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TRACTABILITY OF $\mathbb{L}_2$-APPROXIMATION IN HYBRID FUNCTION SPACES

Peter Kritzer, Helene Laimer, Friedrich Pillichshammer

Abstract: We consider multivariate $\mathbb{L}_2$-approximation in reproducing kernel Hilbert spaces which are tensor products of weighted Walsh spaces and weighted Korobov spaces. We study the minimal worst-case error $e^{\mathbb{L}_2-\mathrm{app},\Lambda}(N,d)$ of all algorithms that use N information evaluations from the class Λ in the d-dimensional case. The two classes Λ considered in this paper are the class Λ^{all} consisting of all linear functionals and the class Λ^{std} consisting only of function evaluations.

The focus lies on the dependence of $e^{\mathbb{L}_2-\mathrm{app},\Lambda}(N,d)$ on the dimension d. The main results are conditions for weak, polynomial, and strong polynomial tractability.

Keywords: multivariate approximation, Walsh space, Korobov space, hybrid function space.

1. Introduction

We consider $\mathbb{L}_2$-approximation of functions in certain reproducing kernel Hilbert spaces $\mathcal{H}(K)$, which are embedded into $\mathbb{L}_2([0,1]^d)$, where K denotes the reproducing kernel. To be more precise, we approximate the embedding operator

$$\mathrm{EMB}_d : \mathcal{H}(K) \to \mathbb{L}_2([0,1]^d), \quad \mathrm{EMB}_d(f) = f,$$

and measure the approximation error in the $\mathbb{L}_2$-norm. Since $\mathcal{H}(K)$ is a reproducing kernel Hilbert space it is known (cf. [16, 20]) that there is no loss of generality when we restrict ourselves to linear approximation algorithms of the form $A_{N,d}(f) = \sum_{k=1}^N a_k L_k(f)$ with coefficients $a_k \in \mathbb{L}_2([0,1]^d)$ and continuous linear functionals L_k on $\mathcal{H}(K)$ from a permissible class of information Λ. Here N is the number of information evaluations.

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We study the problem in the so-called worst-case setting, i.e., we measure the approximation error of an algorithm $A_{N,d}$ by means of the worst-case error,

$$e^{\mathbb{L}_2-\mathrm{app}}(A_{N,d}) = \sup_{\substack{f\in\mathcal{H}(K)\\ \|f\|_{\mathcal{H}(K)}\le 1}} \|\mathrm{EMB}_d(f) - A_{N,d}(f)\|_{\mathbb{L}_2([0,1]^d)}.$$

The N^{th} minimal worst-case error is given by

$$e^{\mathbb{L}_2-\mathrm{app},\Lambda}(N,d) = \inf_{A_{N,d}} e^{\mathbb{L}_2-\mathrm{app}}(A_{N,d}),$$

where the infimum is extended over all linear algorithms $A_{N,d}$ using N information evaluations from the class Λ. We are particularly interested in the dependence of the N^{th} minimal worst-case error on the dimension d. To study this dependence systematically we consider the information complexity $N^{\mathbb{L}_2-\mathrm{app},\Lambda}(\varepsilon,d)$, which is the minimal number N for which there exists an algorithm using N information evaluations from the class $\Lambda \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}$ with an error of at most ε.

We would like to avoid cases where the information complexity $N^{\mathbb{L}_2-\mathrm{app},\Lambda}(\varepsilon,d)$ grows exponentially or even faster with the dimension d or with ε^{-1}. To quantify the behavior of the information complexity we use different notions of tractability, namely strong polynomial tractability, polynomial tractability, and weak tractability (we refer to Section 3 for the precise definitions).

The current state of the art of tractability theory is summarized in the three volumes of the book of Novak and Woźniakowski [16, 17, 18] which we refer to for extensive information on this subject and further references.

In previous papers, several authors have studied similar approximation problems in various reproducing kernel Hilbert spaces, see, e.g., [2, 3, 4, 12, 15, 22]. These investigations have in common that the reproducing kernel Hilbert spaces considered are tensor products of one-dimensional spaces whose kernels are all of the same type (but maybe equipped with different weights). In the current paper we consider the case where the reproducing kernel is a product of kernels of different type. We call such spaces *hybrid spaces*. Some results on tractability in general hybrid spaces can be found in the literature. For example, in [17] multivariate integration is studied for arbitrary reproducing kernels K_d without relation to K_{d+1}. Here we consider as a special instance the tensor product of Walsh and Korobov spaces. The problem of numerical integration in such spaces was recently considered in [11]. The study of a hybrid of Korobov and Walsh spaces could be important in view of functions which are periodic with respect to some of the components and, for example, piece-wise constant with respect to the remaining components. Moreover, it has been pointed out by several scientists (see, e.g., [10, 13]) that hybrid problems may be relevant for certain applications. Indeed, communication with the authors of [10] and [13] have motivated our idea for considering function spaces where we may have very different properties of the elements with respect to different components, as for example regarding smoothness.

From the analytical point of view, it is very challenging to deal with hybrid spaces. The reason for this is the rather complex interplay between the different analytic and algebraic structures of the kernel functions. In the present study we are concerned with Fourier analysis carried out simultaneously with respect to the Walsh and the trigonometric function systems. The problem is also closely related to the study of hybrid point sets which received much attention in recent years (see, for example, [8, 9]). Hence we also have considerable theoretical interest in studying this problem.

The paper is organized as follows. In Section 2 we introduce the Hilbert space under consideration. In Section 3 we specify the problem setting and state our main result. The proofs are presented in Section 4.

2. The hybrid function space

We study a specific reproducing kernel Hilbert space, namely the tensor product of a Korobov space and a Walsh space, that was introduced in [11]. See [1] for general information about reproducing kernel Hilbert spaces.

