Functiones et Approximatio
XXXIII (2005), 73–84

BOUNDEDNESS FOR MULTILINEAR MARCINKIEWICZ OPERATORS ON TRIEBEL-LIZORKIN SPACES

LANZHE LIU

Abstract: In this paper we establish the boundedness in the context of Triebel-Lizorkin spaces for multilinear Marcinkiewicz operators.
Keywords: Marcinkiewicz integral operator; Multilinear Operators; Triebel-Lizorkin space; Lipschitz space.

1. Introduction and results

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, Q will denote a cube in R^n with side parallel to the axes, and for a cube Q, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup\limits_{x\in Q} |Q|^{-1}\int_Q |f(y) - f_Q|dy$. For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Triebel-Lizorkin space. The Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$||f||_{\dot{\Lambda}_\beta} = \sup_{\substack{x,h\in R^n\\ h\neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator (see [13]).

Fix $\lambda > 1$, $\delta > 0$ and $0 < \gamma \leqslant 1$. Suppose that S^{n-1} is the unit sphere of $R^n (n \geqslant 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on S^{n-1} $(0 < \gamma \leqslant 1)$, i.e.

$$|\Omega(x') - \Omega(y')| \leqslant M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} \Omega(x')dx' = 0$.

2001 Mathematics Subject Classification: 42B20, 42B25.
Supported by the NNSF (Grant: 10271071)

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. Let m be a positive integer and A be a function on R^n.The multilinear Marcinkiewicz integral operator is defined by

$$\mu_\lambda^A(f)(x) = \left[\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)dz$$

and

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\leqslant m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

Set

$$F_t(f)(y) = \int_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z)dz.$$

We also define that

$$\mu_\lambda(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is the Marcinkiewicz integral operator (see [15]).

Let H be the Hilbert space $H = \left\{h : \|h\| = \left(\iint_{R_+^{n+1}} |h(t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\mu_\lambda^A(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^A(f)(x,y)\right\|,$$

$$\mu_\lambda(f)(x) = \left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t(f)(y)\right\|.$$

Note that when $m=0$, μ_λ^A is just the commutator of Marcinkiewicz integral operator(see [15]). It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see[2-5][8][12]). In [10][13], Janson and Paluszynski obtain the boundedness of commutators generated by the Calderón-Zygmund operator or fractional integral operator and Lipschitz functions on Triebel-Lizorkin spaces. The main purpose of this paper is to discuss the boundedness properties of the multilinear Marcinkiewicz operators in the context of Triebel-Lizorkin spaces. We shall prove the following theorems in Section 3.

Theorem 1. *Let* $0 \leqslant \delta < n$, $0 < \beta < 1/2$, $1 < p < n/\delta$, $1/p - 1/q = \delta/n$ *and* $D^\alpha A \in \dot{\Lambda}_\beta$ *for* $|\alpha| = m$. *Then* μ_λ^A *maps* $L^p(R^n)$ *continuously into* $\dot{F}_q^{\beta,\infty}(R^n)$.

Theorem 2. *Let* $0 \leqslant \delta < n$, $0 < \beta < 1/2$, $1 < p < n/(\delta + \beta)$, $1/p - 1/q = (\delta + \beta)/n$ *and* $D^\alpha A \in \dot{\Lambda}_\beta$ *for* $|\alpha| = m$. *Then* μ_λ^A *maps* $L^p(R^n)$ *continuously into* $L^q(R^n)$.

Theorem 3. *Let* $0 \leqslant \delta < n$, $0 < \beta < 1/2$, $\delta + \beta < n$ *and* $D^\alpha A \in \dot{\Lambda}_\beta$ *for* $|\alpha| = m$. *Then for any* $\eta > 0$,

$$|\{x \in R^n : \mu_\lambda^A(f)(x) > \eta\}| \leqslant C\Big(\sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} ||f||_{L^1}/\eta\Big)^{n/(n-\delta-\beta)}.$$

2. Some Lemmas

We begin with some preliminary lemmas.

Lemma 1. (see [13, Lemma 1.5]) *Let* $0 < \beta < 1$, $1 < p < \infty$, *then*

$$||f||_{\dot{F}_p^{\beta,\infty}} \approx \left\| \sup_{\cdot \in Q} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p} \approx \left\| \sup_{\cdot \in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right\|_{L^p}.$$

Lemma 2.. (see [13, Lemma 1.5]) *Let* $0 < \beta < 1, 1 \leqslant p \leqslant \infty$, *then*

$$||f||_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p}.$$

Lemma 3. (see [1, Lemma 2]) *Let* $1 \leqslant r < \infty$, $\delta > 0$ *and*

$$M_{\delta,r}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\delta r/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

Suppose $r < p < n/\delta$ *and* $1/q = 1/p - \delta/n$. *Then* $||M_{\delta,r}(f)||_{L^q} \leqslant C||f||_{L^p}$.

Lemma 4. (see [7, p.14]) *Let* $Q_1 \subset Q_2$. *Then*

$$|f_{Q_1} - f_{Q_2}| \leqslant C||f||_{\dot{\Lambda}_\beta} |Q_2|^{\beta/n}.$$

Lemma 5.. (see [5, Lemma]) *Let* A *be a function on* R^n *such that* $D^\alpha A \in L^q(R^n)$ *for* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y)| \leqslant C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ *is the cube centered at* x *and with side length* $5\sqrt{n}|x - y|$.

