Functiones et Approximatio
57.1 (2017), 7–20
doi: 10.7169/facm/1567

ON THE INTEGRAL OF PRODUCTS OF HIGHER-ORDER BERNOULLI AND EULER POLYNOMIALS

Muhammet Cihat Dağli, Mümün Can

Abstract: In this paper, we derive a formula on the integral of products of the higher-order Euler polynomials. By the same method, similar relations are obtained for l higher-order Bernoulli polynomials and r higher-order Euler polynomials. Moreover, we establish a connection between these results and the generalized Dedekind sums and Hardy–Berndt sums.

Keywords: Bernoulli polynomials and numbers, Dedekind sums, integrals, recurrence relations.

1. Introduction

The Bernoulli polynomials $B_m(x)$ and Euler polynomials $E_m(x)$ are usually defined by means of the following generating functions:

$$\frac{ue^{uz}}{e^u-1}=\sum_{m=0}^{\infty}B_m(z)\frac{u^m}{m!}\qquad (|u|<2\pi)$$

and

$$\frac{2e^{uz}}{e^u+1}=\sum_{m=0}^{\infty}E_m(z)\frac{u^m}{m!}\qquad (|u|<\pi)\,.$$

In particular, the rational numbers $B_m=B_m(0)$ and integers $E_m=2^mE_m(1/2)$ are called Bernoulli numbers and Euler numbers, respectively.

As is well known, the Bernoulli and Euler polynomials play important roles in different areas of mathematics such as number theory, combinatorics, special functions and analysis.

This paper is primarily concerned with the higher-order Bernoulli and Euler polynomials. We derive a formula for the integral having r higher-order Euler polynomials and also for l higher-order Bernoulli and r higher-order Euler polynomials. The result is the corresponding generalization of some formulas discovered by Agoh

2010 Mathematics Subject Classification: primary: 11B68; secondary: 11F20

and Dilcher [1], Hu et al [7] and also [3, 6, 10, 12, 13, 14]. From our formula, we establish the connection between the sums of products of Euler (and Bernoulli and Euler) polynomials and the reciprocity formula for generalized Dedekind (and Hardy–Berndt) sums, motivated by Dağlı and Can [5].

We now turn to higher-order analogues. The higher-order Bernoulli polynomials $B_m^{(\alpha)}(z)$ and higher-order Euler polynomials $E_m^{(\alpha)}(z)$, each of degree m in z and in α, are defined by means of the generating functions [13]

$$\left(\frac{u}{e^u-1}\right)^{\alpha} e^{uz} = \sum_{m=0}^{\infty} B_m^{(\alpha)}(z)\frac{u^m}{m!}$$

and

$$\left(\frac{2}{e^u+1}\right)^{\alpha} e^{uz} = \sum_{m=0}^{\infty} E_m^{(\alpha)}(z)\frac{u^m}{m!},$$

respectively. For $\alpha = 1$, we have $B_m^{(1)}(z) = B_m(z)$ and $E_m^{(1)}(z) = E_m(z)$. They possess the differential property

$$\frac{d}{dz}B_m^{(\alpha)}(z) = mB_{m-1}^{(\alpha)}(z), \qquad \frac{d}{dz}E_m^{(\alpha)}(z) = mE_{m-1}^{(\alpha)}(z) \tag{1}$$

and reciprocal relations

$$B_m^{(\alpha)}(\alpha - z) = (-1)^m B_m^{(\alpha)}(z), \qquad E_m^{(\alpha)}(\alpha - z) = (-1)^m E_m^{(\alpha)}(z) \tag{2}$$

which imply $B_m^{(\alpha)}(\alpha/2) = 0$ and $E_m^{(\alpha)}(\alpha/2) = 0$ for odd m.

Also, we need the following expression of the Euler polynomials in terms of Bernoulli polynomials:

$$E_n(x) = \frac{2}{n+1}\left\{B_{n+1}(x) - 2^{n+1}B_{n+1}(x/2)\right\} \tag{3}$$

for $n \geqslant 0$.

We summarize this study as follows: we first give several convolution formulas for higher-order Bernoulli and Euler polynomials applying the generating function methods, motivated by [4]. We also derive an integral formula for higher-order Euler polynomials. By this, we extend the result of Hu et al [7] and Liu et al [9]. By the same method, similar relations are obtained for l higher-order Bernoulli polynomials and r higher-order Euler polynomials, as well. Moreover, we establish a connection between these results and the reciprocity formulas for generalized Dedekind sums $T_r(c,d)$ and generalized Hardy–Berndt sums $s_{3,r}(c,d)$ and $s_{4,r}(c,d)$.

2. Convolutions of higher-order Bernoulli and Euler polynomials

In this section, we give some convolutions involving higher-order Bernoulli and Euler polynomials which we will use in the next section.

