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ON RELATIONS EQUIVALENT TO THE GENERALIZED RIEMANN HYPOTHESIS FOR THE SELBERG CLASS

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Abstract: We prove that the generalized Riemann hypothesis (GRH) for functions in the class $\mathcal{S}^{\sharp\flat}$ containing the Selberg class is equivalent to a certain integral expression of the real part of the generalized Li coefficient $\lambda_F(n)$ associated to $F \in \mathcal{S}^{\sharp\flat}$, for positive integers n. Moreover, we deduce that the GRH is equivalent to a certain expression of $\mathrm{Re}(\lambda_F(n))$ in terms of the sum of the Chebyshev polynomials of the first kind. Then, we partially evaluate the integral expression and deduce further relations equivalent to the GRH involving the generalized Euler-Stieltjes constants of the second kind associated to F. The class $\mathcal{S}^{\sharp\flat}$ unconditionally contains all automorphic L-functions attached to irreducible cuspidal unitary representations of $\mathrm{GL}_N(\mathbb{Q})$, hence, as a corollary we also derive relations equivalent to the GRH for automorphic L-functions.

Keywords: Selberg class, generalized Riemann hypothesis, generalized Li coefficients, Euler-Stieltjes constants of the second kind.

1. Introduction

The Selberg class of functions $\mathcal{S}$, introduced by A. Selberg in [23], is a general class of Dirichlet series F satisfying the following properties:

(i) (Dirichlet series) F possesses a Dirichlet series representation

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}, \tag{1}$$

that converges absolutely for $\mathrm{Re}(s) > 1$.

(ii) (Analytic continuation) There exists an integer $m \geqslant 0$ such that $(s-1)^m F(s)$ is an entire function of finite order. The smallest such number is denoted by m_F and called the polar order of F.

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(iii) (Functional equation) The function F satisfies the functional equation

$$\xi_F(s) = w\overline{\xi_F(1-\bar{s})},$$

where

$$\xi_F(s) = s^{m_F}(1-s)^{m_F}F(s)Q_F^s\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j),$$

with $Q_F > 0$, $r \geqslant 0$, $\lambda_j > 0$, $|w| = 1$, $\mathrm{Re}\mu_j \geqslant 0$, $j = 1, \dots, r$.

(iv) (Ramanujan conjecture) For every $\epsilon > 0$, $a_F(n) \ll n^{\epsilon}$.

(v) (Euler product)

$$\log F(s) = \sum_{n=1}^{\infty}\frac{b_F(n)}{n^s},$$

where $b_F(n) = 0$, for all $n \neq p^m$ with $m \geqslant 1$ and p prime, and $b_F(n) \ll n^{\theta}$, for some $\theta < \frac{1}{2}$.

The extended Selberg class $\mathcal{S}^{\sharp}$, introduced in [12] is a class of functions satisfying conditions (i), (ii) and (iii).

As usual, the non-trivial zeros of $F \in \mathcal{S}$ are zeros of the complete function ξ_F; we denote the set of non-trivial zeros of F by $Z(F)$. One of the most important conjectures about the Selberg class is the generalized Riemann hypothesis (GRH), i.e. the conjecture that for all $F \in \mathcal{S}$, the non trivial zeros of F are located on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. For more details concerning the Selberg class we refer to the surveys of Kaczorowski [11] and Perelli [19].

The main class of this paper is the class $\mathcal{S}^{\sharp\flat} \supseteq \mathcal{S}$, defined in Section 2 below, consisting of functions $F \in \mathcal{S}^{\sharp}$ that possess an Euler sum. The class $\mathcal{S}^{\sharp\flat}$ is similar to the class $\widetilde{\mathcal{S}}$ considered in [1], whose elements under average density hypothesis have uniformly distributed (modulo 1) imaginary parts of zeros. The main reason why we consider the class $\mathcal{S}^{\sharp\flat}$ is that, as proved in [14, Prop. 2.1. and Sec. 4], it contains both the Selberg class $\mathcal{S}$ and (unconditionally) the class of all automorphic L functions attached to automorphic irreducible unitary cuspidal representations of $GL_N(\mathbb{Q})$.

Moreover, the class $\mathcal{S}^{\sharp\flat}$ contains functions that do not necessarily satisfy axioms (iv) and (v) of the Selberg class, hence the GRH may fail for functions in $\mathcal{S}^{\sharp\flat}$, see e.g. [19], p. 28. For example, for $\delta \in (0, 1/4)$ and an odd Dirichlet character χ, the product $F(s) = L(2(s-\delta)-1/2,\chi)L(2(s+\delta)-1/2,\chi)$ of shifted Dirichlet $L-$functions associated to χ is a positive degree function which belongs to the class $\mathcal{S}^{\sharp\flat}$, violates the Ramanujan conjecture and whose zeros are off the critical line. Therefore, it is of interest to deduce criteria equivalent to the GRH in the class $\mathcal{S}^{\sharp\flat}$ which may be implemented both analytically and numerically.

For an integer $n \neq 0$, the generalized nth Li coefficient attached to $F \in \mathcal{S}^{\sharp\flat}$ non-vanishing at zero is defined as the sum

$$\lambda_F(n) = \sum_{\rho\in Z(F)}{}^{*}\left(1-\left(1-\frac{1}{\rho}\right)^{n}\right), \tag{2}$$

where the $*$ means that the sum is taken in the sense of the limit $\lim_{T\to\infty}\sum_{|\mathrm{Im}(\rho)|\leqslant T}$. In the sequel, we denote by $\mathcal{S}_0^{\sharp\flat}$ the set of all $F\in\mathcal{S}^{\sharp\flat}$, non-vanishing at zero. For $t>0$, by $\mathcal{S}_t^{\sharp\flat}$ we denote the set of all $F\in\mathcal{S}^{\sharp\flat}$ such that all eventual non-trivial zeros of F which lie off the critical line have the absolute value of the imaginary part bigger than t.

The existence of coefficients $\lambda_F(n)$, for $n\in\mathbb{Z}\setminus\{0\}$ and $F\in\mathcal{S}_0^{\sharp\flat}$ is proved in [28, Theorem 4.1.], together with the generalized Li criterion, stating that the GRH for $F\in\mathcal{S}_0^{\sharp\flat}$ is equivalent to the non-negativity of the set of numbers $\mathrm{Re}(\lambda_F(n))$ for $n\in\mathbb{N}$ (see [28, Theorem 4.3.]).

Moreover, in [14] and [16] it is proved that for $F\in\mathcal{S}_0^{\sharp\flat}$

$$GRH \iff \lambda_F(-n)=\frac{d_F}{2}n\log n+c_Fn+O(\sqrt{n}\log n),\qquad \text{as } n\to\infty, \tag{3}$$

where

$$c_F=\frac{d_F}{2}(\gamma-1)+\frac{1}{2}\log(\lambda Q_F^2),\qquad \lambda=\prod_{j=1}^{r}\lambda_j^{2\lambda_j},$$

$d_F=2\sum_{j=1}^{r}\lambda_j$ is the degree of F and γ is the classical Euler constant.

The above result was proved also in [13] using the saddle-point method in conjunction with the theory of the Nörlund-Rice integrals.

There exist many relations equivalent to the Riemann hypothesis. An interesting integral criteria, proved by M. Balazard, E. Saias, and M. Yor [2] states that the Riemann hypothesis is equivalent to the equation

$$\int_{-\infty}^{\infty}\frac{\log|\zeta(1/2+it)|}{1+4t^2}=0,$$

where $\zeta(s)$ denotes the Riemann zeta function.

Another integral criteria is obtained by V. Volchkov in [30] and states that the Riemann hypothesis is equivalent to the equation

$$\int_{0}^{\infty}\left(\frac{1-12t^2}{(1+4t^2)^3}\int_{\frac{1}{2}}^{\infty}\log|\zeta(\sigma+it)|d\sigma\right)dt=\pi\frac{3-\gamma}{32}. \tag{4}$$

Both results are generalized by S. K. Sekatskii, S. Beltraminelli and D. Merlini in [21] and [22].

Actually, an elementary computation, based on application of Littlewood's theorem to the function $\log\zeta(s)$ and the appropriate rectangular contour shows that equation (4) is equivalent to the equation

$$\int_{0}^{\infty}\frac{t\arg\zeta(1/2+it)}{(1/4+t^2)^2}dt=\pi\frac{\gamma-3}{2}, \tag{5}$$

hence we call equation (5) the Volchkov criterion for the Riemann hypothesis as well.

The Volchkov criterion and the results of [22] are further generalized in [8], where equation (5) was interpreted in terms of the argument of the Veneziano amplitude.

In this paper, we prove that the GRH for function $F \in \mathcal{S}_0^{\sharp\flat}$ is equivalent to the integral representation

$$\mathrm{Re}(\lambda_F(n)) = 16n \int_0^{\infty} N_F(x) \frac{x}{(4x^2+1)^2} U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right) dx - (1-(-1)^n) N_F(0) \quad (6)$$

of the real part of the nth generalized Li coefficient $\lambda_F(n)$, for all positive integers n. Here, $N_F(x)$ denotes the counting function of the number of non-trivial zeros $\rho \in Z(F)$ such that $|\mathrm{Im}(\rho)| \leqslant x$ and $U_{n-1}(x)$ is the Chebyshev polynomial of the second kind, see Sections 2.2. and 2.5. Obviously, $N_F(0)$ is the number of Siegel zeros of F, i.e. eventual non-trivial real zeros of F.