Fix a prime number b and let $\mathtt{i} = \sqrt{-1}$. For $k \in \mathbb{N}_0$ with b-adic expansion $k = \kappa_a b^a + \cdots + \kappa_1 b + \kappa_0$ with $\kappa_j \in \{0, \ldots, b-1\}$ we define the k^{th} Walsh function $\mathrm{wal}_k : [0,1) \to \mathbb{C}$ by

$$\mathrm{wal}_k(x) = \exp\left(2\pi\mathtt{i}\frac{\xi_1\kappa_0 + \cdots + \xi_{a+1}\kappa_a}{b}\right),$$

for $x \in [0,1)$ with b-adic expansion $x = \frac{\xi_1}{b} + \frac{\xi_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the ξ_i are different from $b-1$). Note that $a = \lfloor \log_b k \rfloor$.

For $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$ the $\boldsymbol{k}^{\text{th}}$ s-variate Walsh function $\mathrm{wal}_{\boldsymbol{k}} : [0,1)^s \to \mathbb{C}$ is given by $\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^s \mathrm{wal}_{k_j}(x_j)$.

Further, for $\boldsymbol{l} \in \mathbb{Z}^t$ we define the t-variate $\boldsymbol{l}^{\text{th}}$ trigonometric function $\mathrm{e}_{\boldsymbol{l}} \colon [0,1)^t \to \mathbb{C}$ as

$$\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) = \exp(2\pi\mathtt{i}\boldsymbol{l}\cdot\boldsymbol{y}),$$

where $\cdot$ denotes the usual Euclidean inner product.

Let now $s, t \in \mathbb{N}$, $\alpha, \beta > 1$ and let $\boldsymbol{\gamma}^{(1)}, \boldsymbol{\gamma}^{(2)}$ be two non-increasing sequences $\boldsymbol{\gamma}^{(i)} = (\gamma_j^{(i)})_{j\ge1}$ for $i \in \{1,2\}$, where $0 < \gamma_j^{(i)} \le 1$. We define two functions $\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}$ and $r_{\beta,\boldsymbol{\gamma}^{(2)}}$ as follows: For $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{l} = (l_1, \ldots, l_t) \in \mathbb{Z}^t$ let

$$\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) = \prod_{j=1}^s \rho_{\alpha,\gamma_j^{(1)}}(k_j) \qquad \text{and} \qquad r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) = \prod_{j=1}^t r_{\beta,\gamma_j^{(2)}}(l_j),$$

where

$$\rho_{\alpha,\gamma_j^{(1)}}(k_j) = \begin{cases} 1 & \text{if } k_j = 0, \\ \gamma_j^{(1)} b^{-\alpha\lfloor \log_b(k_j)\rfloor} & \text{if } k_j \ne 0, \end{cases}$$

and

$$r_{\beta,\gamma_j^{(2)}}(l_j) = \begin{cases} 1 & \text{if } l_j = 0, \\ \gamma_j^{(2)} |l_j|^{-\beta} & \text{if } l_j \neq 0. \end{cases}$$

With the help of these functions one can define so-called Walsh spaces [5, 7] and Korobov spaces [6, 14, 17].

Here we define a hybrid function space as the tensor product of the Walsh and Korobov spaces. The hybrid space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$, where $\boldsymbol{\gamma} = (\boldsymbol{\gamma}^{(1)}, \boldsymbol{\gamma}^{(2)})$, is the reproducing kernel Hilbert space with kernel function given by $K_{s,t,\alpha,\beta,\boldsymbol{\gamma}} : [0,1)^{s+t} \times [0,1)^{s+t} \to \mathbb{C}$,

$$\begin{aligned} &K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}((\boldsymbol{x},\boldsymbol{y}),(\boldsymbol{x}',\boldsymbol{y}')) \\ &\qquad = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \sum_{\boldsymbol{l}\in\mathbb{Z}^t} \rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}')} \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) \overline{\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}')} \end{aligned}$$

and inner product

$$\langle f, g \rangle_{s,t,\alpha,\beta,\boldsymbol{\gamma}} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \sum_{\boldsymbol{l}\in\mathbb{Z}^t} \frac{1}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})} \frac{1}{r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})} \widehat{f}(\boldsymbol{k},\boldsymbol{l}) \overline{\widehat{g}(\boldsymbol{k},\boldsymbol{l})},$$

with

$$\widehat{f}(\boldsymbol{k},\boldsymbol{l}) = \int_{[0,1]^s} \int_{[0,1]^t} f(\boldsymbol{x},\boldsymbol{y}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y})} \, \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y}.$$

The space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ is the tensor product of a Walsh space and a Korobov space. If $s = 0$, then we obtain the Korobov space, if $t = 0$, then we obtain the Walsh space.

Remark 1. For convenience we will in the following use the notation $\int_{[0,1]^d} f(\boldsymbol{x},\boldsymbol{y}) \, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$, where $d = s + t$, by which we mean $\int_{[0,1]^s} \int_{[0,1]^t} f(\boldsymbol{x},\boldsymbol{y}) \, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$.

The hybrid space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ is the space of all absolutely convergent series f of the form

$$f(\boldsymbol{x},\boldsymbol{y}) = \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{N}_0^s\times\mathbb{Z}^t} \widehat{f}(\boldsymbol{k},\boldsymbol{l}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) \qquad \text{with } \|f\|_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})} < \infty,$$

where $\|\cdot\|_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}$ denotes the norm in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$. For further information on the space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ we refer to [11, Section 2.2].

