + c_3 R_3 \mod (p-1), \\ 2c_1 &\equiv c_1 R_2 + c_2 R_3 + c_3 R_4 \mod (p-1), \end{aligned} \tag{3.7}$$

where $5 = g^{c_1}$, $3 = g^{c_2}$, $2 = g^{c_3}$ in $(\mathbb{Z}/p)^\times = \langle g \rangle \cong \mathbb{Z}/(p-1)$. The systems of congruences (3.7) and (3.4) are identical. Hence using the same calculation we obtain the solution R_1, R_2, R_3, R_4.

Hence (1.4) holds with $c = 1$. But Λ and Λ' are not commensurable because they are free $\mathbb{Z}$-modules of rank 5 while their intersection $\Lambda \cap \Lambda'$ is a free $\mathbb{Z}$-module of rank 4.

Remark 3.2. Similarly as in example [Sch, pp. 419-420] we can prove more. Namely consider the reduction map $r_p \times r_p : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/p \times \mathbb{Z}/p$. We have already proved equality (3.2) for $p > 5$. Observe that (3.2) also holds for $p = 2, 3, 5$. Namely the equations (3.3) and (3.6) have the following solutions:

	(3.3) (R_1, R_2, R_3, R_4)	(3.6) (R_1, R_2, R_3, R_4)
$p = 2$	(0, 2, 0, 0)	$(1, 1, 0, 0)$
$p = 3$	(0, 0, 0, 0)	$(0, 0, 0, 2)$
$p = 5$	(2, 2, 0, 0)	$(0, 4, 4, 0)$

Remark 3.3. Let $F = \mathbb{Q}$ as above, $T := \mathbb{G}_m^2$ and $\mathcal{T} := \mathbb{G}_m \times_{\operatorname{spec} \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]} \mathbb{G}_m$. Consider the following 1-motives over $\operatorname{spec} \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]$: $[\Lambda \to \mathcal{T}]$, $[\Lambda' \to \mathcal{T}]$, $[\Lambda \cap \Lambda' \to \mathcal{T}]$ and $[\Lambda \cdot \Lambda' \to \mathcal{T}]$ in the sense of P. Deligne. Changing the base to $\operatorname{spec} \mathbb{F}_p$ and taking the images of the subgroups Λ, Λ', $\Lambda \cap \Lambda'$, $\Lambda \cdot \Lambda'$ in $T_p(\mathbb{F}_p)$ via r_p (for $p \neq 2, 3, 5$), we obtain torsion 1-motives $[r_p(\Lambda) \to T_p]$, $[r_p(\Lambda') \to T_p]$, $[r_p(\Lambda \cap \Lambda') \to T_p]$ and $[r_p(\Lambda \cdot \Lambda') \to T_p]$. The above example shows that these four torsion 1-motives are all equal, for each $p \neq 2, 3, 5$ (for the def. of torsion 1-motive see [B-VRS] and [J2]).

Remark 3.4. A. Schinzel [Sch, pp. 419-420] considered four linearly independent points:

$$\gamma := \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \qquad \gamma_1 := \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \gamma_2 := \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \gamma_3 := \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

in $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}]^\times \times \mathbb{Z}[\frac{1}{2}, \frac{1}{3}]^\times$. This leads to two lattices $\Gamma_0 \subset \Gamma$ as follows:

$$\begin{aligned} \Gamma &:= \gamma^{\mathbb{Z}} \cdot \gamma_1^{\mathbb{Z}} \cdot \gamma_2^{\mathbb{Z}} \cdot \gamma_3^{\mathbb{Z}}, \\ \Gamma_0 &:= \gamma_1^{\mathbb{Z}} \cdot \gamma_2^{\mathbb{Z}} \cdot \gamma_3^{\mathbb{Z}}. \end{aligned}$$

A. Schinzel proved that $r_p(\Gamma) = r_p(\Gamma_0)$ for $p \neq 2, 3$. He also checked that the equality holds for $p = 2, 3$. Our extension of Schinzel's example above gives lattices Λ, Λ' such that $\Lambda \not\subset \Lambda'$ and $\Lambda' \not\subset \Lambda$.

Consider $\mathcal{T} = \mathbb{G}_m \times_{\operatorname{spec} \mathbb{Z}[\frac{1}{2},\frac{1}{3}]} \mathbb{G}_m$. We have two 1-motives $[\Gamma \to \mathcal{T}]$ and $[\Gamma_0 \to \mathcal{T}]$ which give two equal torsion 1-motives $[r_p(\Gamma) \to T_p] = [r_p(\Gamma_0) \to T_p]$ for each $p > 3$. The 1-motive $[\Gamma_0 \to \mathcal{T}]$ was called by P. Jossen [J2] the Schinzel's 1-motive.