We also prove that the GRH for $F \in \mathcal{S}_0^{\sharp\flat}$ is equivalent to the representation

$$\mathrm{Re}(\lambda_F(n)) = \sum_{\rho=\sigma+i\gamma \in Z(F)} \left(1 - T_n\left(\frac{4\gamma^2-1}{4\gamma^2+1}\right)\right) \quad (7)$$

of $\mathrm{Re}(\lambda_F(n))$, for all positive integers n, where $T_n(x)$ is the Chebyshev polynomial of the first kind and the zeros on the right-hand side of (7) are taken according to their multiplicities.

Then, we partially evaluate the integral in (6) and write $\mathrm{Re}(\lambda_F(n))$ as a sum of integral and oscillatory part, which we relate to the generalized Euler-Stieltjes constants of the second kind associated to $F \in \mathcal{S}_0^{\sharp\flat}$. Moreover, we show that the GRH for $F \in \mathcal{S}_0^{\sharp\flat}$ is equivalent to a certain asymptotic integral formula (see formula (35) below).

In the case when $n = 1$ or $n = 2$, under additional assumptions on the location of the imaginary part of the first non-trivial zero of $F \in \mathcal{S}^{\sharp\flat}$, which lies off the critical line, we prove that equations (6) and (7) with $n = 1$ or $n = 2$ are equivalent to the GRH. As a special case of this result and the evaluation of the integral in (6) when $F = \zeta$ and $n = 1$ we deduce equation (5), meaning that the Volchkov criterion for the Riemann Hypothesis is a special case of our results.

Moreover, we prove that the GRH for $F \in \mathcal{S}_{1/\sqrt{3}}^{\sharp\flat}$ is equivalent to a certain integral expression of the real part of the constant term in the Laurent (Taylor) series expression of the logarithmic derivative $F'(s)/F(s)$ at $s = 1$.

The class $\mathcal{S}^{\sharp\flat}$ unconditionally contains the class of $L-$functions attached to irreducible, cuspidal unitary representations of $\mathrm{GL}_N(\mathbb{Q})$, hence, as a corollary, we also derive several relations equivalent to the GRH for automorphic $L-$functions. For example, let $L(s,\pi)$ be an $L-$function associated to a representation π such that $L(s,\pi)$ possesses no zeros off the critical line with absolute value of the imaginary part less than $1/\sqrt{3}$. We prove that the GRH is equivalent to a simple relation

$$\mathrm{Re}\left(\gamma_\pi(0)\right) = 3\delta_\pi + 16 \int_0^{+\infty} \frac{x S_\pi(x)}{(4x^2+1)^2} dx,$$

where $\gamma_\pi(0)$ is the constant term in the Laurent series expansion of $L'(s,\pi)/L(s,\pi)$ at $s=1$, $\delta_\pi=1$, if π is trivial and zero otherwise, and $S_\pi(x)$ is defined in Section 6.

The paper is organized as follows: in Section 2 we recall necessary background results; in Section 3 we derive representations of the real part of the generalized Li coefficient equivalent to the GRH. Section 4 is devoted to partial evaluation of the integral on the right hand side of (6), while in Section 5 we discuss the relation with the generalized Euler-Stieltjes constants of the second kind. In Section 6 we derive results for the automorphic $L-$functions. Concluding remarks are presented in the last section.

2. Preliminaries

2.1. The class $\mathcal{S}^{\sharp\flat}$

Throughout this paper, we focus on the class $\mathcal{S}^{\sharp\flat}$ of functions satisfying axioms (i), (ii) and the following two axioms:

(iii') (Functional equation) The function F satisfies the functional equation

$$\xi_F(s) = w\overline{\xi_F(1-\bar{s})},$$

where

$$\xi_F(s) = s^{m_F}(1-s)^{m_F}F(s)Q_F^s\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j) = \gamma_F(s)F(s)$$

with $Q_F>0$, $r\geqslant 0$, $\lambda_j>0$, $|w|=1$, $\mathrm{Re}\mu_j > -\frac{1}{4}$, $\mathrm{Re}(\lambda_j+2\mu_j)>0$, $j=1,\dots,r$.

(v') (Euler sum) The logarithmic derivative of the function F possesses a Dirichlet series representation

$$\frac{F'}{F}(s) = -\sum_{n=2}^{\infty}\frac{c_F(n)}{n^s}$$

converging absolutely for $\mathrm{Re}(s)>1$.

Let us note that (iii') implies that $\mathrm{Re}(\lambda_j+\mu_j)>0$.

For $F\in\mathcal{S}^{\sharp\flat}$ it is easy to deduce, using the Phragmén Lindelöf principle that the complete function ξ_F is entire function of order one. Moreover, using an explicit formula for the class $\mathcal{S}^{\sharp\flat}$ applied to a suitably chosen test function, it is proved in [14] that for all $F\in\mathcal{S}_0^{\sharp\flat}$ and all $z\notin Z(F)$ one has

$$\frac{\xi_F'}{\xi_F}(z) = \lim_{T\to\infty}\sum_{\rho\in Z(F),|\mathrm{Im}(\rho)|\leqslant T}\frac{1}{z-\rho} = \sum_{\rho\in Z(F)}{}^{*}\left(\frac{1}{z-\rho}\right), \tag{8}$$

where each zero is counted according to its multiplicity.

2.2. Distribution of non-trivial zeros of $F \in \mathcal{S}^{\sharp\flat}$

For $T > 0$, such T and $-T$ are not the ordinates of a non-trivial zero, we denote by $N_F(T)$ the number of non-trivial zeros $\rho = \sigma + it \in Z(F)$ of $F \in \mathcal{S}^{\sharp\flat}$ such that $|t| \leqslant T$. By $N_F^+(T)$ and $N_F(T)^-$ respectively we denote the number of non-trivial zeros of F with $0 \leqslant \mathrm{Im}\rho \leqslant T$, respectively $-T \leqslant \mathrm{Im}\rho \leqslant 0$.

In the case when T or $-T$ is the ordinate of a non-trivial zero, we define $N_F(T) = N_F(T+0)$ and $N_F^\pm(T) = N_F^\pm(T+0)$.

Vertical distribution of zeros of a function $F \in \mathcal{S}^\sharp$ satisfying axiom (v') was discussed in [28, Sec. 5.1.] where it was proved that

$$N_F^+(T) = \frac{d_F}{2\pi} T \log T + C_F T + a_F \log T + m_F + O(1/T) + S_F^+(T), \qquad \text{as } T \to \infty, \tag{9}$$

where

$$C_F = \frac{1}{2\pi}(\log q_F - d_F(\log(2\pi)+1)), \qquad q_F = (2\pi)^{d_F} Q_F^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$$

is the conductor of F, $a_F = \frac{1}{\pi}\sum_{j=1}^r \mathrm{Im}(\mu_j)$ and, in the case when T is not the ordinate of the non-trivial zero, $S_F^+(T)$ is the value of the function $\frac{1}{\pi}\arg F(s)$ obtained by continuous variation along the straight lines joining points 2, $2+iT$ and $1/2+iT$. When T is the ordinate of a non-trivial zero of F, $S_F^+(T)$ is by definition equal to $S_F^+(T+0)$. Moreover, it was proved that $S_F^+(T) = O_F(\log T)$, as $T \to \infty$.

Proceeding analogously as in [28], applying the argument principle to the complete function $\xi_F(s)$ and the rectangle with vertices $-1-iT$, $2-iT$, $2+iT$ and $-1+iT$, we get

$$\begin{aligned} N_F(T) &= S_F(T) + 2m_F + \frac{2T\log Q_F}{\pi} \\ &+ \frac{1}{\pi}\mathrm{Im}\left(\sum_{j=1}^{r}\left(\log\Gamma\left(\lambda_j\left(\frac{1}{2}+iT\right)+\mu_j\right) + \log\Gamma\left(\lambda_j\left(\frac{1}{2}+iT\right)+\overline{\mu_j}\right)\right)\right), \end{aligned} \tag{10}$$

where, in the case when $\pm T$ is not the ordinate of the non-trivial zero, $S_F(T)$ is the value of the function $\frac{1}{\pi}\arg F(s)$ obtained by continuous variation along the straight lines joining points $1/2-iT$, $2-iT$, $2+iT$ and $1/2+iT$ and, in the case when $\pm T$ is the ordinate of a non-trivial zero of F, we define $S_F(T)$ to be equal to $S_F(T+0)$.

Moreover, it is easy to see that the bound $S_F(x) = O_F(\log T)$, as $T \to \infty$ holds true in the case when the axiom (iii) is replaced by (iii').

In the case when the coefficients $a_F(n)$ in the Dirichlet series representation (1) of $F \in \mathcal{S}^{\sharp\flat}$ are real and the function has no Siegel zeros, application of the reflection principle and the functional equation axiom (iii') yields that non-trivial zeros come in conjugate pairs, hence $N_F^+(T) = N_F^-(T)$. Moreover, in this case the function $S_F(T)$ is equal to the value of the function $\frac{2}{\pi}\arg F(s)$ obtained by continuous variation along the straight lines joining points 2, $2+iT$ and $1/2+iT$, which is actually equal to $\frac{2}{\pi}\arg F(1/2+iT)$.