3. $\mathbb{L}_2$-approximation

Our goal is now to approximate the embedding from the hybrid space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ to the space $\mathbb{L}_2([0,1]^{s+t})$, i.e.,

$$\mathrm{EMB}_{s,t} : \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}) \to \mathbb{L}_2([0,1]^{s+t}), \qquad \mathrm{EMB}_{s,t}(f) = f.$$

As already mentioned, it is enough to consider linear algorithms $A_{N,s,t}$ of the form

$$A_{N,s,t}(f) = \sum_{k=1}^{N} a_k L_k(f), \tag{1}$$

with $a_k \in \mathbb{L}_2([0,1]^{s+t})$ and continuous linear functionals L_k on $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ from a permissible class of information Λ. We consider two classes:

- $\Lambda = \Lambda^{\mathrm{all}}$, the class of all continuous linear functionals defined on $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$. Since $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ is a Hilbert space, for every $L_k \in \Lambda^{\mathrm{all}}$ there exists a function f_k from $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ such that $L_k(f) = \langle f, f_k\rangle_{d,\alpha,\beta,\boldsymbol{\gamma}}$ for all $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$.
- $\Lambda = \Lambda^{\mathrm{std}}$, the class of standard information consisting only of function evaluations. That is, $L_k \in \Lambda^{\mathrm{std}}$ if there exists $(\boldsymbol{x}_k, \boldsymbol{y}_k) \in [0,1]^{s+t}$ such that $L_k(f) = f(\boldsymbol{x}_k, \boldsymbol{y}_k)$ for all $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$.

Since $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ is a reproducing kernel Hilbert space, function evaluations are continuous linear functionals, and therefore $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$. More precisely,

$$L_k(f) = f(\boldsymbol{x}_k, \boldsymbol{y}_k) = \langle f, K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}(\cdot, (\boldsymbol{x}_k, \boldsymbol{y}_k))\rangle_{s,t,\alpha,\beta,\boldsymbol{\gamma}}$$

and

$$\|L_k\| = \|K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}\|_{s,t,\alpha,\beta,\boldsymbol{\gamma}} = K^{1/2}_{s,t,\alpha,\beta,\boldsymbol{\gamma}}((\boldsymbol{x}_k, \boldsymbol{y}_k), (\boldsymbol{x}_k, \boldsymbol{y}_k)).$$

The worst-case error in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ of a linear algorithm as in (1) is

$$e^{\mathbb{L}_2-\mathrm{app}}(A_{N,s,t}) = \sup_{\substack{f \in \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}) \\ \|f\|_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})} \le 1}} \|\mathrm{EMB}_{s,t}(f) - A_{N,s,t}(f)\|_{\mathbb{L}_2([0,1]^{s+t})}.$$

The N^{th} minimal worst-case error is given by

$$e^{\mathbb{L}_2-\mathrm{app},\Lambda}(N,s,t) = \inf_{A_{N,s,t}} e^{\mathrm{app}}(A_{N,s,t}),$$

where the infimum is extended over all linear algorithms $A_{N,s,t}$ using information from the class Λ. The information complexity is given as

$$N^{\mathbb{L}_2-\mathrm{app},\Lambda}(\varepsilon,s,t) := \min\{N : e^{\mathbb{L}_2-\mathrm{app},\Lambda}(N,s,t) \le \varepsilon\}.$$

Since $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$, it follows that $N^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{all}}}(\varepsilon,s,t) \le N^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{std}}}(\varepsilon,s,t)$.

We say that the $\mathbb{L}_2$-approximation problem $\mathrm{EMB} = (\mathrm{EMB}_{s,t})_{s,t\ge 1}$ is:

- weakly tractable, if

$$\lim_{s+t+\varepsilon^{-1}\to\infty} \frac{\log N^{\mathbb{L}_2-\mathrm{app},\Lambda}(\varepsilon,s,t)}{s+t+\varepsilon^{-1}} = 0;$$

- polynomially tractable, if we can find constants $C, \tau_1, \tau_2 \ge 0$ such that

$$N^{\mathbb{L}_2-\mathrm{app},\Lambda}(\varepsilon,s,t) \le C\varepsilon^{-\tau_1}(s+t)^{\tau_2} \qquad \text{for all } \varepsilon \in (0,1) \text{ and all } s,t \in \mathbb{N};$$

- strongly polynomially tractable, if we can find constants $C, \tau_1 \geq 0$ such that

$$N^{\mathbb{L}_2-\mathrm{app},\Lambda}(\varepsilon, s, t) \leq C\varepsilon^{-\tau_1} \qquad \text{for all } \varepsilon \in (0,1) \text{ and all } s,t \in \mathbb{N}. \tag{2}$$

  The infimum $\tau^*(\Lambda)$ of the real numbers τ_1 such that (2) holds is called the ε-exponent of strong polynomial tractability.

For $\boldsymbol{\gamma} = (\boldsymbol{\gamma}^{(1)}, \boldsymbol{\gamma}^{(2)})$ we define the sum exponent

$$s_{\boldsymbol{\gamma}} = \inf \left\{ \kappa > 0 \ : \ \sum_{j=1}^{\infty} (\gamma_j^{(1)})^{\kappa} < \infty \text{ and } \sum_{j=1}^{\infty} (\gamma_j^{(2)})^{\kappa} < \infty \right\} \tag{3}$$

with the convention that $\inf \emptyset = \infty$.

Our main goal in this paper is to show the following theorem.