Differentiating the generating function of higher-order Euler polynomials as follows

$$\begin{aligned}\frac{d}{du}\left(\left(\frac{2}{e^u+1}\right)^n e^{uz}\right) &= z\frac{2^n e^{uz}}{(e^u+1)^n} - n\frac{2^n e^{u(z+1)}}{(e^u+1)^{n+1}}\\ &= 2^n \frac{d}{du}\frac{e^{uz}}{(e^u+1)^n},\end{aligned}$$

we have

$$\frac{n2^{n+1}e^{u(z+1)}}{(e^u+1)^{n+1}} = \frac{2^{n+1}ze^{uz}}{(e^u+1)^n} - 2^{n+1}\frac{d}{du}\frac{e^{uz}}{(e^u+1)^n}.$$

Taking $z = x+y-1$ and $n = \beta+\gamma-1$ leads to

$$\begin{aligned}\frac{2^{n+1}e^{u(z+1)}}{(e^u+1)^{n+1}} &= \left(\sum_{m=0}^{\infty} E_m^{(\beta)}(x)\frac{u^m}{m!}\right)\left(\sum_{k=0}^{\infty} E_k^{(\gamma)}(y)\frac{u^k}{k!}\right)\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}E_k^{(\beta)}(x)E_{m-k}^{(\gamma)}(y)\frac{u^m}{m!},\\ z\frac{2^{n+1}e^{uz}}{(e^u+1)^n} &= 2(x+y-1)\sum_{m=0}^{\infty}E_m^{(\beta+\gamma-1)}(x+y-1)\frac{u^m}{m!}\end{aligned}$$

and

$$2^{n+1}\frac{d}{du}\frac{e^{uz}}{(e^u+1)^n} = 2\sum_{m=0}^{\infty}E_{m+1}^{(\beta+\gamma-1)}(x+y-1)\frac{u^m}{m!}.$$

By equating the coefficients of $\frac{u^m}{m!}$, we get the convolution formula

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}E_k^{(\beta)}(x)E_{m-k}^{(\gamma)}(y)\\ &\quad= 2\frac{x+y-1}{\beta+\gamma-1}E_m^{(\beta+\gamma-1)}(x+y-1) - \frac{2}{\beta+\gamma-1}E_{m+1}^{(\beta+\gamma-1)}(x+y-1) \qquad (4)\\ &\quad= E_m^{(\beta+\gamma)}(x+y).\end{aligned}$$

Similarly, for higher-order Bernoulli polynomials, we have

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}B_k^{(\beta)}(x)B_{m-k}^{(\gamma)}(y)\\ &\quad= m\frac{x+y-1}{\beta+\gamma-1}B_{m-1}^{(\beta+\gamma-1)}(x+y-1) + \frac{\gamma+\beta-1-m}{\beta+\gamma-1}B_m^{(\beta+\gamma-1)}(x+y-1)\\ &\quad= B_m^{(\gamma+\beta)}(x+y)\end{aligned}$$

(see [11, p. 145 and p. 129]). From the generating functions of the higher-order Bernoulli and Euler polynomials, we can write

$$\left(\frac{u}{e^u-1}\right)^n e^{xu}\left(\frac{2}{e^u+1}\right)^n e^{yu}=\left(\frac{2u}{e^{2u}-1}\right)^n e^{u(x+y)},$$

which gives the following

$$\sum_{k=0}^{m}\binom{m}{k}B_{m-k}^{(n)}(x)E_k^{(n)}(y)=2^m B_m^{(n)}\left(\frac{x+y}{2}\right). \tag{5}$$

3. Integral of products of higher-order Bernoulli and Euler polynomials

This section is devoted to obtain the integral of products of r higher-order Euler polynomials. Also, we derive a formula for the integral of products of l higher-order Bernoulli polynomials and r higher-order Euler polynomials. Furthermore, we relate these results to the reciprocity formulas for generalized Dedekind sums $T_r(c,d)$ and Hardy–Berndt sums $s_{3,r}(c,d)$ and $s_{4,r}(c,d)$.

3.1. Euler polynomials

Let b_s, y_s, α_s be arbitrary real numbers with $b_s\neq 0$, $1\leqslant s\leqslant r$, and

$$\begin{aligned}\widehat{I}_{n_1,\dots,n_r}(x;b;y)&=\widehat{I}_{n_1,\dots,n_r}(x;b_1,\dots,b_r;y_1,\dots,y_r)\\&=\frac{1}{n_1!\cdots n_r!}\int_0^x\prod_{s=1}^{r}E_{n_s}^{(\alpha_s)}(b_s z+y_s)\,dz,\\\widehat{C}_{n_1,\dots,n_r}(x;b;y)&=\widehat{C}_{n_1,\dots,n_r}(x;b_1,\dots,b_r;y_1,\dots,y_r)\\&=\frac{1}{n_1!\cdots n_r!}\left(\prod_{s=1}^{r}E_{n_s}^{(\alpha_s)}(b_s x+y_s)-\prod_{s=1}^{r}E_{n_s}^{(\alpha_s)}(y_s)\right).\end{aligned}$$

Let $\binom{M}{n_1,\dots,n_r}$ denote the multinomial coefficients defined by

$$\binom{M}{n_1,\dots,n_r}=\frac{M!}{n_1!\cdots n_r!},\qquad n_1+\cdots+n_r=M\ \text{ and }\ n_1,\dots,n_r\geqslant 0.$$

Then, we have the following formula:

Theorem 3.1. For any $\mu \geqslant 0$,

$$\begin{aligned}
&\widehat{I}_{n_1,\ldots,n_r}(x;b;y)\\
&\quad=\sum_{a=0}^{\mu}(-1)^a\sum_{j_1+\cdots+j_{r-1}=a}\binom{a}{j_1,\ldots,j_{r-1}}b_1^{j_1}\cdots b_{r-1}^{j_{r-1}}b_r^{-a-1}\\
&\quad\quad\times\widehat{C}_{n_1-j_1,\ldots,n_{r-1}-j_{r-1},n_r+a+1}(x;b;y)\\
&\quad\quad+\frac{(-1)^{\mu+1}}{(n+\mu+1)!}\int_0^x\left(\prod_{s=1}^{r-1}E_{n_s}^{(\alpha_s)}(b_s z+y_s)\right)^{(\mu+1)}E_{n_r+\mu+1}^{(\alpha_r)}(b_r z+y_r)\,dz.
\end{aligned}$$

In particular if $\mu = n_1+\cdots+n_{r-1}$, we have

$$\begin{aligned}
\widehat{I}_{n_1,\ldots,n_r}(x;b;y)&=\sum_{a=0}^{\mu}(-1)^a\sum_{j_1+\cdots+j_{r-1}=a}\binom{a}{j_1,\ldots,j_{r-1}}b_1^{j_1}\cdots b_{r-1}^{j_{r-1}}\\
&\quad\times b_r^{-a-1}\widehat{C}_{n_1-j_1,\ldots,n_{r-1}-j_{r-1},n_r+a+1}(x;b;y).
\end{aligned}\tag{6}$$

Proof. Let

$$f(z)=E_{n_1}^{(\alpha_1)}(b_1 z+y_1)\cdots E_{n_{r-1}}^{(\alpha_{r-1})}(b_{r-1}z+y_{r-1}).$$

Then

$$\begin{aligned}
&\frac{1}{n_r!}\int_0^x f(z)E_{n_r}^{(\alpha_r)}(b_r z+y_r)\,dz\\
&=\left[\frac{1}{b_r(n_r+1)!}f(z)E_{n_r+1}^{(\alpha_r)}(b_r z+y_r)\right]_0^x-\frac{1}{(n_r+1)!}\int_0^x f'(z)E_{n_r+1}^{(\alpha_r)}(b_r z+y_r)\,dz.
\end{aligned}$$

Using μ additional integrations by parts, we find that

$$\begin{aligned}
\frac{1}{n_r!}\int_0^x f(z)E_{n_r}^{(\alpha_r)}(b_r z+y_r)\,dz&=\sum_{a=0}^{\mu}\frac{(-1)^a}{(n_r+a+1)!}\left[f^{(a)}(z)E_{n_r+a+1}^{(\alpha_r)}(b_r z+y_r)\right]_0^x\\
&\quad+\frac{(-1)^{\mu+1}}{(n_r+\mu+1)!}\int_0^x f^{(\mu+1)}(z)E_{n_r+\mu+1}^{(\alpha_r)}(b_r z+y_r)\,dz.
\end{aligned}$$

Using the property of derivative

$$(f_1(z)\cdots f_m(z))^{(a)}=\sum_{j_1+\cdots+j_m=a}\binom{a}{j_1,\ldots,j_m}f_1{}^{(j_1)}(z)\cdots f_m^{(j_m)}(z),$$

and (1), we get the desired result. ■

Setting $x=1$ and $b_s=\alpha_s-2y_s$ with $y_s\neq\alpha_s/2$, $1\leqslant s\leqslant r$, in (6) we have

$$\begin{aligned}&\widehat{I}_{n_1,\ldots,n_r}\left(1;\alpha_1-2y_1,\ldots,\alpha_r-2y_r;y_1,...,y_r\right)\\&=\sum_{a=0}^{n_1+\cdots+n_{r-1}}(-1)^a\sum_{j_1+\cdots+j_{r-1}=a}\binom{a}{j_1,...,j_{r-1}}\frac{\left((-1)^{n_1+\cdots+n_r+1}-1\right)}{(n_1-j_1)!\cdots(n_r+a+1)!}\\&\quad\times b_1^{j_1}\cdots b_{r-1}^{j_{r-1}}b_r^{-a-1}E_{n_1-j_1}^{(\alpha_1)}\left(y_1\right)\cdots E_{n_{r-1}-j_{r-1}}^{(\alpha_{r-1})}\left(y_{r-1}\right)E_{n_r+a+1}^{(\alpha_r)}\left(y_r\right)\end{aligned}$$

since $E_{n_s-j_s}^{(\alpha_s)}\left(b_s+y_s\right)=E_{n_s-j_s}^{(\alpha_s)}\left(\alpha_s-y_s\right)=(-1)^{n_s-j_s}E_{n_s-j_s}^{(\alpha_s)}\left(y_s\right)$ and $j_1+\cdots+j_{r-1}=a$. Therefore, if $n_1+\cdots+n_r+1$ is even, then

$$\widehat{I}_{n_1,\ldots,n_r}\left(1;\alpha_1-2y_1,\ldots,\alpha_r-2y_r;y_1,...,y_r\right)=0,$$

and if $n_1+\cdots+n_r+1$ is odd, then

$$\begin{aligned}&\widehat{I}_{n_1,\ldots,n_r}\left(1;\alpha_1-2y_1,\ldots,\alpha_r-2y_r;y_1,...,y_r\right)\\&\qquad=-2\sum_{a=0}^{n_1+\cdots+n_{r-1}}(-1)^a\frac{\left(\alpha_r-2y_r\right)^{-a-1}}{(n_r+a+1)!}E_{n_r+a+1}^{(\alpha_r)}\left(y_r\right)\\&\qquad\quad\times\sum_{j_1+\cdots+j_{r-1}=a}\binom{a}{j_1,...,j_{r-1}}\prod_{s=1}^{r-1}\frac{\left(\alpha_s-2y_s\right)^{j_s}}{(n_s-j_s)!}E_{n_s-j_s}^{(\alpha_s)}\left(y_s\right).\end{aligned}$$