2.3. An arithmetic formula for the generalized Li coefficients

The symmetry $\rho \leftrightarrow (1-\overline{\rho})$ in the set $Z(F)$ of non-trivial zeros of $F \in \mathcal{S}_0^{\sharp\flat}$ implies that $\lambda_F(-n) = \overline{\lambda_F(n)}$, for all $n \in \mathbb{N}$, hence, $\mathrm{Re}(\lambda_F(n)) = \mathrm{Re}(\lambda_F(-n))$, for all $n \in \mathbb{N}$.

An arithmetic formula for $\lambda_F(n)$ is obtained in [14, Theorem 2.4.], where the authors prove that for $F \in \mathcal{S}_0^{\sharp\flat}$ and positive integers n the Li coefficients $\lambda_F(-n)$ can be expressed as

$$\lambda_F(-n) = m_F + \sum_{l=1}^{n} \binom{n}{l} \gamma_F(l-1) + n \log Q_F + \sum_{l=1}^{n} \binom{n}{l} \eta_F(l-1), \tag{11}$$

where

$$\eta_F(0) = \sum_{j=1}^{r} \lambda_j \psi(\lambda_j + \mu_j)$$

and

$$\eta_F(l-1) = \sum_{j=1}^{r} (-\lambda_j)^l \sum_{k=0}^{\infty} \frac{1}{(\lambda_j + \mu_j + k)^l}, \qquad \text{for } \; l \geqslant 2.$$

Here, $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$ denotes the digamma function and $\gamma_F(k)$ are the coefficients in the Laurent (Taylor) series expansion of $\frac{F'}{F}(s)$ around its possible pole at $s=1$. The coefficients $\gamma_F(k)$ are called the generalized Euler-Stieltjes constants of the second kind (see e.g. [15] for a more detailed explanation).

In this paper we need a slightly different representation of the generalized Li coefficient $\lambda_F(-n)$, $n \in \mathbb{N}$, given in the following Lemma.

Lemma 1. *For $F \in \mathcal{S}_0^{\sharp\flat}$ the generalized Li coefficient $\lambda_F(-n)$ for $n \in \mathbb{N}$ can be expressed as*

$$\begin{aligned}\lambda_F(-n) = m_F &+ \sum_{l=1}^{n} \binom{n}{l} \gamma_F(l-1) + n \log Q_F + n \sum_{j=1}^{r} \lambda_j \psi(\lambda_j + \mu_j) \\ &+ \sum_{j=1}^{r} \sum_{l=2}^{n} \binom{n}{l} \frac{\lambda_j^l}{(l-1)!} \psi^{(l-1)}(\lambda_j + \mu_j).\end{aligned} \tag{12}$$

Proof. Using the relation

$$\zeta(l,z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^l} = \frac{(-1)^l}{(l-1)!} \psi^{(l-1)}(z)$$

between the Hurwitz zeta function $\zeta(s,z)$ and the derivatives of digamma function for integers $l \geqslant 2$ we may write the equation (11) as (12). ■

The sum

$$S_\infty(F,n) := n\log Q_F + n\sum_{j=1}^{r}\lambda_j\psi(\lambda_j+\mu_j) + \sum_{j=1}^{r}\sum_{l=2}^{n}\binom{n}{l}\frac{\lambda_j^l}{(l-1)!}\psi^{(l-1)}(\lambda_j+\mu_j)$$

arises from the gamma factors in the functional equation (iii'), hence it is called the archimedean part of the generalized Li coefficient $\lambda_F(-n)$. Analogously, the sum

$$S_{\mathrm{NA}}(F,n) := m_F + \sum_{l=1}^{n}\binom{n}{l}\gamma_F(l-1) \tag{13}$$

is called the non-archimedean part of $\lambda_F(-n)$.

In [14] and [16] the authors compute the full asymptotic expansion of the archimedean part of the generalized Li coefficient $\lambda_F(-n)$ attached to $F \in \mathcal{S}_0^{\sharp\flat}$ as $n \to \infty$, and prove that the GRH is equivalent to the asymptotic bound

$$\sum_{l=1}^{n}\binom{n}{l}\mathrm{Re}\left(\gamma_F(l-1)\right) = O_F(\sqrt{n}\log n), \qquad \text{as } \ n\to\infty. \tag{14}$$

2.4. Automorphic *L*-functions

The (finite) automorphic L-function $L(s,\pi)$ attached to an irreducible unitary cuspidal representation π of $GL_N(\mathbb{Q})$ is given for $\mathrm{Re}(s) > 1$ by the absolutely convergent product over primes p of its local factors

$$L(s,\pi) = \prod_{p}\prod_{j=1}^{N}(1-\alpha_{p,j}(\pi)p^{-s}) = \sum_{n=1}^{\infty}\frac{a_n(\pi)}{n^s}. \tag{15}$$

The completed L-function

$$\Lambda(s,\pi) = Q(\pi)^{s/2}L_\infty(s,\pi)L(s,\pi),$$

where $Q(\pi)$ is the conductor of π and

$$L_\infty(s,\pi) = \prod_{j=1}^{N}\Gamma_{\mathbb{R}}(s+k_j(\pi)) = \prod_{j=1}^{N}\pi^{-(s+k_j(\pi))/2}\Gamma\left(\frac{1}{2}(s+k_j(\pi))\right)$$

is the archimedean factor satisfies the functional equation

$$\Lambda(s,\pi) = \epsilon(\pi)\overline{\Lambda(1-\overline{s},\pi)} \tag{16}$$

with the constant $\epsilon(\pi)$ of absolute value 1.

Using the results of [6], [9], [10], [20] and [24]-[27], it is proved in [14, Sec. 4] that the $L-$function $L(s,\pi)$ belongs to the class $\mathcal{S}^{\sharp\flat}$.

2.5. The Chebyshev polynomials of the first and the second kind

The Chebyshev polynomials of the first kind are defined for $x \in [-1,1]$ by the recurrence relations $T_0(x) = 1,\ T_1(x) = x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, see e.g. [7, pages 993-996]. Putting $x = \cos(\theta)$, $\theta \in [0,\pi]$, we may write $T_n(x) = \cos(n\theta)$.

The generating function for the sequence $\{T_n(x)\}$ of the Chebyshev polynomials of the first kind is

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-tx}{1-2tx+t^2}, \qquad t \in (0,1). \tag{17}$$

The Chebyshev polynomials of the second kind are defined for $x \in [-1,1]$ by the recurrence relations $U_0(x) = 1, U_1(x) = 2x$ and $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$. The generating function for the sequence $\{U_n(x)\}$ of the Chebyshev polynomials of the second kind is

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1-2tx+t^2}, \qquad t \in (0,1).$$

There are many relations between the Chebyshev polynomials of the first and the second kind (see [7, Section 8.94]). In the sequel, we will use relations

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x) \tag{18}$$

and

$$(1-x^2)U_{n-1}(x) = xT_n(x) - T_{n+1}(x), \qquad n \geqslant 1. \tag{19}$$

3. Representations of $\mathrm{Re}(\lambda_F(n))$ equivalent to the GRH

In this section we derive representations of the real part of the nth generalized Li coefficient attached to $F \in \mathcal{S}_0^{\sharp\flat}$ equivalent to the GRH for F.

In Theorem 2 below we prove that the GRH is equivalent to (7) and (6), for all positive integers n. Then, we prove that, under certain mild assumptions on the location of the imaginary part of the first eventual non-trivial zero of $F \in \mathcal{S}^{\sharp\flat}$ which is off the critical line, equations (7) and (6) with $n=1$ or $n=2$ actually yield the GRH for F.

Theorem 2. *The GRH for $F \in \mathcal{S}_0^{\sharp\flat}$ is equivalent to representation* (7) *of* $\mathrm{Re}(\lambda_F(n))$ *for all positive integers n, or, equivalently, to representation* (6) *of* $\mathrm{Re}(\lambda_F(n))$ *for all positive integers n.*

Let us note here that each zero in the sum (7) should be taken according to its multiplicity.

Proof. First, we assume that the GRH holds true, hence each $\rho \in Z(F)$ can be represented as $\rho = 1/2 + i\gamma$, $\gamma \in \mathbb{R}$.

We start with the definition of the Li coefficient and the formula $\lambda_F(n) = \overline{\lambda_F(-n)}$, hence

$$\begin{aligned}
2\mathrm{Re}(\lambda_F(n)) &= \lambda_F(n) + \overline{\lambda_F(n)} = \lambda_F(n) + \lambda_F(-n) \\
&= \lim_{T\to+\infty} \sum_{|\gamma|\leqslant T} \left(2 - \left(1 - \frac{1}{1/2+i\gamma}\right)^n - \left(1 - \frac{1}{1/2+i\gamma}\right)^{-n}\right) \\
&= \sum_{\rho=1/2+i\gamma} \left(2 - \left(\frac{2\gamma+i}{2\gamma-i}\right)^n - \left(\frac{2\gamma+i}{2\gamma-i}\right)^{-n}\right).
\end{aligned}$$

It is obvious that

$$\left|\left(\frac{2\gamma+i}{2\gamma-i}\right)\right| = 1$$

for all $\gamma = \mathrm{Im}(\rho)$, hence

$$\left(\frac{2\gamma+i}{2\gamma-i}\right)^n + \left(\frac{2\gamma+i}{2\gamma-i}\right)^{-n} = 2\cos(n\theta(\gamma)) = 2T_n(\cos(\theta(\gamma))),$$

where $\theta(\gamma)$ is the argument of $\frac{2\gamma+i}{2\gamma-i}$, when $\gamma \geqslant 0$, or minus the argument of $\frac{2\gamma+i}{2\gamma-i}$, when $\gamma < 0$. (Here, we take the principal value of the argument which lies in the interval $(-\pi, \pi]$.)