In a similar way we can consider the case of abelian varieties.

3.2. Counterexample for abelian varieties

Let E_d be the elliptic curve over $\mathbb{Q}$ given by the equation $y^2 = x^3 - d^2x$. It has CM by $\mathbb{Z}[i]$. The rank of $E_d(\mathbb{Q})$ can reach 6 see [RS, Table 2, p. 464]. Assume that the rank of $E_d(\mathbb{Q})$ is at least 3. Then $\operatorname{rank}_{\mathbb{Z}} E_d(\mathbb{Q}(i)) = 2 \operatorname{rank}_{\mathbb{Z}}(E_d(\mathbb{Q}))$. So the $\operatorname{rank}_{\mathbb{Z}[i]} E_d(\mathbb{Q}(i))$ is at least 3. Define the abelian surface $A_d := E_d \times E_d = E_d^2$ over $\mathbb{Q}(i)$. Observe that A_d does not satisfy the condition (3.1) since $\dim_{\mathbb{Q}(i)} H_1(E_d(\mathbb{C}), \mathbb{Q}) = 1$.

Let $Q_1, Q_2, Q_3 \in E_d(\mathbb{Q}(i))$ be independent over $\mathbb{Z}[i]$. Consider the following points in $A_d(\mathbb{Q}(i))$:

$$P := \begin{bmatrix} 0 \\ Q_1 \end{bmatrix},\ P_1 := \begin{bmatrix} Q_1 \\ 0 \end{bmatrix},\ P_2 := \begin{bmatrix} Q_2 \\ Q_1 \end{bmatrix},\ P_3 := \begin{bmatrix} Q_3 \\ Q_2 \end{bmatrix},\ P_4 := \begin{bmatrix} 0 \\ Q_3 \end{bmatrix},\ P' := \begin{bmatrix} Q_3 \\ 0 \end{bmatrix}.$$

Consider the following two free $\mathbb{Z}[i]$-submodules of $A_d(\mathbb{Q}(i))$:

$$\Lambda := \mathbb{Z}[i]P + \mathbb{Z}[i]P_1 + \mathbb{Z}[i]P_2 + \mathbb{Z}[i]P_3 + \mathbb{Z}[i]P_4,$$

$$\Lambda' := \mathbb{Z}[i]P_1 + \mathbb{Z}[i]P_2 + \mathbb{Z}[i]P_3 + \mathbb{Z}[i]P_4 + \mathbb{Z}[i]P'.$$

Then

$$\Lambda \cap \Lambda' = \mathbb{Z}[i]P_1 + \mathbb{Z}[i]P_2 + \mathbb{Z}[i]P_3 + \mathbb{Z}[i]P_4.$$

In the same way as in the case of tori above and using [BK, Proposition 5.6] we prove that $r_v(\Lambda) = r_v(\Lambda \cap \Lambda') = r_v(\Lambda')$ for all primes $v \nmid 2d \in \mathbb{Z}[i]$. Hence property (1.4) holds with $c = 1$. But Λ and Λ' are not commensurable because they are free $\mathbb{Z}[i]$-modules of rank 5 and their intersection $\Lambda \cap \Lambda'$ is a free $\mathbb{Z}[i]$-module of rank 4.

Remark 3.5. Let S be the set of primes of bad reduction for A. Let $\mathcal{A}$ be the Néron model over $\mathcal{O}_{F,S}$. Consider the following 1-motives over $\operatorname{spec} \mathcal{O}_{F,S}$: $[\Lambda \to \mathcal{A}]$, $[\Lambda' \to \mathcal{A}]$, $[\Lambda \cap \Lambda' \to \mathcal{A}]$ and $[\Lambda + \Lambda' \to \mathcal{A}]$, in the sense of P. Deligne [Del]. Changing the base to $\operatorname{spec} k_v$ and taking the images of the subgroups Λ, Λ', $\Lambda \cap \Lambda'$, $\Lambda + \Lambda'$ in $A_v(k_v)$ via the reduction maps r_v (for $v \notin S$), we obtain torsion 1-motives $[r_v(\Lambda) \to A_v]$, $[r_v(\Lambda') \to A_v]$, $[r_v(\Lambda \cap \Lambda') \to A_v]$ and $[r_v(\Lambda + \Lambda') \to A_v]$. The above example shows that for each $v \notin S$ these four torsion 1-motives are all equal.