Theorem 1. *Consider the approximation problem* EMB. *Then we have:*

1. *Strong polynomial tractability and polynomial tractability in the class Λ^{all} are equivalent, and they hold if and only if $s_{\boldsymbol{\gamma}} < \infty$, where $s_{\boldsymbol{\gamma}}$ is defined in* (3). *In this case the exponent of strong polynomial tractability is $\tau^*(\Lambda^{\mathrm{all}}) = 2\max(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha}, \frac{1}{\beta})$.*
2. *The problem is weakly tractable in the class Λ^{all} if and only if*
$$\lim_{s+t\to\infty} \frac{\sum_{j=1}^{s} \gamma_j^{(1)} + \sum_{j=1}^{t} \gamma_j^{(2)}}{s+t} = 0. \tag{4}$$
3. *The problem is strongly polynomially tractable in the class Λ^{std} if*
$$\sum_{j=1}^{\infty} \gamma_j^{(1)} < \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} \gamma_j^{(2)} < \infty.$$
The exponent of strong polynomial tractability in the class Λ^{std} satisfies
$$\tau^*(\Lambda^{\mathrm{std}}) \in [2\max(\tfrac{1}{\alpha}, \tfrac{1}{\beta}, s_{\boldsymbol{\gamma}}), 4 + 2\max(\tfrac{1}{\alpha}, \tfrac{1}{\beta}, s_{\boldsymbol{\gamma}})].$$
4. *The problem is polynomially tractable in the class Λ^{std} if*
$$\limsup_{s\to\infty} \frac{\sum_{j=1}^{s} \gamma_j^{(1)}}{\log s} < \infty \qquad \text{and} \qquad \limsup_{t\to\infty} \frac{\sum_{j=1}^{t} \gamma_j^{(2)}}{\log t} < \infty.$$
5. *The problem is weakly tractable in the class Λ^{std} if and only if*
$$\lim_{s+t\to\infty} \frac{\sum_{j=1}^{s} \gamma_j^{(1)} + \sum_{j=1}^{t} \gamma_j^{(2)}}{s+t} = 0.$$

Remark 2. Since it can easily be verified that integration in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ is not harder than approximation, all sufficient conditions stated in Theorem 1 for approximation are sufficient for integration in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ as well. These conditions coincide with the ones given in [11] for QMC integration.

4. Proof of Theorem 1

We recall that strong polynomial tractability implies polynomial tractability which in turn implies weak tractability. Furthermore, all sufficient conditions for the class $\Lambda^{\rm std}$ are also sufficient for the class $\Lambda^{\rm all}$ with $\tau^*(\Lambda^{\rm all}) \leq \tau^*(\Lambda^{\rm std})$ in the case of strong polynomial tractability. All necessary conditions for the class $\Lambda^{\rm all}$ are also necessary for the class $\Lambda^{\rm std}$.

4.1. Proof of Item 1

In order to give a necessary and sufficient condition for strong polynomial tractability for $\Lambda^{\rm all}$ we use a criterion from [16, Section 5.1]. Let us consider the self-adjoint operator $W_{s,t} := \mathrm{EMB}_{s,t}^* \mathrm{EMB}_{s,t} : \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}) \to \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$, which in our case is given by

$$W_{s,t} f = \sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t} \rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) \widehat{f}(\boldsymbol{k},\boldsymbol{l}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}).$$

The eigenvalues are then given by the collection of the numbers

$$\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) \qquad \text{for} \ \ (\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t.$$

Furthermore, the largest eigenvalue is $\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{0}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{0}) = 1$.

From [16, Theorem 5.2] we find that the problem EMB is polynomially tractable for $\Lambda^{\rm all}$ if and only if there exist $\nu > 0$ and $q \geq 0$ such that

$$\sup_{s,t} \left(\sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t} \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{\nu} \right)^{1/\nu} (s+t)^{-q} < \infty. \tag{5}$$

Furthermore, we have strong polynomial tractability if and only if (5) holds with $q = 0$.

It is easy to check that we require $\nu > \max(\frac{1}{\alpha}, \frac{1}{\beta})$ in order for (5) to hold with $q = 0$. Let us now assume that ν is indeed bigger than $\max(\frac{1}{\alpha}, \frac{1}{\beta})$. For the sum in (5) we have

$$\begin{aligned} &\sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t} \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{\nu} \\ &\qquad = \prod_{j=1}^{s} \left(1 + (\gamma_j^{(1)})^{\nu} \mu(\alpha\nu)\right) \prod_{j=1}^{t} \left(1 + (\gamma_j^{(2)})^{\nu} 2\zeta(\beta\nu)\right), \end{aligned} \tag{6}$$

where $\mu(x) = \frac{b^x(b-1)}{b^x - b}$ for $x > 1$ and $\zeta(x)$ is the Riemann zeta function.

Now, using arguments outlined in [19] (see also [14, Section 4.5]), it is easy to see that the existence of some $\nu > \max(\frac{1}{\alpha}, \frac{1}{\beta})$ with

$$\sum_{j=1}^{\infty} (\gamma_j^{(1)})^{\nu} < \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} (\gamma_j^{(2)})^{\nu} < \infty$$

is a necessary and sufficient condition for (5) with $q = 0$ and therefore for strong polynomial tractability of the problem EMB.

Again according to [16, Theorem 5.2], the exponent of strong polynomial tractability is $2\max(\frac{1}{\alpha}, \frac{1}{\beta}, s_{\boldsymbol{\gamma}})$, where $s_{\boldsymbol{\gamma}}$ is defined in (3).