In both cases, due to the parity of the cosine function, we have

$$\cos(\theta(\gamma)) = \mathrm{Re}\left(\frac{2\gamma+i}{2\gamma-i}\right) = \frac{4\gamma^2-1}{4\gamma^2+1},$$

thus, (7) holds true.

Now, we prove the equation

$$\begin{aligned}
\sum_{\rho=\sigma+i\gamma\in Z(F)} \left(1 - T_n\left(\frac{4\gamma^2-1}{4\gamma^2+1}\right)\right) = 16n \int_0^\infty \frac{xN_F(x)}{(4x^2+1)^2} U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right) dx \\
- (1-(-1)^n)N_F(0). \qquad (20)
\end{aligned}$$

We start with

$$\sum_{\rho=\sigma+i\gamma\in Z(F)} \left(1 - T_n\left(\frac{4\gamma^2-1}{4\gamma^2+1}\right)\right) = \lim_{T\to\infty} \int_0^T \left(1 - T_n\left(\frac{4x^2-1}{4x^2+1}\right)\right) dN_F(x). \qquad (21)$$

A simple computation shows that

$$1 - T_n\left(\frac{4x^2-1}{4x^2+1}\right) \sim \frac{n^2}{x^2} + O_n(x^{-4}), \qquad \text{as } x \to \infty, \qquad (22)$$

hence, integrating by parts in (21), employing equation (18), relation $T_n(-1) = (-1)^n$ and having in mind that $N_F(x) = O(x \log x)$, as $x \to \infty$, we immediately deduce equation (20).

It is left to prove the converse, i.e. that equation (7) for all positive n implies the GRH.

This follows trivially from the fact that $T_n(x) = \cos(n\theta)$, for $x = \cos(\theta)$, hence the right-hand side of equation (7) is always non-negative. Therefore, equation (7) yields that $\mathrm{Re}(\lambda_F(n)) \geqslant 0$ for all positive n, a condition equivalent to the GRH. This completes the proof. ■

In some cases, we may deduce that more general statement holds true. Namely, we have the following theorem treating the cases $n = 1$ and $n = 2$.

Theorem 3.

(i) *The GRH for $F \in \mathcal{S}^{\sharp\flat}_{1/\sqrt{3}}$ is equivalent to the equation*

$$\mathrm{Re}(\lambda_F(1)) = 16 \int_0^\infty \frac{x}{(4x^2+1)^2} N_F(x)dx$$

or to the equation

$$\mathrm{Re}(\lambda_F(1)) = \sum_{\rho=\sigma+i\gamma \in Z(F)} \frac{2}{4\gamma^2+1}. \tag{23}$$

(ii) *If, additionally, $F \in \mathcal{S}^{\sharp\flat}_{\sqrt{6}}$, then, the GRH for F is also equivalent to the equation*

$$\mathrm{Re}(\lambda_F(2)) = 64 \int_0^\infty \frac{x(4x^2-1)}{(4x^2+1)^3} N_F(x)dx$$

or to the equation

$$\mathrm{Re}(\lambda_F(2)) = \sum_{\rho=\sigma+i\gamma \in Z(F)} \frac{32\gamma^2}{(4\gamma^2+1)^2}. \tag{24}$$

Proof. From the proof of Theorem 2 and the fact that $T_1(x) = x$ we see that, in order to prove the first statement, it is sufficient to prove that (23) yields the GRH. Analogously, in order to prove the second statement, it is sufficient to prove that (24) yields the GRH.

Let us denote by $\rho = \sigma + i\gamma$ the non-trivial zeros of F. Since the non-trivial zeros of F come in pairs ρ and $1 - \overline{\rho}$, we have

$$2\mathrm{Re}(\lambda_F(n)) = \lambda_F(n) + \lambda_F(-n) = \sum_{\rho \in Z(F)} \left(2 - \left(\frac{\sigma - 1 + i\gamma}{\sigma + i\gamma}\right)^n - \left(\frac{1 - \sigma + i\gamma}{-\sigma + i\gamma}\right)^n\right).$$

The two terms on the right hand side of the above equation are complex conjugates, hence we get

$$\mathrm{Re}(\lambda_F(n)) = \sum_{\rho \in Z(F)} \left(1 - \left(\frac{(\sigma-1)^2+\gamma^2}{\sigma^2+\gamma^2}\right)^{\frac{n}{2}} T_n\left(\frac{\sigma(\sigma-1)+\gamma^2}{\sqrt{(\sigma^2+\gamma^2)((\sigma-1)^2+\gamma^2)}}\right)\right).$$

We put

$$g_{n,\gamma}(\sigma)=\left(\frac{(\sigma-1)^2+\gamma^2}{\sigma^2+\gamma^2}\right)^{n/2}T_n\left(\frac{\sigma(\sigma-1)+\gamma^2}{\sqrt{(\sigma^2+\gamma^2)((\sigma-1)^2+\gamma^2)}}\right)$$

and let $\gamma_F>0$ be such that all eventual non-trivial zeros of F that lie off the critical line have the absolute value of the imaginary part bigger than γ_F. It is obvious that

$$g_{n,\gamma}(1/2)=T_n\left(\frac{4\gamma^2-1}{4\gamma^2+1}\right),$$

hence, it remains to prove that the equation

$$\sum_{|\gamma|>\gamma_F}(1-g_{n,\gamma}(\sigma))=\sum_{|\gamma|>\gamma_F}\left(1-T_n\left(\frac{4\gamma^2-1}{4\gamma^2+1}\right)\right), \tag{25}$$

yields that σ must be equal to $1/2$.

Employing the fact that $Z(F)=1-\overline{Z(F)}$ and that $g_{n,-\gamma}(\sigma)=g_{n,\gamma}(\sigma)$, we may write the sum on the left hand side of equation (25) as

$$\frac{1}{2}\sum_{|\gamma|>\gamma_F}\left(2-(g_{n,\gamma}(\sigma)+g_{n,\gamma}(1-\sigma))\right),$$

hence it is sufficient to prove that

$$\frac{1}{2}\sum_{|\gamma|>\gamma_F}\left(2-(g_{n,\gamma}(\sigma)+g_{n,\gamma}(1-\sigma))\right)=\sum_{|\gamma|>\gamma_F}(1-g_{n,\gamma}(1/2))$$

if and only if $\sigma=1/2$, for $n=1,2$.

Now we consider the two cases.

(i) When $n=1$, by the assumption of the theorem, we have $\gamma_F=1/\sqrt{3}$ and the second derivative of $1-g_{1,\gamma}(\sigma)$ is equal to a product of a positive factor and $\sigma^2-3\gamma^2$, which is negative for $\sigma\in[0,1]$ and $|\gamma|>1/\sqrt{3}$. Hence, the function $1-g_{1,\gamma}(\sigma)$ is strictly concave, meaning that $2-(g_{1,\gamma}(\sigma)+g_{1,\gamma}(1-\sigma))\leqslant 2\,(1-g_{1,\gamma}(1/2))$ for all $\sigma\in[0,1]$, $|\gamma|>1/\sqrt{3}$, and the equality holds true if and only if $\sigma=1/2$. Therefore, equation (23) yields the GRH.

(ii) In the case $n=2$, we have $\gamma_F=\sqrt{6}$. The second derivative of $1-g_{2,\gamma}(\sigma)$ is equal to a product of a positive factor and the expression $-[2(3-2\sigma)\sigma^4+4\gamma^2\sigma^2(-9+2\sigma)+6\gamma^4(1+2\sigma)]$, which is negative for $|\gamma|>\sqrt{6}$ and $\sigma\in[0,1]$. Hence, the function $1-g_{2,\gamma}(\sigma)$ is strictly concave, meaning that $2-(g_{2,\gamma}(\sigma)+g_{2,\gamma}(1-\sigma))\leqslant 2\,(1-g_{2,\gamma}(1/2))$ for all $\sigma\in[0,1]$, $|\gamma|>\sqrt{6}$, and the equality holds true if and only if $\sigma=1/2$. Therefore, equation (24) yields the GRH.

The proof is complete. ∎

Remark 4. From the recurrence relations satisfied by the Chebyshev polynomials of the first and the second kind, it is obvious that, for all $n \geqslant 2$ we can write $T_n(x) = 2^{n-1}x + P_{n-2}(x)$ and $U_n(x) = 2^n x + Q_{n-2}(x)$, for some polynomials $P_{n-2}(x)$ and $Q_{n-2}(x)$ of degree $n-2$. Then, a short computation using the relations (18) and (19) yields that $(g_{n,\gamma}(\sigma))''$, the second derivative of the function $g_{n,\gamma}(\sigma)$ defined in the proof of Theorem 3, is equal to a product of positive factors and a polynomial in γ of degree $2n$ with positive leading coefficient. Therefore, for any positive integer n, there exists a real number $t(n)$ such that for all γ satisfying the inequality $|\gamma| > t(n)$ and all $\sigma \in [0,1]$ we have $(1 - g_{n,\gamma}(\sigma))'' < 0$.