For example, we have

$$\int_0^1 E_2^{(3)}\left(7z-2\right)E_3^{(1/2)}\left(-\frac{3}{2}z+1\right)E_{10}^{(5)}\left(4z+1/2\right)dz=0$$

and

$$\frac{1}{2!10!}\int_0^1 E_2^{(3)}\left(3z\right)E_{10}^{(5)}\left(-3z+4\right)dz=\frac{2}{3}\sum_{a=0}^{2}\frac{E_{2-a}^{(3)}\left(0\right)}{(2-a)!}\frac{E_{11+a}^{(5)}\left(4\right)}{(11+a)!}.$$

It is seen from the definition of the integral $\widehat{I}_{n_1,\ldots,n_r}(x;b;y)$ that the left-hand side of (6) is invariant under interchanging the order of the integrands. That is, for $r=2$,

$$\begin{aligned}&\sum_{a=0}^{n}(-1)^a\binom{m+n+1}{n-a}b_1^ab_2^{-a-1}\\&\times\left(E_{n-a}^{(\gamma)}\left(b_1x+y_1\right)E_{m+a+1}^{(\beta)}\left(b_2x+y_2\right)-E_{n-a}^{(\gamma)}\left(y_1\right)E_{m+a+1}^{(\beta)}\left(y_2\right)\right)\\&=\sum_{a=0}^{m}(-1)^a\binom{m+n+1}{m-a}b_2^ab_1^{-a-1}\\&\times\left(E_{m-a}^{(\gamma)}\left(b_2x+y_2\right)E_{n+a+1}^{(\beta)}\left(b_1x+y_1\right)-E_{m-a}^{(\gamma)}\left(y_2\right)E_{n+a+1}^{(\beta)}\left(y_1\right)\right).\end{aligned}\tag{7}$$

So, we may investigate the reciprocity relation for sums of products of higher-order Euler polynomials as follows: Let

$$\begin{aligned}T &:= \sum_{a=0}^{n}(-1)^{a}\binom{m+n+1}{n-a}b_1^{a}b_2^{-a-1}E_{n-a}^{(\gamma)}(y_1)\,E_{m+a+1}^{(\beta)}(y_2)\\ &\quad-\sum_{a=0}^{m}(-1)^{a}\binom{m+n+1}{m-a}b_2^{a}b_1^{-a-1}E_{m-a}^{(\gamma)}(y_2)\,E_{n+a+1}^{(\beta)}(y_1)\,.\end{aligned}$$

We first rewrite this as

$$\begin{aligned}T &= \sum_{a=0}^{n}(-1)^{n-a}\binom{m+n+1}{a}b_1^{n-a}b_2^{a-n-1}E_{a}^{(\gamma)}(y_1)\,E_{m+n+1-a}^{(\beta)}(y_2)\\ &\quad-\sum_{a=0}^{m}(-1)^{m-a}\binom{m+n+1}{a}b_2^{m-a}b_1^{a-m-1}E_{a}^{(\gamma)}(y_2)\,E_{m+n+1-a}^{(\beta)}(y_1)\,.\end{aligned}\tag{8}$$

Without loss of generality we may assume that $n \geqslant m$; in this case we separate the sum from 0 to m and $m+1$ to n on the first summation in (8), and rewrite these as

$$\begin{aligned}&\sum_{a=0}^{m}(-1)^{n-a}\binom{m+n+1}{a}b_1^{n-a}b_2^{a-n-1}E_{a}^{(\gamma)}(y_1)\,E_{m+n+1-a}^{(\beta)}(y_2)\\ &\qquad=\sum_{a=n+1}^{m+n+1}(-1)^{m+1-a}\binom{m+n+1}{a}b_1^{a-m-1}b_2^{m-a}E_{m+n+1-a}^{(\gamma)}(y_1)\,E_{a}^{(\beta)}(y_2)\end{aligned}$$

and

$$\begin{aligned}&\sum_{a=m+1}^{n}(-1)^{n-a}\binom{m+n+1}{a}b_1^{n-a}b_2^{a-n-1}E_{a}^{(\gamma)}(y_1)\,E_{m+n+1-a}^{(\beta)}(y_2)\\ &\qquad=\sum_{a=m+1}^{n}(-1)^{m+1-a}\binom{m+n+1}{a}b_1^{a-m-1}b_2^{m-a}E_{m+n+1-a}^{(\gamma)}(y_1)\,E_{a}^{(\beta)}(y_2)\,.\end{aligned}$$

Thus, we have

$$\begin{aligned}T &= \frac{1}{b_1^{m+1}b_2^{n+1}}\sum_{a=0}^{m+n+1}(-1)^{m+1-a}\binom{m+n+1}{a}\\ &\quad\times b_1^{a}b_2^{m+n+1-a}E_{m+n+1-a}^{(\gamma)}(y_1)\,E_{a}^{(\beta)}(y_2)\,.\end{aligned}\tag{9}$$