It remains to show the equivalence of strong polynomial and polynomial tractability. Of course, it suffices to show that polynomial tractability implies strong polynomial tractability. So assume that the problem EMB is polynomially tractable for the class Λ^{all}. Then we obtain polynomial tractability also for the embedding problem in the pure Walsh space $\mathcal{H}(K_{s,0,\alpha,\beta,\boldsymbol{\gamma}})$ and in the pure Korobov space $\mathcal{H}(K_{0,t,\alpha,\beta,\boldsymbol{\gamma}})$. According to [21, Theorem 2] this is equivalent to strong polynomial tractability for the embedding problem in the pure Walsh space $\mathcal{H}(K_{s,0,\alpha,\beta,\boldsymbol{\gamma}})$ and in the pure Korobov space $\mathcal{H}(K_{0,t,\alpha,\beta,\boldsymbol{\gamma}})$. According to [3] and [12] this implies the existence of $\nu_1 > 0$ such that $\sum_{j\geq 1}(\gamma_j^{(1)})^{\nu_1} < \infty$ and the existence of $\nu_2 > 0$ such that $\sum_{j\geq 1}(\gamma_j^{(2)})^{\nu_2} < \infty$. Hence we have $s_{\boldsymbol{\gamma}} < \infty$ and this in turn implies strong polynomial tractability for the class Λ^{all}, as shown above. This completes the proof of Item 1.

4.2. Proof of Item 2

Sufficiency of Condition (4) follows by Item 5 of the Theorem which we show in the next section.

For showing necessity of Condition (4), we use [16, Theorem 5.3] in the following. To keep notation simple, we shall frequently write again d instead of $s + t$. Theorem 5.3 in [16] states that our approximation problem is weakly tractable for Λ^{all} if and only if

- $\lim_{j\to\infty} \lambda_{d,j} \log^2 j = 0$ for all $d \in \mathbb{N}$ and
- there exists some function $f\colon (0, \frac{1}{2}] \to \mathbb{N}$ such that

$$\sup_{\eta\in(0,\frac{1}{2}]} \frac{1}{\eta^2} \sup_{d\geq f(\eta)} \sup_{j\geq\lceil\exp(d\sqrt{\eta})\rceil+1} \lambda_{d,j} \log^2 j < \infty, \tag{7}$$

where $\lambda_{d,j} = \lambda_{s+t,j}$ denotes the j^{th} eigenvalue of $W_{s,t}$ in non-increasing order.

Let us now assume that the approximation problem is weakly tractable for Λ^{all}. This then in particular implies that

$$\lim_{j\to\infty} \lambda_{d,j} \log^2 j = 0 \qquad \text{for all} \;\; d \in \mathbb{N}. \tag{8}$$

We are now going to show that (8) implies (4). To this end, recall that the eigenvalues of $W_{s,t}$ are of the form

$$\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) \qquad \text{for } \ (\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t.$$

Note that we have $\lambda_{d,1} = 1$; furthermore, note that $\rho_{\alpha,\gamma_j^{(1)}}(1) = \gamma_j^{(1)}$ for any $j \in \mathbb{N}$, and $r_{\beta,\gamma_i^{(2)}}(1) = \gamma_i^{(2)}$ for any $i \in \mathbb{N}$. Hence, by choosing all components of $(\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t$ but one equal to zero, and the remaining equal to one, we see that

$$\gamma_1^{(1)}, \dots, \gamma_s^{(1)} \qquad \text{and} \qquad \gamma_1^{(2)}, \dots, \gamma_t^{(2)}$$

are eigenvalues of $W_{s,t}$. Consequently,

$$\sum_{j=1}^{s} \gamma_j^{(1)} + \sum_{j=1}^{t} \gamma_j^{(2)} \leq \sum_{j=1}^{d} \lambda_{d,j},$$

and hence

$$\lim_{s+t\to\infty} \frac{\sum_{j=1}^{s} \gamma_j^{(1)} + \sum_{j=1}^{t} \gamma_j^{(2)}}{s+t} \leq \lim_{d\to\infty} \frac{\sum_{j=1}^{d} \lambda_{d,j}}{d}.$$

However, due to (8), it follows that the latter limit is 0, which shows that indeed (4) holds.

4.3. Proof of Items 3–5

Any $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ can be displayed as

$$f(\boldsymbol{x},\boldsymbol{y}) = \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{N}_0^s\times\mathbb{Z}^t} \widehat{f}(\boldsymbol{k},\boldsymbol{l}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}).$$

In order to approximate $\widehat{f}(\boldsymbol{k},\boldsymbol{l})$, we are going to use quasi-Monte Carlo algorithms based on classical and on polynomial lattice point sets.

Classical lattice point sets. For $N \in \mathbb{N}$ and $\boldsymbol{z} = (z_1, \dots, z_t) \in Z_N^t$, where $Z_N := \{z \in \{1, \dots, N-1\} : \gcd(z,N) = 1\}$, the lattice point set $\{\boldsymbol{q}_v\}_{v=0}^{N-1}$ with generating vector $\boldsymbol{z}$ is defined by

$$\boldsymbol{q}_v = \left(\left\{ \frac{vz_1}{N} \right\}, \dots, \left\{ \frac{vz_t}{N} \right\} \right) \qquad \text{for all } \ 0 \leq v \leq N-1.$$

Here $\{\cdot\}$ denotes the fractional part of a real number.