This implies that for $|\gamma| > t(n)$ and $\sigma \in [0,1]$ we have $2 - (g_{n,\gamma}(\sigma) + g_{n,\gamma}(1-\sigma)) \leqslant 2\,(1 - g_{n,\gamma}(1/2))$ and the equality holds true if and only if $\sigma = 1/2$. Hence, proceeding analogously as in the proof of Theorem 3 we may obtain more formulas equivalent to the GRH for functions $F \in \mathcal{S}^{\sharp\flat}_{t(n)}$. In other words, we have proved that for $F \in \mathcal{S}^{\sharp\flat}_{t(n)}$, the equation

$$\operatorname{Re}(\lambda_F(n)) = 16n \int_0^{\infty} \frac{x N_F(x)}{(4x^2+1)^2} U_{n-1}\left(\frac{(4x^2-1)}{(4x^2+1)}\right) dx$$

or, equivalently, the equation

$$\operatorname{Re}(\lambda_F(n)) = \sum_{\rho=\sigma+i\gamma \in Z(F)} \left(1 - T_n\left(\frac{4\gamma^2-1}{4\gamma^2+1}\right)\right)$$

both yield the GRH.

The sequence $t(n)$ is increasing, hence the assumption that $F \in \mathcal{S}^{\sharp\flat}_{t(n)}$ becomes more restrictive for large n.

An immediate consequence of the first part of Theorem 3 is the following corollary.

Corollary 5. *The GRH for $F \in \mathcal{S}^{\sharp\flat}_{1/\sqrt{3}}$ is equivalent to the equation*

$$\operatorname{Re}\left(\frac{\xi_F'}{\xi_F}(0)\right) = -2 \sum_{\rho=\sigma+i\gamma \in Z(F)} \frac{1}{4\gamma^2+1}$$

or to the equation

$$\operatorname{Re}\left(\frac{\xi_F'}{\xi_F}(0)\right) = -16 \int_0^{\infty} \frac{x}{(4x^2+1)^2} N_F(x) dx$$

Proof. The proof follows from equation (8) with $z = 0$, which implies that

$$\frac{\xi_F'}{\xi_F}(0) = -\sum_{\rho \in Z(F)}{}^{*} \frac{1}{\rho} = -\lambda_F(1) \tag{26}$$

and the first part of Theorem 3. ■

Example 6. Let a, b be positive real numbers such that $a+b=1$, $a>1/2$, and, for an arbitrary integer $m>1$ define the function $G_m(s)=(1-m^{a-s})(1-m^{b-s})$. Function $G_2(s)$ is introduced in [19], p. 28 as an example of function satisfying axioms (i)-(iv) but violating GRH. It is obvious that for $\mathrm{Re}(s)>1$ logarithmic derivative $G'_m(s)/G_m(s)$ possesses an absolutely convergent Euler sum representation

$$\frac{G'_m}{G_m}(s)=\log m\sum_{k=1}^{\infty}\frac{m^{ak}+m^{bk}}{(m^k)^s},$$

hence $G_m\in\mathcal{S}^{\sharp\flat}$. Moreover, G_m satisfies the functional equation $G_m(s)=m\cdot m^{-2s}G_m(1-s)$, hence $\xi_{G_m}(s)=m^sG_m(s)$.

An immediate computation shows that

$$\lambda_{G_m}(1)=-\frac{\xi'_{G_m}}{\xi_{G_m}}(0)=\log m\left(\frac{m^a}{m^a-1}+\frac{m^b}{m^b-1}-1\right)=\frac{m^a(m-1)\log m}{(m^a-1)(m-m^a)}>0,$$

hence the Li criterion for $n=1$ is fulfilled and $\lambda_{G_m}(1)$ is a function of $a\in(1/2,1)$. On the other hand, imaginary parts of zeros of $G_m(s)$ are obviously independent of a, hence the positive sum which appears in (23) is apparently different from the above expression, thus showing that function G violates GRH.

If one takes $a=b=1/2$ and defines $G_m(s)=(1-m^{1/2-s})^2$, then $G_m(s)\in\mathcal{S}_0^{\sharp\flat}$ and satisfies generalized Riemann hypothesis, since, obviously, the zeros of G_m are $1/2+2k\pi i/\log m$, where $k\in\mathbb{Z}$ and each zero has multiplicity two. Moreover,

$$-\frac{\xi'_{G_m}}{\xi_{G_m}}(0)=\frac{\sqrt{m}+1}{\sqrt{m}-1}\log m,$$

hence, formula (23) yields an interesting summation formula

$$\frac{\sqrt{m}+1}{\sqrt{m}-1}=4\sum_{k=-\infty}^{+\infty}\frac{\log m}{16k^2\pi^2+\log^2 m},$$

valid for all integers $m>1$.

Example 7. Let χ be an odd Dirichlet character and, proceeding as in [19], p. 28, we consider the function $G(s)=L(2s-1/2,\chi)$, where $L(s,\chi)$ is the Dirichlet $L-$function associated to primitive odd character χ of modulus q. For $\delta\in(0,1/4)$ one may easily show that the function $F(s)=G(s-\delta)G(s+\delta)$ belongs to the class $\mathcal{S}^{\sharp\flat}$ and violates the Ramanujan conjecture.

Moreover, $\rho\in Z(F)$ if and only if $\rho-\delta$ or $\rho+\delta$ belongs to $Z(G)$. A simple computation shows that the completed functions $\xi_F(s)$ and $\xi_G(s)$ associated to F and G respectively are related by $\xi_F(s)=\xi_G(s-\delta)\xi_G(s+\delta)$, hence

$$\frac{\xi'_F}{\xi_F}(0)=\frac{\xi'_G}{\xi_G}(\delta)+\frac{\xi'_G}{\xi_G}(-\delta)=\sum_{\rho\in Z(G)}\frac{-2\rho}{(\rho-\delta)(\rho+\delta)}.$$

It is interesting to notice that functions $F(s)$ and $G(s)^2$ have the same degree and conductor, hence the same asymptotic behavior of the archimedean part of the nth Li coefficient (which grows as $2n\log n+2[(\gamma-1)+\log(q/\pi)]n+O(1)$, as $n\to\infty$). Moreover, under GRH for $L(s,\chi)$ it is obvious that $F(s)$ violates GRH.

In this example we show how to deduce that $F(s)$ violates GRH using our results, under assumption that the smallest positive imaginary part y_0 of the non-trivial zero $x_0+iy_0\in Z(G)$ is such that $y_0>\sqrt{3}/4$. Then, $\delta^2+3y^2>x^2$ for all zeros $x+iy\in Z(G)$, which yields inequality

$$\frac{x(x^2+y^2-\delta^2)}{(x^2+y^2)^2+\delta^2(\delta^2+2y^2-2x^2)}<\frac{x}{x^2+y^2},$$

for all $x+iy\in Z(G)$.

Moreover, for a fixed $y\geqslant y_0>\sqrt{3}/4$, function $g(x)=\frac{x}{x^2+y^2}$ is such that $g''(x)<0$ for all $x\in(1/4,3/4)$, hence $\frac{1}{2}g(x)+\frac{1}{2}g(1-x)<g(1/2)$. Therefore, we conclude that

$$\begin{aligned}\operatorname{Re}(\lambda_F(1))=\operatorname{Re}\left(-\frac{\xi_F'}{\xi_F}(0)\right)&=2\sum_{x+iy\in Z(G)}\frac{x(x^2+y^2-\delta^2)}{(x^2+y^2)^2+\delta^2(\delta^2+2y^2-2x^2)}\\&<2\sum_{x+iy\in Z(G)}\frac{x}{x^2+y^2}<4\sum_{x+iy\in Z(G)}\frac{1}{1+4y^2}=2\sum_{\sigma+i\gamma\in Z(F)}\frac{1}{1+4\gamma^2},\end{aligned}$$

hence, Theorem 2 with $n=1$ yields that $F(s)$ violates GRH.

Remark 8. When all coefficients $a_F(n)$ in the Dirichlet series representation (1) are real, the reflection principle and the functional equation imply that $Z(F)=1-Z(F)=1-\overline{Z(F)}$, hence, the generalized Li coefficients are real.

Moreover, under the GRH, it is possible to apply methods from [17] (in the case of Dirichlet $L-$functions with real coefficients) and [18] (in the case of Hecke L-functions) and obtain that, for any $F\in\mathcal{S}^{\sharp\flat}$ with real coefficients $a_F(n)$ and such that $N_F(0)=0$, we have

$$\lambda_F(n)=32n\int_0^\infty\frac{x}{(4x^2+1)^2}N_F^+(x)U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right)dx$$

and

$$\lambda_F(n)=2\sum_{\gamma>0}\left(1-T_n\left(\frac{4\gamma^2-1}{4\gamma^2+1}\right)\right),$$

where each zero in the last sum is counted according to its multiplicity.

In this paper, we prove a more general statement, i.e., we show that the above representations of $\lambda_F(n)$ are equivalent to the GRH. Moreover, the method of proof of our Theorems 2 and 3 is different from the methods given in [17] and [18] (under the GRH).

4. An integral formula for the generalized Li coefficients

In this section we partially evaluate the integral in (6) and write equation (6) as a sum of integral and oscillatory part. Namely, we prove the following theorem.