Combining (7) and (9) gives the following reciprocity relation for sums of products of higher-order Euler polynomials:

Corollary 3.2.

$$\begin{aligned}
&\sum_{a=0}^{n}(-1)^{a}\binom{m+n+1}{n-a}b_1^a b_2^{-a-1}E_{n-a}^{(\gamma)}\left(b_1x+y_1\right)E_{m+a+1}^{(\beta)}\left(b_2x+y_2\right)\\
&-\sum_{a=0}^{m}(-1)^{a}\binom{m+n+1}{m-a}b_2^a b_1^{-a-1}E_{m-a}^{(\gamma)}\left(b_2x+y_2\right)E_{n+a+1}^{(\beta)}\left(b_1x+y_1\right) \qquad (10)\\
&=\frac{1}{b_1^{m+1}b_2^{n+1}}\sum_{a=0}^{m+n+1}(-1)^{m+1-a}\binom{m+n+1}{a}b_1^a b_2^{m+n+1-a}E_{m+n+1-a}^{(\gamma)}\left(y_1\right)E_{a}^{(\beta)}\left(y_2\right).
\end{aligned}$$

In particular for $y_1=\gamma/2$, $y_2=\beta/2$ and even $(m+n)$, the right-hand side of (10) vanishes.

Remark 3.3. Beginning from the left-hand side of (10) and using the arguments in the proof of (9), the right-hand side of (10) turns into

$$\begin{aligned}
&\frac{1}{b_1^{m+1}b_2^{n+1}}\sum_{a=0}^{m+n+1}(-1)^{m+1-a}\binom{m+n+1}{a}\\
&\qquad\times b_1^a b_2^{m+n+1-a}E_{m+n+1-a}^{(\gamma)}\left(b_1x+y_1\right)E_{a}^{(\beta)}\left(b_2x+y_2\right).
\end{aligned}$$

So it follows that for all x,

$$\begin{aligned}
&\sum_{a=0}^{m+n+1}(-1)^{a}\binom{m+n+1}{a}b_1^a b_2^{m+n+1-a}E_{m+n+1-a}^{(\gamma)}\left(b_1x+y_1\right)E_{a}^{(\beta)}\left(b_2x+y_2\right)\\
&\qquad=\sum_{a=0}^{m+n+1}(-1)^{a}\binom{m+n+1}{a}b_1^a b_2^{m+n+1-a}E_{m+n+1-a}^{(\gamma)}\left(y_1\right)E_{a}^{(\beta)}\left(y_2\right).
\end{aligned}$$

- Let $b_1=b_2=1$ in (10). Then the right-hand side becomes, with the use of (2),

$$(-1)^{n}T=\sum_{a=0}^{m+n+1}\binom{m+n+1}{a}E_{m+n+1-a}^{(\gamma)}\left(\gamma-y_1\right)E_{a}^{(\beta)}\left(y_2\right).$$

  Now using (4) by taking $x=y_2$ and $y=\gamma-y_1$, (10) reduces to

$$\begin{aligned}
&\sum_{a=0}^{n}(-1)^{a}\binom{m+n+1}{n-a}E_{n-a}^{(\gamma)}\left(x+y_1\right)E_{m+a+1}^{(\beta)}\left(x+y_2\right)\\
&-\sum_{a=0}^{m}(-1)^{a}\binom{m+n+1}{m-a}E_{m-a}^{(\gamma)}\left(x+y_2\right)E_{n+a+1}^{(\beta)}\left(x+y_1\right)\\
&=2\left(-1\right)^{n}\frac{y_2-y_1+\gamma-1}{\gamma+\beta-1}E_{m+n+1}^{(\gamma+\beta-1)}\left(y_2-y_1+\gamma-1\right)\\
&\quad-2\frac{\left(-1\right)^{n}}{\gamma+\beta-1}E_{m+n+2}^{(\gamma+\beta-1)}\left(y_2-y_1+\gamma-1\right)\\
&=\left(-1\right)^{n}E_{m+n+1}^{(\gamma+\beta)}\left(y_2-y_1+\gamma\right).
\end{aligned}$$

- Setting $b_1 = 1$, $b_2 = -1$ and using (4), (10) becomes

$$\begin{aligned}
&\sum_{a=0}^{n} \binom{m+n+1}{n-a} E_{n-a}^{(\gamma)}\left(x+y_1\right) E_{m+a+1}^{(\beta)}\left(y_2-x\right) \\
&+\sum_{a=0}^{m} \binom{m+n+1}{m-a} E_{m-a}^{(\gamma)}\left(y_2-x\right) E_{n+a+1}^{(\beta)}\left(x+y_1\right) \\
&= 2\frac{y_2+y_1-1}{\gamma+\beta-1} E_{m+n+1}^{(\gamma+\beta-1)}\left(y_2+y_1-1\right) - \frac{2}{\gamma+\beta-1} E_{m+n+2}^{(\gamma+\beta-1)}\left(y_2+y_1-1\right) \\
&= E_{m+n+1}^{(\gamma+\beta)}\left(y_2+y_1\right).
\end{aligned}$$