Polynomial lattice point sets. Let $\mathbb{F}_b$ be the finite field of prime order b, $\mathbb{F}_b[x]$ be the set of polynomials over $\mathbb{F}_b$, and let $\mathbb{F}_b((x^{-1}))$ be the field of formal Laurent series over $\mathbb{F}_b$. The latter contains the field of rational functions as a subfield. Given $m \in \mathbb{N}$, set $G_{b,m} := \{a \in \mathbb{F}_b[x] : \deg(a) < m\}$ and define a mapping $\nu_m : \mathbb{F}_b((x^{-1})) \to [0,1)$ by

$$\nu_m\left(\sum_{l=z}^{\infty} t_l x^{-l}\right) := \sum_{l=\max(1,z)}^{m} t_l b^{-l}.$$

Let $f \in \mathbb{F}_b[x]$ with $\deg(f) = m$ and $\boldsymbol{g} = (g_1, \ldots, g_s) \in \mathbb{F}_b[x]^s$. The polynomial lattice point set $(\boldsymbol{p}_v)_{v \in G_{b,m}}$ with generating vector $\boldsymbol{g}$ is defined by

$$\boldsymbol{p}_v := \left(\nu_m\left(\frac{v(x)g_1(x)}{f(x)}\right), \ldots, \nu_m\left(\frac{v(x)g_s(x)}{f(x)}\right)\right) \qquad \text{for all } \ v \in G_{b,m}.$$

Note that we can associate the polynomial $v(x) = \sum_{r=0}^{m-1} v_r x^r \in G_{b,m}$ with the integer $v = \sum_{r=0}^{m-1} v_r b^r$, where, with a slight abuse of notation, the element $v_r \in \mathbb{F}_b$ is associated with the integer $v_r \in \{0, 1 \ldots, b-1\}$. In this way we can index the points of a polynomial lattice point set by integers ranging from 0 to $b^m - 1$.

Now suppose that N is of the form b^m for some $m \in \mathbb{N}$, and let $\mathrm{PL} = \{\boldsymbol{p}_0, \ldots, \boldsymbol{p}_{N-1}\} \subseteq [0,1)^s$ be a polynomial lattice point set and $\mathrm{L} = \{\boldsymbol{q}_0, \ldots, \boldsymbol{q}_{N-1}\} \subseteq [0,1)^t$ be a lattice point set. We consider the point set $(\mathrm{PL}, \mathrm{L}) = \{(\boldsymbol{p}, \boldsymbol{q})_v = (\boldsymbol{p}_v, \boldsymbol{q}_v) \colon v = 0, \ldots, N-1\}$.

For $M \geq 1$ define the set

$$\mathcal{A}_M = \{(\boldsymbol{k}, \boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t : (\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}))^{-1} (r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}))^{-1} \leq M\}. \tag{9}$$

In order to approximate the embedding $\mathrm{EMB}_{s,t}(f) = f$ for $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ we use the algorithm

$$A_{N,s,t,M}(f)(\boldsymbol{x}, \boldsymbol{y}) = \sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathcal{A}_M} \left(\frac{1}{N} \sum_{v=0}^{N-1} f((\boldsymbol{p}, \boldsymbol{q})_v) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{p}_v) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{q}_v)}\right) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}). \tag{10}$$

It can easily be checked that $A_{N,s,t,M}$ is a linear algorithm of the form (1) with

$$a_v(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{N} \sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathcal{A}_M} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x} \ominus \boldsymbol{p}_v) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y} - \boldsymbol{q}_v)$$

and

$$L_v(f) = f((\boldsymbol{p}, \boldsymbol{q})_v), \qquad 0 \leq v \leq N-1.$$

The error of approximation for given $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ is then

$$\begin{aligned}(f - A_{N,s,t,M}(f))(\boldsymbol{x},\boldsymbol{y}) \\ &= \sum_{(\boldsymbol{k},\boldsymbol{l})\notin\mathcal{A}_M} \widehat{f}(\boldsymbol{k},\boldsymbol{l})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) \\ &\quad + \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M} \left(\widehat{f}(\boldsymbol{k},\boldsymbol{l}) - \frac{1}{N}\sum_{v=0}^{N-1} f((\boldsymbol{p},\boldsymbol{q})_v)\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{p}_v)\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{q}_v)}\right)\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}). \quad (11)\end{aligned}$$

We use (11) and Parseval's identity to obtain

$$\|\mathrm{EMB}_{s,t}(f) - A_{N,s,t,M}(f)\|^2_{\mathbb{L}_2([0,1]^{s+t})} = S_1 + S_2,$$

where

$$S_1 := \sum_{(\boldsymbol{k},\boldsymbol{l})\notin\mathcal{A}_M} |\widehat{f}(\boldsymbol{k},\boldsymbol{l})|^2,$$

and

$$S_2 := \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M} \left|\int_{[0,1]^{s+t}} f_{(\boldsymbol{k},\boldsymbol{l})}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} - \frac{1}{N}\sum_{v=0}^{N-1} f_{(\boldsymbol{k},\boldsymbol{l})}((\boldsymbol{p},\boldsymbol{q})_v)\right|^2,$$

with

$$f_{(\boldsymbol{k},\boldsymbol{l})}(\boldsymbol{x},\boldsymbol{y}) := f(\boldsymbol{x},\boldsymbol{y})\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y})}.$$

From the definition of $\mathcal{A}_M$ it follows easily that

$$S_1 < \frac{1}{M}\|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}.$$

Let us now consider S_2. The term in-between the absolute value signs in S_2 is the integration error of the QMC rule using the nodes (PL,L) for the function $f_{(\boldsymbol{k},\boldsymbol{l})}(\boldsymbol{x},\boldsymbol{y})$. Since the product of two Walsh functions is again a Walsh function, and the analogue is true for trigonometric functions, it can easily be verified that $f_{(\boldsymbol{k},\boldsymbol{l})} \in \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$. Hence we can bound S_2 by

$$S_2 \le (e^{\mathrm{int}}(\mathrm{PL},\mathrm{L}))^2 \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M} \|f_{(\boldsymbol{k},\boldsymbol{l})}\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})},$$

where $e^{\mathrm{int}}(\mathrm{PL},\mathrm{L})$ is the worst-case integration error in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ of the QMC rule based on the nodes (PL,L), i.e.,

$$e^{\mathrm{int}}(\mathrm{PL},\mathrm{L}) = \sup_{\substack{f\in\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})\\ \|f\|_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}\le 1}} \left|\int_{[0,1]^{s+t}} f(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} - \frac{1}{N}\sum_{v=0}^{N-1} f((\boldsymbol{p},\boldsymbol{q})_v)\right|.$$