Theorem 9. *The GRH for $F \in \mathcal{S}_0^{\sharp\flat}$ is equivalent to the formula*

$$\begin{aligned}\operatorname{Re}(\lambda_F(n)) = n\log Q_F + (1-(-1)^n)(2m_F - N_F(0)) + \mathcal{I}_F(n) \\ + 16n\int_0^{+\infty} \frac{xS_F(x)}{(4x^2+1)^2} U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right)dx,\end{aligned} \tag{27}$$

for all positive integers n, where the function $S_F(x)$ is defined in Section 2.2. and

$$\mathcal{I}_F = \operatorname{Re}\left(\sum_{j=1}^{r}\left[n\lambda_j\psi(\lambda_j+\mu_j) + \sum_{k=2}^{n}\binom{n}{k}\frac{\lambda_j^k}{(k-1)!}\psi^{(k-1)}(\lambda_j+\mu_j)\right]\right).$$

In the case when $n=1$ the second sum in the above equation is equal to zero.

Proof. Let us put

$$G_n(x) := 1 - T_n\left(\frac{4x^2-1}{4x^2+1}\right).$$

Then, using equation (10) we get

$$\begin{aligned}\operatorname{Re}(\lambda_F(n)) &= -\int_0^{+\infty} N_F(x)G_n'(x)dx - (1-(-1)^n)N_F(0) \\ &= I_1(n) + I_2(n) + I_3(n) - (1-(-1)^n)N_F(0),\end{aligned}$$

where

$$I_1(n) := -\int_0^{+\infty} G_n'(x)\left(2m_F + \frac{2\log Q_F}{\pi}x\right)dx,$$

$$\begin{aligned}I_2(n) := -\frac{1}{\pi}\sum_{j=1}^{r}\operatorname{Im}\Bigg(\int_0^{+\infty} G_n'(x)\Bigg(\log\Gamma\left(\lambda_j\left(\frac12+ix\right)+\mu_j\right) \\ +\log\Gamma\left(\lambda_j\left(\frac12+ix\right)+\overline{\mu_j}\right)\Bigg)dx\Bigg)\end{aligned}$$

and

$$I_3(n) := -\int_0^{+\infty} G_n'(x)S_F(x)dx = 16n\int_0^{+\infty}\frac{xS_F(x)}{(4x^2+1)^2}U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right)dx.$$

Now, we evaluate integrals $I_1(n)$ and $I_2(n)$. We start with the generating function (17) for the sequence of Chebyshev polynomials of the first kind and define

$$G(x,t) = \sum_{n=0}^{\infty} G_n(x)t^n = \frac{2t(1+t)}{(1-t)}\cdot\frac{1}{(2(1-t)x)^2+(1+t)^2}.$$

Then, $G(x,t)$ is differentiable with respect to $x \in \mathbb{R}$ for all $t \in (0,1)$ and $G'_x(x,t) = \sum_{n=0}^{\infty} G'_n(x)t^n$.

Moreover, equation (22) together with the recurrence relation (19) yield that

$$G'_n(x) = -\frac{16nx}{(4x^2+1)^2} U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right) = O_n\left(\frac{x}{(4x^2+1)^2}\right),$$

as $x \to \pm\infty$, where the implied constant depends polynomially upon n, hence we may interchange the sum and the integral in order to deduce that

$$\sum_{n=0}^{\infty} I_1(n)t^n = -\int_0^{+\infty} G'_x(x,t)\left(2m_F + \frac{2\log Q_F}{\pi}x\right)dx \tag{28}$$

for all $t \in (0,1)$.

Analogously, since $|\log\Gamma(z)| \sim |z|\log|z|$, as $|z| \to \infty$, interchanging the sum and the integral, we get

$$\begin{aligned}\sum_{n=0}^{\infty} I_2(n)t^n = -\frac{1}{\pi}\sum_{j=1}^{r}\operatorname{Im}\Bigg(\int_0^{+\infty} G'_x(x,t)\Bigg(\log\Gamma\left(\lambda_j\left(\frac{1}{2}+ix\right)+\mu_j\right)\\ +\log\Gamma\left(\lambda_j\left(\frac{1}{2}+ix\right)+\overline{\mu_j}\right)\Bigg)dx\Bigg),\end{aligned} \tag{29}$$

for all $t \in (0,1)$.

Employing equation (28) we get

$$\begin{aligned}\sum_{n=0}^{\infty} I_1(n)t^n &= -2m_F\int_0^{+\infty} G'_x(x,t)dx - \frac{2\log Q_F}{\pi}\int_0^{+\infty} G'_x(x,t)xdx\\ &= 4m_F\frac{t}{(1+t)(1-t)} + \log Q_F\frac{t}{(1-t)^2}\\ &= \sum_{n=1}^{\infty}\left(n\log Q_F + 2m_F(1-(-1)^n)\right)t^n,\end{aligned}$$

hence

$$I_1(n) = n\log Q_F + 2m_F(1-(-1)^n). \tag{30}$$

Therefore, in order to prove the theorem, it is left to show that $I_2(n) = \mathcal{I}_F(n)$. First, we note that

$$\lim_{x\to\infty}\left(G(x,t)\log\Gamma\left(\lambda_j\left(\frac{1}{2}+ix\right)+\mu_j\right)\right) = 0,$$

for all $j = 1, ..., r$. Moreover, we have

$$\operatorname{Im}\left(G(0,t)\sum_{j=1}^{r}\left(\log\Gamma\left(\lambda_j/2+\mu_j\right)+\log\Gamma\left(\lambda_j/2+\overline{\mu_j}\right)\right)\right) = 0,$$

hence, integration by parts in (29) yields the equation

$$\sum_{n=0}^{\infty} I_2(n)t^n = \frac{1}{\pi}\sum_{j=1}^{r}\lambda_j \mathrm{Re}\left(\int_0^{+\infty} G(x,t)\left(\psi\left(\lambda_j\left(\frac{1}{2}+ix\right)+\mu_j\right)\right.\right.$$
$$\left.\left.+\psi\left(\lambda_j\left(\frac{1}{2}+ix\right)+\overline{\mu_j}\right)\right)dx\right).$$

Next, we use the fact that the function $G(x,t)$ is even function of x to deduce that

$$\sum_{n=0}^{\infty} I_2(n)t^n = \frac{1}{\pi}\sum_{j=1}^{r}\lambda_j \mathrm{Re}\left(\int_{-\infty}^{+\infty} G(x,t)\psi\left(\lambda_j\left(\frac{1}{2}+ix\right)+\mu_j\right)dx\right), \tag{31}$$

for all $t \in (0,1)$.

Since the function $\psi(z)$ grows as $\log|z|$, as $|z| \to \infty$, in order to compute the integral on the right hand side of (31), we integrate the function

$$g_j(z,t) := G(z,t)\psi\left(\lambda_j\left(\frac{1}{2}+iz\right)+\mu_j\right)$$

along the negatively oriented contour C_R consisting of the segment of the real line from $-R$ to R and the half circle $\{z \in \mathbb{C} : |z| = R,\ \arg z \in [-\pi, 0]\}$, and then let $R \to +\infty$. Inside the contour C_R (and along C_R), the imaginary part of the argument z is non-positive, hence

$$\mathrm{Re}\left(\lambda_j\left(\frac{1}{2}+iz\right)+\mu_j\right) \geqslant \mathrm{Re}(\lambda_j/2+\mu_j) > 0$$

(by the axiom (iii')). Therefore, the function $\psi\left(\lambda_j\left(\frac{1}{2}+iz\right)+\mu_j\right)$ is holomorphic inside C_R and along C_R. The function $G(z,t)$ is holomorphic along C_R and possesses a simple pole at $z_t = -\frac{1+t}{2(1-t)}i$ inside C_R, hence the function $g_j(z,t)$ is holomorphic on C_R and inside C_R except for a simple pole at z_t with the residue

$$\mathrm{Res}_{z=z_t} g_j(z,t) = \frac{it}{2(1-t)^2}\psi\left(\frac{\lambda_j}{(1-t)}+\mu_j\right).$$

The contour C_R is negatively oriented, hence, applying the calculus of residues we get

$$\int_{C_R} G(z,t)\psi\left(\lambda_j\left(\frac{1}{2}+iz\right)+\mu_j\right)dz = -2\pi i\mathrm{Res}_{z=z_t} g_j(z,t)$$
$$= \frac{\pi t}{(1-t)^2}\psi\left(\frac{\lambda_j}{(1-t)}+\mu_j\right).$$

The integral over C_R is equal to the sum of the integral over the segment of the real line from $-R$ to R and the integral along the half-circle $|z| = R$, where arg(z) takes values from 0 to $-\pi$. Since $|\psi(z)| \sim \log|z|$, as $|z| \to \infty$, it is obvious that the integral along the half-circle tends to zero as $R \to +\infty$, hence, letting $R \to +\infty$ we deduce that

$$\int_{-\infty}^{+\infty} G(x,t)\psi\left(\lambda_j\left(\frac{1}{2}+ix\right)+\mu_j\right)dx = \frac{\pi t}{(1-t)^2}\psi\left(\frac{\lambda_j}{(1-t)}+\mu_j\right).$$

This together with (31) yields

$$\sum_{n=0}^{\infty} I_2(n)t^n = \frac{t}{(1-t)^2}\mathrm{Re}\left(\sum_{j=1}^{r}\lambda_j\psi\left(\frac{\lambda_j}{(1-t)}+\mu_j\right)\right), \tag{32}$$

for all $t \in (0,1)$.