- Set $\beta = \gamma = 1$, $b_1 = 2$ and $b_2 = -1$ in (10). In view of [4, Theorem 6], (10) becomes

$$\begin{aligned}
&\sum_{a=0}^{n} \binom{m+n+1}{n-a} 2^{m+1+a} E_{n-a}\left(2x+y_1\right) E_{m+a+1}\left(-x+y_2\right) \\
&+\sum_{a=0}^{m} \binom{m+n+1}{m-a} 2^{m-a} E_{m-a}\left(-x+y_2\right) E_{n+a+1}\left(2x+y_1\right) \\
&= \sum_{a=0}^{m+n+1} \binom{m+n+1}{a} 2^{a} E_{a}\left(y_2\right) E_{m+n+1-a}\left(y_1\right) \\
&= E_{m+n+1}(2y_2+y_1) + 2^{m+n+1} E_{m+n+1}\left(\frac{2y_2+y_1}{2}\right) \\
&- 2^{m+n+1} E_{m+n+1}\left(\frac{2y_2+y_1+1}{2}\right).
\end{aligned}$$

- Let $\gamma = \beta = 1$ and $y_1 = y_2 = 0$ in (10). Then,

$$\begin{aligned}
&\sum_{a=0}^{n} (-1)^{a} \binom{m+n+1}{n-a} b_1^{a} b_2^{-a-1} E_{n-a}\left(b_1 x\right) E_{m+a+1}\left(b_2 x\right) \\
&-\sum_{a=0}^{m} (-1)^{a} \binom{m+n+1}{m-a} b_2^{a} b_1^{-a-1} E_{m-a}\left(b_2 x\right) E_{n+a+1}\left(b_1 x\right) \\
&= \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^{a} b_2^{m+n+1-a} \\
&\times E_{m+n+1-a}\left(0\right) E_{a}\left(0\right). \qquad (11)
\end{aligned}$$

From the property $B_{2n+1}(0) = 0$, $n \geqslant 1$ and (3) for $x = 0$, we have $(-1)^{a} E_a(0) = -E_a(0)$ for $a > 0$. Then, the right-hand side of (11) can be written

$$T = \frac{(-1)^m}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} \times E_{m+n+1-a}(0) E_a(0) - 2\frac{(-b_2)^m}{b_1^{m+1}} E_{m+n+1}(0). \tag{12}$$

Remark 3.4. Kim and Son [8] proved the reciprocity formula for generalized Dedekind sums $T_r(c,d)$ as

$$cd^r T_r(c,d) + dc^r T_r(d,c) = -\frac{1}{2}\sum_{a=0}^{r}\binom{r}{a} d^{a-1} c^{r-1-a} \overline{E}_a(0)\overline{E}_{r-a}(0) + \overline{E}_{r+1}(0), \tag{13}$$

where $T_r(d,c)$ is defined by

$$T_r(c,d) = \sum_{j=0}^{|d|-1} (-1)^j\, \overline{E}_1\left(\frac{j}{d}\right) \overline{E}_r\left(\frac{cj}{d}\right)$$

in which

$$\begin{aligned} \overline{E}_r(x) &= E_r(x), && 0 \leqslant x < 1, \\ \overline{E}_r(x+p) &= (-1)^p\, \overline{E}_r(x), && p \in \mathbb{Z}. \end{aligned}$$

It is seen from (12) and (13) that the reciprocity formula of the generalized Dedekind sum $T_r(c,d)$ can be written in terms of the reciprocity relation of Euler polynomials.

3.2. Bernoulli and Euler polynomials

Theorem 3.5. Let b_s and y_s, $1 \leqslant s \leqslant l+r$ be arbitrary real numbers with $b_s \neq 0$. Let $N = n_1! \cdots n_l! m_1! \cdots m_r!$ and

$$\begin{aligned} J_{n_1,\ldots,m_r}(x;b;y) &= J_{n_1,\ldots,m_r}(x; b_1,\ldots,b_{l+r}; y_1,\ldots,y_{l+r}) \\ &= \frac{1}{N}\int_0^x \prod_{s=1}^{l} B_{n_s}^{(\gamma_s)}(b_s z + y_s) \prod_{i=1}^{r} E_{m_i}^{(\beta_i)}(b_{l+i} z + y_{l+i})\, dz, \\ D_{n_1,\ldots,m_r}(x;b;y) &= D_{n_1,\ldots,m_r}(x; b_1,\ldots,b_{l+r}; y_1,\ldots,y_{l+r}) \\ &= \frac{1}{N}\prod_{s=1}^{l} B_{n_s}^{(\gamma_s)}(b_s x + y_s) \prod_{i=1}^{r} E_{m_i}^{(\beta_i)}(b_{l+i} x + y_{l+i}) \\ &\quad - \frac{1}{N}\prod_{s=1}^{l} B_{n_s}^{(\gamma_s)}(y_s) \prod_{i=1}^{r} E_{m_i}^{(\beta_i)}(y_{l+i}). \end{aligned}$$