From [11, Theorem 3] it then follows that

$$S_2 \leq \frac{2}{N}\left(\prod_{j=1}^{s}(1+\gamma_j^{(1)}2\mu(\alpha))\right)\left(\prod_{j=1}^{t}(1+\gamma_j^{(2)}4\zeta(\beta))\right)\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M}\|f_{(\boldsymbol{k},\boldsymbol{l})}\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}. \tag{12}$$

Next we find an estimate for $\|f_{(\boldsymbol{k},\boldsymbol{l})}\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}$ for $(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M$. From the easily seen fact that $\widehat{f}_{(\boldsymbol{k},\boldsymbol{l})}(\boldsymbol{h},\boldsymbol{m})=\widehat{f}(\boldsymbol{k}\oplus\boldsymbol{h},\boldsymbol{l}+\boldsymbol{m})$ we obtain

$$\begin{aligned}
&\|f_{(\boldsymbol{k},\boldsymbol{l})}\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}\\
&\qquad=\sum_{\boldsymbol{h}\in\mathbb{N}_0^s}\sum_{\boldsymbol{m}\in\mathbb{Z}^t}\frac{|\widehat{f}(\boldsymbol{k}\oplus\boldsymbol{h},\boldsymbol{l}+\boldsymbol{m})|^2}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{m})}\\
&\qquad=\sum_{\boldsymbol{h}\in\mathbb{N}_0^s}\sum_{\boldsymbol{m}\in\mathbb{Z}^t}\frac{|\widehat{f}(\boldsymbol{k}\oplus\boldsymbol{h},\boldsymbol{l}+\boldsymbol{m})|^2}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})}\frac{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{m})}.
\end{aligned}$$

Combining results from [3] and [12] we find

$$\begin{aligned}
\frac{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{m})}&\leq\frac{1}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})}\prod_{j=1}^{t}\max(1,2^\beta\gamma_j^{(2)})\\
&\leq M\prod_{j=1}^{t}\max(1,2^\beta\gamma_j^{(2)}),
\end{aligned}$$

and hence, after applying an index shift,

$$\begin{aligned}
\|f_{(\boldsymbol{k},\boldsymbol{l})}\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}&\leq M\prod_{j=1}^{t}\max(1,2^\beta\gamma_j^{(2)})\sum_{\boldsymbol{h}\in\mathbb{N}_0^s}\sum_{\boldsymbol{m}\in\mathbb{Z}^t}\frac{|\widehat{f}(\boldsymbol{k}\oplus\boldsymbol{h},\boldsymbol{l}+\boldsymbol{m})|^2}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})}\\
&=M\|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}\prod_{j=1}^{t}\max(1,2^\beta\gamma_j^{(2)}).
\end{aligned}$$

Plugging this into (12) we obtain

$$\begin{aligned}
S_2\leq{}&\frac{2}{N}\left(\prod_{j=1}^{s}(1+\gamma_j^{(1)}2\mu(\alpha))\right)\left(\prod_{j=1}^{t}(1+\gamma_j^{(2)}4\zeta(\beta))\right)\\
&\times\|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}M|\mathcal{A}_M|\prod_{j=1}^{t}\max(1,2^\beta\gamma_j^{(2)}).
\end{aligned}\tag{13}$$

Next we study the cardinality of the set $\mathcal{A}_M$.

Lemma 1. *Let $\theta = \min(\alpha, \beta)$. For arbitrary $\kappa > 1/\theta = \max(\frac{1}{\alpha}, \frac{1}{\beta})$ we have*

$$|\mathcal{A}_M| \le M^\kappa \prod_{j=1}^{s} \left(1 + 2\zeta(\theta\kappa)(b^\alpha \gamma_j^{(1)})^\kappa\right) \prod_{j=1}^{t} \left(1 + 2\zeta(\theta\kappa)(\gamma_j^{(2)})^\kappa\right).$$

Proof. For $k \in \mathbb{N}$ we have

$$\frac{1}{\rho_{\alpha,\gamma}(k)} = \frac{b^{\alpha\lfloor \log_b k \rfloor}}{\gamma} \ge \frac{b^{\alpha(-1+\log_b k)}}{\gamma} = \frac{k^\alpha}{\gamma b^\alpha} = \frac{1}{r_{\alpha,\gamma b^\alpha}(k)}.$$

Then we have

$$\begin{aligned}
\mathcal{A}_M &= \left\{(\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t : \frac{1}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})} \frac{1}{r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})} \le M\right\} \\
&\subseteq \left\{(\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t : \frac{1}{r_{\alpha,\boldsymbol{\gamma}^{(1)} b^\alpha}(\boldsymbol{k})} \frac{1}{r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})} \le M\right\} \\
&\subseteq \left\{(\boldsymbol{k},\boldsymbol{l}) \in \mathbb{Z}^s \times \mathbb{Z}^t : \frac{1}{r_{\theta,\boldsymbol{\gamma}^{(1)} b^\alpha}(\boldsymbol{k})} \frac{1}{r_{\theta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})} \le M\right\}
\end{aligned}$$

from which the result follows immediately from [12, Lemma 1]. ∎

Considering Lemma 1, for any $\kappa > 1/\nu$ we obtain

$$S_2 \le c_{s,t,\alpha,\beta,\boldsymbol{\gamma},\kappa} \frac{M^{1+\kappa}}{N} \|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})},$$

where

$$\begin{aligned}
c_{s,t,\alpha,\beta,\boldsymbol{\gamma},\kappa} := 2 &\left(\prod_{j=1}^{s}(1 + \gamma_j^{(1)} 2\mu(\alpha))\right) \left(\prod_{j=1}^{t}(1 + \gamma_j^{(2)} 4\zeta(\beta))\right) \prod_{j=1}^{t} \max(1, 2^\beta \gamma_j^{(2)}) \\
&\times \prod_{j=1}^{s} \left(1 + 2\zeta(\theta\kappa)(b^\alpha \gamma_j^{(1)})^\kappa\right) \prod_{j=1}^{t} \left(1 + 2\zeta(\theta\kappa)(\gamma_j^{(2)})^\kappa\right).
\end{aligned} \tag{14}$$