4. Commensurability and finite number of reductions

4.1. Abelian variety case

Let $B = A$ be as in Corollary 3.1. The first author and P. Krasoń proved [BK, Theorem 6.4] that for such an abelian variety we can use only a finite set $S_{P,\Lambda}^{\mathrm{fin}}$ of primes $v \in \mathcal{O}_F$ to check whether $P \in \Lambda + A(F)_{\mathrm{tor}}$. This set depends on A, P, Λ and a choice of a $\mathbb{Z}$-basis of $A(F)/A(F)_{\mathrm{tor}}$. The proof of [BK, Theorem 6.4] used the effective Chebotarev theorem [LO] and the height pairing that is a symmetric, positive definite bilinear form over $\mathbb{R}$ (see [HS], [Sil]). In addition $S^{\mathrm{fin}}_{P,\Lambda}$ can be constructed in an effective way when one can choose a $\mathbb{Z}$-basis of $A(F)/A(F)_{\mathrm{tor}}$ that is constructed in an effective way (cf. [BK, Remark 6.5]). If the Tate-Shafarevich group of A is finite then an effective construction of a $\mathbb{Z}$-basis of $A(F)/A(F)_{\mathrm{tor}}$ is possible.

Remark 4.1. Let $B = A$ be as in Corollary 3.1. Let Λ, $\Lambda' \subset A(F)$ be two subgroups. Let $P_1, \ldots, P_r$ and $P'_1, \ldots, P'_s$ be generators of Λ and Λ' respectively. By theorem [BK, Theorem 6.4] there exist finite sets $S^{\mathrm{fin}}_{P_i,\Lambda\cap\Lambda'}$ and $S^{\mathrm{fin}}_{P'_j,\Lambda\cap\Lambda'}$ for each $i = 1, \ldots, r$ and $j = 1, \ldots, s$ such that the following properties hold:

$$\begin{aligned}&\text{if} \quad r_v(P_i) \in r_v(\Lambda \cap \Lambda') \quad \text{for all } v \in S^{\mathrm{fin}}_{P_i,\Lambda\cap\Lambda'} \quad \text{then} \quad P_i \in \Lambda \cap \Lambda' + A(F)_{\mathrm{tor}},\\ &\text{if} \quad r_v(P'_j) \in r_v(\Lambda \cap \Lambda') \quad \text{for all } v \in S^{\mathrm{fin}}_{P'_j,\Lambda\cap\Lambda'} \quad \text{then} \quad P'_j \in \Lambda \cap \Lambda' + A(F)_{\mathrm{tor}}.\end{aligned} \tag{4.1}$$

Let

$$S^{\mathrm{fin}}_{\Lambda,\Lambda'} := \bigcup_{i=1}^{r} S^{\mathrm{fin}}_{P_i,\Lambda\cap\Lambda'} \cup \bigcup_{j=1}^{s} S^{\mathrm{fin}}_{P'_j,\Lambda\cap\Lambda'}.$$

Then by (4.1) we obtain the following criterion for strong commensurability in terms of a finite number of reductions.

Corollary 4.2. *Let Λ, $\Lambda' \subset A(F)$ be two subgroups. Assume that*

$$r_v(\Lambda) \subset r_v(\Lambda \cap \Lambda') \quad \textit{and} \quad r_v(\Lambda') \subset r_v(\Lambda \cap \Lambda') \qquad \textit{for all } \ v \in S^{\mathrm{fin}}_{\Lambda,\Lambda'}.$$

Then Λ and Λ' are strongly commensurable.

4.2. 1-dimensional torus case

In this section we will show that an analogue of Corollary 4.2 holds also for S-units. Let S be a finite set of places in $\mathcal{O}_F$ containing the Archimedean ones. Let $U(S) := \mathcal{O}^{\times}_{F,S}/(\mathcal{O}^{\times}_{F,S})_{\mathrm{tor}}$. There is a bilinear regulator pairing:

$$\begin{aligned}\langle\, .\, ,\, .\,\rangle &: U(S) \times U(S) \to \mathbb{R},\\ \langle u,\, u'\rangle &:= \sum_{v} \log |u|_v \, \log |u'|_v,\end{aligned}$$

where $\sum_v$ means that summation is over set S with one Archimedean place removed. The pairing $\langle\, .\, ,\, .\,\rangle$ is symmetric and semi-positive definite. It is also non-degenerate because the regulator is non-zero (cf. [Bart, pp. 219-221] for different approach to regulator pairing). Hence $\langle\, .\, ,\, .\,\rangle$ is positive definite.