Then, for $\mu = n_1 + \cdots + n_l + m_1 + \cdots + m_{r-1}$,

$$J_{n_1,\dots,m_r}(x;b;y) = \sum_{a=0}^{\mu} (-1)^a \sum_{j_1+\cdots+j_{l+r-1}=a} \binom{a}{j_1,\dots,j_{l+r-1}} \times b_1^{j_1} \cdots b_{l+r-1}^{j_{l+r-1}} b_{l+r}^{-a-1} D_{n_1-j_1,\dots,m_{r-1}-j_{l+r-1},m_r+a+1}(x;b;y).$$

Proof. The proof can be obtained by using the arguments in the proof of Theorem 3.1. ∎

In order to obtain the reciprocity relation for sums of products of higher-order Bernoulli and Euler polynomials, similar to T, we define

$$T_1 := \sum_{a=0}^{n} (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B_{n-a}^{(\gamma)}(y_1) E_{m+a+1}^{(\beta)}(y_2) - \sum_{a=0}^{m} (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} E_{m-a}^{(\gamma)}(y_2) B_{n+a+1}^{(\beta)}(y_1).$$

Similarly, we have

$$\begin{aligned} T_1 &= \sum_{a=0}^{n} (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B_{n-a}^{(\gamma)}(b_1x+y_1) E_{m+a+1}^{(\beta)}(b_2x+y_2) \\ &\quad - \sum_{a=0}^{m} (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} E_{m-a}^{(\gamma)}(b_2x+y_2) B_{n+a+1}^{(\beta)}(b_1x+y_1) \\ &= \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} \\ &\quad \times b_1^a b_2^{m+n+1-a} E_a^{(\beta)}(y_2) B_{m+n+1-a}^{(\gamma)}(y_1). \end{aligned} \tag{14}$$

Notice that the right-hand side of (14) vanishes for $y_1 = \gamma/2$, $y_2 = \beta/2$ and even $m+n$.

- Setting $\beta = \gamma = 1$ and $b_1 = 2$, $b_2 = -1$ in (14), we get

$$\begin{aligned} 2^{m+1} T_1 &= -\sum_{a=0}^{m+n+1} \binom{m+n+1}{a} 2^a E_a(y_2) B_{m+n+1-a}(y_1) \\ &= -B_{m+n+1}(2y_2+y_1) + 2^{m+n-1}(m+n+1) E_{m+n}\left(\frac{2y_2+y_1+1}{2}\right) \\ &\quad - 2^{m+n-1}(m+n+1) E_{m+n}\left(\frac{2y_2+y_1}{2}\right) \end{aligned}$$

  by [4, Theorem 10]. After similar manipulations to T, we have for $\gamma = \beta$ and $b_2 = b_1 = 1$

$$(-1)^n T_1 = \sum_{a=0}^{m+n+1} \binom{m+n+1}{a} B_{m+n+1-a}^{(\gamma)}(\gamma - y_1) E_a^{(\gamma)}(y_2).$$

In view of (5) for $x=\gamma-y_1$, $y=y_2$, we get

$$T_1=(-1)^n\,2^{m+n+1}B_{m+n+1}^{(\gamma)}\left(\frac{\gamma-y_1+y_2}{2}\right).$$

- For $y_1=y_2=0$ and $\gamma=\beta=1$, T_1 can be written as

$$\begin{aligned}T_1&=\sum_{a=0}^{n}(-1)^a\binom{m+n+1}{n-a}b_1^ab_2^{-a-1}B_{n-a}(0)\,E_{m+a+1}(0)\\&\quad-\sum_{a=0}^{m}(-1)^a\binom{m+n+1}{m-a}b_2^ab_1^{-a-1}E_{m-a}(0)\,B_{n+a+1}(0)\\&=\frac{(-1)^{m+1}}{b_1^{m+1}b_2^{n+1}}\sum_{a=0}^{m+n+1}(-1)^a\binom{m+n+1}{a}b_1^ab_2^{m+n+1-a}B_{m+n+1-a}(0)\,E_a(0).\end{aligned}$$

Using (3) for $x=0$, we get

$$\begin{aligned}T_1=\frac{(-1)^{m+1}}{b_1^{m+1}b_2^{n+1}}&\sum_{a=1}^{m+n+2}(-1)^{a-1}\binom{m+n+1}{a-1}\\&\times b_1^{a-1}b_2^{m+n+2-a}B_{m+n+2-a}\frac{2}{a}\left(1-2^a\right)B_a.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\sum_{a=0}^{n}(-1)^a\binom{m+n+1}{n-a}b_1^ab_2^{-a-1}B_{n-a}E_{m+a+1}(0)\\&-\sum_{a=0}^{m}(-1)^a\binom{m+n+1}{m-a}b_2^ab_1^{-a-1}E_{m-a}(0)\,B_{n+a+1}\\=&\frac{(-1)^m}{b_1^{m+2}b_2^{n+1}}\frac{2}{m+n+2}\sum_{a=1}^{m+n+2}(-1)^a\binom{m+n+2}{a}\\&\times b_1^ab_2^{m+n+2-a}\left(1-2^a\right)B_{m+n+2-a}B_a.\end{aligned}\tag{15}$$