Summing up we have

$$\|\mathrm{EMB}_{s,t}(f) - A_{N,s,t,M}(f)\|^2_{\mathbb{L}_2([0,1]^{s+t})} \le \left(\frac{1}{M} + c_{s,t,\alpha,\beta,\boldsymbol{\gamma},\kappa} \frac{M^{1+\kappa}}{N}\right) \|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}.$$

Choosing $M = M(N) = (N/c_{s,t,\alpha,\beta,\boldsymbol{\gamma},\kappa})^{1/(2+\kappa)}$ and taking the square root we obtain the following proposition and its corollary, which then concludes the proof of Theorem 1.

Proposition 1. *Let $\kappa > 1/\min(\alpha,\beta)$ and let $c_{s,t,\alpha,\beta,\boldsymbol{\gamma},\kappa}$ be defined as in* (14)*. The worst-case error of the algorithm $A_{N,s,t,M}$ as defined in* (10) *using a point set* (PL, L) *constructed by [11, Algorithm 1] and with $M = (N/c_{s,t,\alpha,\beta,\boldsymbol{\gamma},\kappa})^{1/(2+\kappa)}$ satisfies*

$$e^{\mathbb{L}_2-\mathrm{app}}(A_{N,s,t,M}) \le \sqrt{2}\left(\frac{c_{s,t,\alpha,\beta,\boldsymbol{\gamma},\kappa}}{N}\right)^{\frac{1}{4+2\kappa}}.$$

Corollary 1. *Consider the approximation problem* EMB *with information from the class Λ^{std}.*

- *If $\sum_{j=1}^{\infty}\gamma_j^{(1)} < \infty$ and $\sum_{j=1}^{\infty}\gamma_j^{(2)} < \infty$, then* EMB *is strongly polynomially tractable with ε-exponent at most $4 + 2\max(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha}, \frac{1}{\beta})$;*
- *if $\limsup_{s\to\infty}\sum_{j=1}^{s}\frac{\gamma_j^{(1)}}{\log(s+1)} < \infty$ and $\limsup_{t\to\infty}\sum_{j=1}^{t}\frac{\gamma_j^{(2)}}{\log(t+1)} < \infty$, then* EMB *is polynomially tractable;*
- *if $\lim_{s+t\to\infty}\frac{\sum_{j=1}^{s}\gamma_j^{(1)}+\sum_{j=1}^{t}\gamma_j^{(2)}}{s+t} = 0$, then* EMB *is weakly tractable.*

Proof. Employing Proposition 1, the result follows by the same arguments as used in [11, Section 5.2]. We only show the first item: Let $\kappa = 1$. Since $\log(1+x) \le x$ we obtain

$$c_{s,t,\alpha,\beta,\boldsymbol{\gamma},1} \le 2\exp\left(u_1(\alpha,\beta)\sum_{j=1}^{s}\gamma_j^{(1)} + u_2(\alpha,\beta)\sum_{j=1}^{s}\gamma_j^{(2)}\right) \le c_{\infty,\infty,\alpha,\beta,\boldsymbol{\gamma},1} < \infty,$$

where $u_1(\alpha,\beta) = 2\mu(\alpha) + 2\zeta(\theta)b^{\alpha}$ and $u_2(\alpha,\beta) = 4\zeta(\beta) + 2^{\beta} + 2\zeta(\theta)$. Then Proposition 1 with $\kappa = 1$ implies that

$$e^{\mathbb{L}_2-\mathrm{app}}(N, s+t) \le \sqrt{2}\left(\frac{c_{\infty,\infty,\alpha,\beta,\boldsymbol{\gamma},1}}{N}\right)^{1/6}.$$

Recall that N is of the form b^m. Now, for $\varepsilon > 0$ choose $m \in \mathbb{N}$ such that $b^{m-1} < \lceil 8c_{\infty,\infty,\alpha,\beta,\boldsymbol{\gamma},1}\varepsilon^{-6}\rceil$
$=: N' \le b^m$. Then we have $e^{\mathbb{L}_2-\mathrm{app}}(b^m, s+t) \le \varepsilon$ and hence

$$N^{\mathbb{L}_2-\mathrm{app},\Lambda^{\mathrm{std}}}(\varepsilon, s+t) \le b^m < bN' = b\lceil 8c_{\infty,\infty,\alpha,\beta,\boldsymbol{\gamma},1}\varepsilon^{-6}\rceil.$$

This is strong polynomial tractability. The result for the ε-exponent can be shown easily by similar arguments. ■

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Addresses: Peter Kritzer and Helene Laimer: Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenbergerstr. 69, 4040 Linz, Austria;
Friedrich Pillichshammer: Institut für Finanzmathematik und Angewandte Zahlentheorie, Johannes Kepler Universität Linz, Altenbergerstr. 69, 4040 Linz, Austria.

E-mail: peter.kritzer@oeaw.ac.at, helene.laimer@oeaw.ac.at, friedrich.pillichshammer@jku.at

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