Let $(u_1, \ldots, u_m)$ be a basis of $U(S)$. Let γ, γ' be any elements of $U(S) \otimes \mathbb{R}$. Then γ and γ' can be uniquely written as $\gamma = \sum_{i=1}^{m} u_i \otimes c_i$ and $\gamma' = \sum_{j=1}^{m} u_j \otimes c'_j$

(using additive notation for the group $U(S)$). Consider extension of coefficients to $\mathbb{R}$ as follows:

$$\langle\,.\,,\,.\,\rangle \otimes \mathbb{R} : U(S) \otimes \mathbb{R} \,\times\, U(S) \otimes \mathbb{R} \longrightarrow \mathbb{R},$$
$$\langle \gamma, \gamma' \rangle := \sum_{i,j} c_i c_j' \sum_v \log |u_i|_v \log |u_j|_v.$$

This bilinear pairing is symmetric. For $\gamma = \sum_{i=1}^m u_i \otimes c_i \in U(S) \otimes \mathbb{R}$ we have

$$\langle \gamma, \gamma \rangle = \sum_v \left(\sum_{i,j} c_i \log |u_i|_v \, c_j \log |u_j|_v \right) = \sum_v \left(\sum_i c_i \log |u_i|_v \right)^2 \geq 0.$$

Hence $\langle\,.\,,\,.\,\rangle \otimes \mathbb{R}$ is semi-positive definite. It is also non-degenerate since its matrix in the basis $(u_1 \otimes 1, \ldots, u_m \otimes 1)$ is the same as the matrix of $\langle\,.\,,\,.\,\rangle$ in the basis $(u_1, \ldots, u_m)$. Hence $\langle\,.\,,\,.\,\rangle \otimes \mathbb{R}$ is positive definite.

Recall that for 1-dimensional tori the Detecting Property holds by [Sch, Theorem 2, p. 398] without torsion ambiguity. By [BK, Remark 5.1], using the effective Chebotarev theorem, the bilinear pairing $\langle\,.\,,\,.\,\rangle \otimes \mathbb{R}$ and the methods of the proof of [BK, Theorem 6.4] one obtains the analogue of [BK, Theorem 6.4] for 1-dimensional tori.

Theorem 4.3. *Let $\lambda \in \mathcal{O}_{F,S}^{\times}$ and Λ be a subgroup of $\mathcal{O}_{F,S}^{\times}$. There is a finite set $S_{\lambda,\Lambda}^{\text{fin}}$ of primes v of $\mathcal{O}_{F,S}$ depending on λ, Λ and the basis $u_1, \ldots, u_m$ of $U(S)$, such that the following property holds:*

$$\textit{if} \quad r_v(\lambda) \in r_v(\Lambda) \quad \textit{for all } v \in S_{\lambda,\Lambda}^{\text{fin}} \quad \textit{then} \quad \lambda \in \Lambda \cdot (\mathcal{O}_{F,S}^{\times})_{\text{tor}}.$$

Remark 4.4. The set $S_{\lambda,\Lambda}^{\text{fin}}$ can be constructed effectively because the generators of $U(S)$ can be constructed effectively (cf. [Le, p. 234]).

Remark 4.5. Let Λ, $\Lambda' \subset \mathcal{O}_{F,S}^{\times}$ be two subgroups. Let $\lambda_1, \ldots, \lambda_r$ and $\lambda_1', \ldots, \lambda_s'$ be generators of Λ and Λ' respectively. In the same way as in Remark 4.1, using Theorem 4.3, we obtain a finite set $S_{\Lambda,\Lambda'}^{\text{fin}}$ such that the following result holds.

Corollary 4.6. *Let Λ, $\Lambda' \subset \mathcal{O}_{F,S}^{\times}$ be two subgroups. Assume that*

$$r_v(\Lambda) \subset r_v(\Lambda \cap \Lambda') \quad \textit{and} \quad r_v(\Lambda') \subset r_v(\Lambda \cap \Lambda') \qquad \textit{for all } v \in S_{\Lambda,\Lambda'}^{\text{fin}}.$$

Then Λ and Λ' are strongly commensurable.

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Address: Grzegorz Banaszak and Dorota Blinkiewicz: Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland.

E-mail: banaszak@amu.edu.pl, dorota.blinkiewicz@amu.edu.pl

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