Remark 3.6. Observe that the sum on the right-hand side of (15) is the reciprocity formula for the Hardy–Berndt sums $s_{3,r}(c,d)$ and $s_{4,r}(c,d)$ given by [2]

$$\begin{aligned}(r+1)&\left(cd^rs_{3,r}(c,d)-2^{-2}d\,(2c)^r\,s_{4,r}(d,c)\right)\\&=2\sum_{a=1}^{r+1}\binom{r+1}{a}(-1)^a\,c^ad^{r+1-a}\left(1-2^a\right)B_aB_{r+1-a},\end{aligned}\tag{16}$$

where d and r are odd and

$$s_{3,r}(c,d)=\sum_{j=1}^{d-1}(-1)^j\,\overline{B}_r\left(\frac{cj}{d}\right),\qquad s_{4,r}(c,d)=-4\sum_{j=1}^{d-1}\overline{B}_r\left(\frac{cj}{2d}\right).$$

Thus, the reciprocity formulas given by (15) and (16) can be associated as

$$\begin{aligned}
&\sum_{a=0}^{n}(-1)^{m-a}\binom{m+n+1}{n-a}b_1^{m+2+a}b_2^{n-a}B_{n-a}E_{m+a+1}(0)\\
&-\sum_{a=0}^{m}(-1)^{m-a}\binom{m+n+1}{m-a}b_2^{n+1+a}b_1^{m+1-a}E_{m-a}(0)\,B_{n+a+1}\\
=\;& b_1b_2^r s_{3,r}(b_1,b_2)-2^{-2}b_2(2b_1)^r s_{4,r}(b_2,b_1)\\
=\;& \frac{2}{r+1}\sum_{a=1}^{r+1}(-1)^a\binom{r+1}{a}b_1^a b_2^{r+1-a}\left(1-2^a\right)B_{r+1-a}B_a
\end{aligned}$$

for odd integers $r=(m+n+1)$ and b_2.

From this relationship, (15) can be evaluated for some special cases. Since $s_{3,r}(d,1)=0$ and $s_{4,r}(d,1)=0$, we have

$$\begin{aligned}
&\sum_{a=0}^{n}(-1)^{a}\binom{m+n+1}{n-a}b^{-a-1}B_{n-a}E_{m+a+1}(0)\\
&-\sum_{a=0}^{m}(-1)^{a}\binom{m+n+1}{m-a}b^{a}E_{m-a}(0)\,B_{n+a+1}\\
=\;& (-1)^m b^m s_{3,m+n+1}(1,b)
\end{aligned}$$

for odd integers $(m+n+1)$ and b, and

$$\begin{aligned}
&\sum_{a=0}^{n}(-1)^{a}\binom{m+n+1}{n-a}b^{a}B_{n-a}E_{m+a+1}(0)\\
&-\sum_{a=0}^{m}(-1)^{a}\binom{m+n+1}{m-a}b^{-a-1}E_{m-a}(0)\,B_{n+a+1}\\
=\;& (-1)^{m+1}2^{m+n-1}b^{n-1}s_{4,m+n+1}(1,b)
\end{aligned}$$

for odd integer $(m+n+1)$.

References

[1] T. Agoh, K. Dilcher, *Integrals of products of Bernoulli polynomials,* J. Math. Anal. Appl. **381** (2011), 10–16.

[2] M. Can, M. Cenkci and V. Kurt, *Generalized Hardy-Berndt sums,* Proc. Jangjeon Math. Soc. **9** (2006), 19–38.

[3] L. Carlitz, *Note on the integral of the product of several Bernoulli polynomials,* J. London Math. Soc. **34** (1959), 361–363.

[4] W. Chu, R.R. Zhou, *Convolution of Bernoulli and Euler polynomials,* Sarajevo J. Math. **6**(18) (2010), 147–163.

[5] M.C. Dağlı and M. Can, *On reciprocity formula of character Dedekind sums and the integral of products of Bernoulli polynomials,* J. Number Theory **156** (2015), 105–124.

[6] O. Espinosa, V.H. Moll, *The evaluation of Tornheim double sums,* J. Number Theory **116** (2006), 200–229.

[7] S. Hu, D. Kim, M.-S. Kim, *On the integral of the product of four and more Bernoulli polynomials,* Ramanujan J. **33** (2014), 281–293.

[8] M.S. Kim, J.W. Son, *On generalized Dedekind sums involving quasi-periodic Euler functions,* J. Number Theory **144** (2014), 267–280.

[9] J. Liu, H. Pan, Y. Zhang, *On the integral of the product of the Appell polynomials,* Integral Transforms Spec. Funct. **25** (2014), no. 9, 680–685.

[10] M. Mikolás, *Integral formulae of arithmetical characteristics relating to the zeta-function of Hurwitz,* Publ. Math. Debrecen **5** (1957), 44–53.

[11] L.M. Milne-Thomson, *The Calculus of Finite Differences,* Macmillan, London, 1933.

[12] L.J. Mordell, *Integral formulae of arithmetical character,* J. London Math. Soc. **33** (1958), 371–375.

[13] N.E. Nörlund, *Vorlesungen über Differenzenrechnung,* Springer-Verlag, Berlin, 1924.

[14] J.C. Wilson, *On Franel–Kluyver integrals of order three,* Acta Arith. **66** (1994), 71–87.

Address: Muhammet Cihat Dağlı and Mümün Can: Department of Mathematics, Akdeniz University, Antalya, TR-07058, Turkey.

E-mail: mcihatdagli@akdeniz.edu.tr, mcan@akdeniz.edu.tr

Received: 29 October 2015; **revised:** 21 December 2016