Now, we compute the Taylor series expansion of the right hand side of (32) at $t=0$. We put $y_j = \lambda_j(1-t)^{-1}+\mu_j$, $j=1,...,r$ and get

$$\begin{aligned}\frac{d}{dy_j}\log\Gamma(y_j) &= \psi(\lambda_j+\mu_j) + \sum_{k=1}^{\infty}\frac{\psi^{(k)}(\lambda_j+\mu_j)}{k!}(y_j-\lambda_j-\mu_j)\\ &= \psi(\lambda_j+\mu_j) + \sum_{k=1}^{\infty}\frac{\psi^{(k)}(\lambda_j+\mu_j)}{k!}\frac{\lambda_j^k t^k}{(1-t)^k},\end{aligned}$$

hence

$$\begin{aligned}\frac{\lambda_j t}{(1-t)^2}\frac{d}{dt}\log\Gamma\left(\frac{\lambda_j}{(1-t)}+\mu_j\right) &= \frac{\lambda_j t}{(1-t)^2}\psi(\lambda_j+\mu_j)\\ &\quad + \sum_{k=1}^{\infty}\frac{\psi^{(k)}(\lambda_j+\mu_j)}{k!}\frac{\lambda_j^{k+1}t^{k+1}}{(1-t)^{k+2}}.\end{aligned}$$

Using the identity

$$\frac{1}{(1-t)^{k+2}} = \sum_{i=0}^{\infty}\frac{(i+k+1)!}{i!(k+1)!}t^i,$$

for $k \geqslant 0$ we obtain

$$\begin{aligned}\frac{\lambda_j t}{(1-t)^2}\frac{d}{dt}\log\Gamma\left(\frac{\lambda_j}{(1-t)}+\mu_j\right) &= \lambda_j\sum_{n=1}^{\infty}n\psi(\lambda_j+\mu_j)t^n\\ &\quad + \sum_{i=0}^{\infty}\sum_{k=1}^{\infty}\frac{\psi^{(k)}(\lambda_j+\mu_j)}{k!}\lambda_j^{k+1}\frac{(i+k+1)!}{i!(k+1)!}t^{i+k+1}.\end{aligned} \tag{33}$$

Putting $k+1=l$ and taking $n=i+l$, we may write the second sum in equation (33) as

$$\sum_{n=2}^{\infty}\left(\sum_{l=2}^{n}\frac{\psi^{(l-1)}(\lambda_j+\mu_j)}{(l-1)!}\lambda_j^l\binom{n}{l}\right)t^n.$$

Inserting the above equation in (33), together with (32) yields that $I_2(n) = \mathcal{I}_F(n)$. The proof is complete. ∎

In the case when $n = 1$, from the above theorem and the first part of Theorem 3 we deduce the following generalization of the Volchkov criterion.

Corollary 10. *The GRH for $F \in \mathcal{S}^{\sharp\flat}_{1/\sqrt{3}}$ is equivalent to the formula*

$$\int_0^{+\infty} \frac{xS_F(x)}{(x^2+1/4)^2}dx = -\mathrm{Re}\left(\frac{\xi_F'}{\xi_F}(0)\right) - \log Q_F - 4m_F - \mathrm{Re}\left(\sum_{j=1}^{r} \lambda_j \psi(\lambda_j + \mu_j)\right).$$

Proof. An immediate consequence of (27) with $n = 1$, Theorem 3 and relation (26). ■

Example 11. In this example we show how to derive the Volchkov criterion (5) for the Riemann zeta function as a special case of Corollary 10.

For the Riemann zeta function, we have $r = 1$, $\lambda_1 = 1/2$, $\mu_1 = 0$, $Q_\zeta = \pi^{-1/2}$, $m_\zeta = 1$, $N_\zeta(0) = 0$ and the first non-trivial zero (which is on the critical line) has imaginary part bigger than 14, hence assumptions of Corollary 10 are satisfied. Furthermore, we have $I_2(1) = \frac{1}{2}\psi(\frac{1}{2}) = -\frac{\gamma}{2} - \log 2$.

The coefficients in the Dirichlet series representation of $\zeta(s)$ are real, hence $\lambda_\zeta(1)$ is real and, according to the last paragraph in Section 2.2., $S_\zeta(x)$ is equal to $\frac{2}{\pi}\arg\zeta(1/2+ix)$. Therefore, Corollary 10 yields that the Riemann Hypothesis is equivalent to the statement

$$\frac{1}{\pi}\int_0^{+\infty} \frac{2x\arg\zeta(\frac{1}{2}+ix)}{(x^2+1/4)^2}dx = -\frac{\xi_\zeta'}{\xi_\zeta}(0) - 4 + \frac{\gamma}{2} + \log(2\sqrt{\pi}).$$

Now, using that

$$-\frac{\xi_\zeta'}{\xi_\zeta}(0) = \frac{\gamma}{2} + 1 - \frac{1}{2}\log(4\pi),$$

we deduce that the Riemann Hypothesis is equivalent to (5).

5. The GRH and the generalized Euler-Stieltjes constants of the second kind

The sum of generalized Euler–Stieltjes constants of the second kind is closely related to the Li criterion for the GRH, as it appears in the non-archimedean part (13) of the Li coefficient $\lambda_F(-n)$, $n \geqslant 1$.

As an immediate consequence of Theorems 3 and 9, employing the relation $\mathrm{Re}(\lambda_F(n)) = \mathrm{Re}(\lambda_F(-n))$, we obtain an integral formula for the sum of $\mathrm{Re}(\gamma_F(l-1))$ appearing in (11) and equivalent to the GHR, which we state as the following corollary.

Corollary 12. *The GRH for* $F \in \mathcal{S}_0^{\sharp\flat}$ *is equivalent to the formula*

$$\sum_{l=1}^{n} \binom{n}{l} \mathrm{Re}\,(\gamma_F(l-1)) = (1-(-1)^n)(2m_F - N_F(0)) - m_F \\ + 16n \int_0^{+\infty} \frac{xS_F(x)}{(4x^2+1)^2} U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right) dx, \tag{34}$$

for all positive integers n*, where* $S_F(x)$ *is defined in Section 2.2.*

Specially, for functions $F \in \mathcal{S}_{1/\sqrt{3}}^{\sharp\flat}$ *the GRH is equivalent to the formula*

$$\mathrm{Re}\,(\gamma_F(0)) = 3m_F + \int_0^{+\infty} \frac{xS_F(x)}{(x^2+1/4)^2} dx.$$

Moreover, Corollary 12 together with asymptotic relation (14) yield the following asymptotic integral criteria for the GRH.

Corollary 13. *The GRH for* $F \in \mathcal{S}_0^{\sharp\flat}$ *is equivalent to the formula*

$$\int_0^{+\infty} \frac{xS_F(x)}{(x^2+1/4)^2} U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right) dx = O\left(\frac{\log n}{\sqrt{n}}\right), \qquad \textit{as } n \to \infty. \tag{35}$$

In the special case, when coefficients in the Dirichlet series expansion (1) of $F \in \mathcal{S}^{\sharp\flat}$ are real and the function F possesses no non-trivial zeros on the real line, the coefficients $\gamma_F(l)$ are real and $S_F(x) = \frac{2}{\pi} \arg F(1/2+ix)$. In this case we have the following corollary.

Corollary 14. *Let* $F \in \mathcal{S}^{\sharp\flat}$ *be a function with real coefficients in the Dirichlet series representation* (1) *that has no Siegel zeros. Then, the GRH is equivalent to the formula*

$$\sum_{l=1}^{n} \binom{n}{l} \gamma_F(l-1) = m_F(1-2(-1)^n) \\ + \frac{32n}{\pi} \int_0^{+\infty} \frac{x \arg F(1/2+ix)}{(4x^2+1)^2} U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right) dx, \tag{36}$$

for all positive integers n*. Under additional assumption that* $F \in \mathcal{S}_{1/\sqrt{3}}^{\sharp\flat}$*, the GRH is equivalent to the relation*

$$\gamma_F(0) = 3m_F + \frac{2}{\pi} \int_0^{+\infty} \frac{x \arg F(1/2+ix)}{(x^2+1/4)^2} dx. \tag{37}$$

Remark 15. Assuming that the coefficients in the Dirichlet series representation (1) of the function F are real, under the GRH, the above formula provides an integral representation of the coefficients $\gamma_F(l)$ appearing in the Laurent (Taylor) series expansion of the function $F'(s)/F(s)$ at $s = 1$.

For example, when $n = 1$, under the GRH, the constant term $\gamma_F(0)$ in the Laurent (Taylor) series expansion of the function $F'(s)/F(s)$ at $s = 1$ possesses the integral representation (37). Plugging in $n = 2$ into the formula (36) we get, under GRH that

$$\gamma_F(1) = -7m_F + \frac{64}{\pi} \int_0^{+\infty} \frac{x(4x^2-3)\arg F(1/2+ix)}{(4x^2+1)^3} dx.$$

Proceeding inductively in n, using formula (36) it is possible to obtain an integral representation of all generalized Euler-Stieltjes constants of the second kind, under the GRH.

Remark 16. Equation (37) is equivalent to the Volchkov criteria (5) for the Riemann Hypothesis, since $\gamma_\zeta(0) = \gamma$, $m_\zeta = 1$ and $S_\zeta(x) = \frac{2}{\pi} \arg \zeta(1/2 + ix)$, hence Corollary 12 can also be viewed as a generalization of the Volchkov criteria to a larger class of functions.

6. An application to automorphic L-functions

In this section, we derive relations equivalent to the GRH for automorphic L-functions attached to irreducible, cuspidal unitary representations of $GL_N(\mathbb{Q})$.

In [14] it was shown that the (finite) automorphic L-function $L(s, \pi)$, defined by the product of its local factors (15) belongs to the class $\mathcal{S}^{\sharp\flat}$. Moreover, for $F(s) = L(s, \pi)$ (which is a function in $\mathcal{S}^{\sharp\flat}$) we have $r = N$, $Q_F = Q(\pi)^{1/2}\pi^{-N/2}$, $\lambda_j = 1/2$, $\mu_j = \frac{1}{2}k_j(\pi)$, $j = 1, ..., N$ and $d_F = N$. Furthermore, when $N = 1$ and π is trivial, $F(s) = L(s, \pi)$ reduces to the Riemann zeta function, hence, in this case $m_F = 1$. When $N \neq 1$ or π is not trivial, the function $F(s) = L(s, \pi)$ is holomorphic at $s = 1$, hence $m_F = 0$.

We put $\delta_\pi = 1$ if $N = 1$ and π is trivial and $\delta_\pi = 0$, otherwise. We denote by $S_\pi(T)$ the value of the function $\frac{1}{\pi} \arg L(s, \pi)$ obtained by continuous variation along the straight lines joining points $1/2 - iT$, $2 - iT$, $2 + iT$ and $1/2 + iT$.

The completed $L-$function $\Lambda(s, \pi)$ is non-vanishing on the line $\text{Re}(s) = 1$ (see [6]), hence, by the functional equation (16), $0 \notin Z(L(s, \pi))$, meaning that $F(s) = L(s, \pi) \in \mathcal{S}_0^{\sharp\flat}$.

The application of Theorem 9, Corollary 10, Corollary 12 and Corollary 13 to $F(s) = L(s, \pi)$ yields the following corollary.

Corollary 17. *We have*

(i) *The GRH for the function $L(s, \pi)$ is equivalent to the formula*

$$\text{Re}(\lambda_\pi(n)) = \frac{n}{2} \log\left(\frac{Q(\pi)}{\pi^N}\right) + (1 - (-1)^n)(2\delta_\pi - N_\pi(0)) + \mathcal{I}_\pi(n)$$
$$+ \ 16n \int_0^{+\infty} \frac{xS_\pi(x)}{(4x^2+1)^2} U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right) dx,$$

for all positive integers n, where

$$\mathcal{I}_\pi(n) = \mathrm{Re}\left(\sum_{j=1}^{N}\left[\frac{n}{2}\psi\left(\frac{1+k_j}{2}\right) + \sum_{k=2}^{n}\binom{n}{k}\frac{1}{2^k(k-1)!}\psi^{(k-1)}\left(\frac{1+k_j}{2}\right)\right]\right).$$

(ii) *The GRH for $L(s,\pi)$ is equivalent to the formula*

$$\sum_{l=1}^{n}\binom{n}{l}\mathrm{Re}\left(\gamma_\pi(l-1)\right) = (1-(-1)^n)(2\delta_\pi - N_\pi(0)) - \delta_\pi + 16n\int_0^{+\infty}\frac{x\ S_\pi(x)}{(4x^2+1)^2}U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right)dx,$$

for all positive integers n, where the coefficients $\gamma_\pi(l)$ denote the Euler-Stieltjes constants of the second kind associated to $L(s,\pi)$.

(iii) *The GRH for $L(s,\pi)$ is equivalent to the asymptotic formula*

$$\int_0^{+\infty}\frac{x\ S_\pi(x)}{(x^2+1/4)^2}U_{n-1}\left(\frac{4x^2-1}{4x^2+1}\right)dx = O\left(\frac{\log n}{\sqrt{n}}\right), \qquad as\ \ n\to\infty.$$

(iv) *Under additional assumption that $L(s,\pi)$ possesses no zeros off the critical line with the absolute value of the imaginary part less than or equal to $1/\sqrt{3}$, the GRH for $L(s,\pi)$ is equivalent to the equation*

$$\int_0^{+\infty}\frac{xS_\pi(x)}{(x^2+1/4)^2}dx = \mathrm{Re}(\lambda_\pi(1)) - \frac{1}{2}\log\left(\frac{Q(\pi)}{\pi^N}\right) - 4\delta_\pi - \mathrm{Re}\left(\sum_{j=1}^{N}\frac{1}{2}\psi\left(\frac{1}{2}(1+k_j)\right)\right),$$

or, equivalently, to the equation

$$\mathrm{Re}\left(\gamma_\pi(0)\right) = 3\delta_\pi + 16\int_0^{+\infty}\frac{xS_\pi(x)}{(4x^2+1)^2}dx.$$

7. Concluding remarks

In this section we present some problems that will be considered in a sequel to this article.

7.1. Numerical computations

The asymptotic relation (35) which is equivalent to the GRH seems to be very useful for the numerical investigations, due to oscillatory nature of the Chebyshev polynomials of the second kind. It would be interesting to see what would be results of such numerical computations in the case of Dirichlet $L-$functions associated to a certain character, or in the case of Hecke $L-$functions.

7.2. Generalization of results to the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$

The class $\mathcal{S}^{\sharp\flat}$ which is the main class of functions considered in this paper does not unconditionally contain all types of $L-$functions for which the GRH is assumed to hold true. In [5], the authors introduce the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1) \supseteqq \mathcal{S}^{\sharp\flat}$, as the set of all Dirichlet series $F(s)$ converging in some half-plane $\mathrm{Re}(s) > \sigma_0 \geqslant 1$, such that the meromorphic continuation of $F(s)$ to $\mathbb{C}$ is a meromorphic function of a finite order with at most finitely many poles, satisfying a functional equation relating values $F(s)$ with $\overline{F(\sigma_1 - \overline{s})}$ up to multiplicative gamma factors and such that the logarithmic derivative $F'(s)/F(s)$ has a Dirichlet series representation converging in the half plane $\mathrm{Re}(s) > \sigma_0 \geqslant 1$.

The assumptions posed on $F \in \mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ imply that all its non-trivial zeros lie in the strip $\sigma_1 - \sigma_0 \leqslant \mathrm{Re}(s) \leqslant \sigma_0$. The class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ contains products of suitable shifts of $L-$functions from $\mathcal{S}^{\sharp\flat}$, as well as products of shifts of certain $L-$functions possessing an Euler product representation that are not in $\mathcal{S}^{\sharp\flat}$ (such as the Rankin-Selberg $L-$functions).

We believe that results of this paper may be derived for functions in the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$, under some additional assumptions on the gamma factors appearing in the functional equation axiom.

7.3. The generalized τ-Li coefficients

The generalized τ-Li coefficients, for $\tau \in [1,2]$ were introduced by A. Droll in [3] as the $*-$convergent series

$$\lambda_F(n,\tau) = \sum_{\rho \in Z(F)}^{*} \left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right)$$

with the property that the inequality $\mathrm{Re}(\lambda_F(n,\tau)) \geqslant 0$ for all $n \in \mathbb{N}$ is equivalent to non-vanishing of $F \in \mathcal{S}^{\sharp\flat}$ in the half-plane $\mathrm{Re}(s) \geqslant \tau/2$.

We believe that it is possible to establish results similar to Theorem 2 for the generalized $\tau-$Li coefficients in the sense that non-vanishing of $F \in \mathcal{S}^{\sharp\flat}$ in the half-plane $\mathrm{Re}(s) \geqslant \tau/2$ is equivalent to a formula similar to (7) or, equivalently, (6). Furthermore, we expect that $\lambda_F(n,\tau)$ can be written as a sum of integral and oscillatory part which, in the case when $\tau = 1$ coincides with the expression (27) obtained in Theorem 9.

7.4. Deriving further relations equivalent to the GRH

Using Littlewood's theorem (see, e.g. [29, pages. 132-133]) and starting with with relations given in Theorem 3 or with equation (10), it is possible to obtain more relations equivalent to the GRH.

For example, integration by parts in (10) and evaluation of the function

$$I_F(x) = \int_0^x \arg F(1/2 + it)dt$$

using Littlewood's theorem along the rectangle with vertices $1/2$, $1/2+T$, $1/2+T+iT$ and $1/2+iT$, after letting $T \to +\infty$ immediately yields a generalization of equation (4) to a more general class of functions.

Moreover, there might be a connection between equation (10) and the integral

$$\frac{1}{2\pi}\int_{\mathrm{Re}(s)=1/2}\frac{\log|F(s)|}{|s|^2}|ds| \tag{38}$$

which would yield a generalization of the integral criteria of Balazard, Saias and Yor [2] to the class $\mathcal{S}_0^{\sharp\flat}$.

Finally, we believe that for $F \in \mathcal{S}^{\sharp\flat}$ with real Dirichlet series coefficients $a_F(n)$ and no Siegel zeros, the integral (38) taken along the line $\mathrm{Re}(s)=a$, for $a \in (0,1]$, can be represented as a mathematical expectation of a random variable $\log|F(X_a)|$, where X_a is a complex-valued random variable whose imaginary part has the symmetric Cauchy distribution with scale a. In this way it is possible to obtain a relation equivalent to the GRH for F in terms of Cesàro means of a certain ergodic transform, see [4] for a similar result in the case when $F(s)$ is the Riemann zeta function.

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