Lemma 6. (see [2, Theorem 2.3]) *Let T_A be the multilinear operator defined by*

$$T_A(f)(x) = \int_{R^n} \frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} f(y)dy.$$

If $0<\beta<1$, $0\leqslant\delta<n$, $1<p<n/(\beta+\delta)$, $1/q=1/p-(\beta+\delta)/n$ and $D^\alpha A\in\dot{\Lambda}_\beta$ for $|\alpha|=m$. Then T_A is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is

$$||T_A(f)||_{L^q}\leqslant C||f||_{L^p}.$$

Lemma 7. *Let $0<\beta<1$, $0\leqslant\delta<n$, $1<p<n/(\beta+\delta)$, $1/q=1/p-(\beta+\delta)/n$ and $D^\alpha A\in\dot{\Lambda}_\beta$ for $|\alpha|=m$. Then μ_λ^A maps $L^p(R^n)$ continuously into $L^q(R^n)$, that is*

$$||\mu_\lambda^A(f)||_{L^q}\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

Proof. Note that $|x-z|\leqslant 2t$, $|y-z|\geqslant|x-z|-t\geqslant|x-z|-3t$ and $|x-z|\leqslant t(1+2^{k+1})\leqslant 2^{k+2}t$, $|y-z|\geqslant|x-z|-2^{k+3}t$ when $|x-y|\leqslant t$, $|y-z|\leqslant t$, $|x-y|\leqslant 2^{k+1}t$ and $|y-z|\leqslant t$. By the Minkowski inequality, we get

$$\begin{aligned}
&\mu_\lambda^A(f)(x)\\
&\leqslant\int_{R^n}\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\left(\frac{|\Omega(y-z)||R_{m+1}(A;x,z)||f(z)|}{|y-z|^{n-\delta-1}|x-z|^m}\right)^2\chi_{\Gamma(z)}(y,t)\frac{dydt}{t^{n+3}}\right]^{\frac12}dz\\
&\leqslant C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}\left[\int_0^\infty\int_{|x-y|\leqslant t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-3t)^{2n-2\delta-2}}\frac{dydt}{t^{n+3}}\right]^{\frac12}dz\\
&\quad+C\int_{R^n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m}\\
&\quad\times\left[\int_0^\infty\sum_{k=0}^\infty\int_{2^kt<|x-y|\leqslant 2^{k+1}t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\frac{\chi_{\Gamma(z)}(y,t)t^{-n-3}dydt}{(|x-z|-2^{k+3}t)^{2n-2\delta-2}}\right]^{\frac12}dz\\
&\leqslant C\int_{R_n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}\left[\int_{|x-z|/2}^\infty\frac{dt}{(|x-z|-3t)^{2n-2\delta}}\right]^{\frac12}dz\\
&\quad+C\int_{R_n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+1/2}}\left[\sum_{k=0}^\infty\int_{2^{-2-k}|x-z|}^\infty 2^{-kn\lambda}(2^kt)^nt^{-n}\frac{2^kdt}{(|x-z|-2^{k+3}t)^{2n-2\delta}}\right]^{\frac12}dz
\end{aligned}$$

$$\leqslant C\int_{R_n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n-\delta}}dz+C\int_{R_n}\frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n-\delta}}dz\left[\sum_{k=0}^{\infty}2^{kn(1-\lambda)}\right]^{\frac{1}{2}}$$
$$=C\int_{R_n}\frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}}|f(z)|dz,$$

thus, the lemma follows from Lemma 6.

3. Proofs of theorems

Proof of Theorem 1. Fix a cube $Q=Q(x_0,l)$ and $\tilde{x}\in Q$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha$, then $R_m(A;x,y)=R_m(\tilde{A};x,y)$ and $D^\alpha\tilde{A}=D^\alpha A-(D^\alpha A)_{\tilde{Q}}$ for $|\alpha|=m$. We write, for $f_1=f\chi_{\tilde{Q}}$ and $f_2=f\chi_{R^n\setminus\tilde{Q}}$,

$$F_t^A(f)(x,y)$$
$$=\int_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-\delta-1}}\frac{R_{m+1}(\tilde{A};x,z)}{|x-z|^m}f_2(z)dz+\int_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-\delta-1}}\frac{R_m(\tilde{A};x,z)}{|x-z|^m}f_1(z)dz$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-\delta-1}}\frac{(x-z)^\alpha}{|x-z|^m}D^\alpha\tilde{A}(z)f_1(z)dz,$$

then

$$\left|\mu_\lambda^A(f)(x)-\mu_\lambda^{\tilde{A}}(f_2)(x_0)\right|$$
$$=\left|\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t^A(f)(x,y)\right\|-\left\|\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x_0,y)\right\|\right|$$
$$\leqslant\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(y)\right\|$$
$$+\sum_{|\alpha|=m}\frac{1}{\alpha!}\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m}D^\alpha\tilde{A}f_1\right)(y)\right\|$$
$$+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x,y)-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}F_t^{\tilde{A}}(f_2)(x_0,y)\right\|$$
$$=I(x)+II(x)+III(x),$$

thus

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|\mu_\lambda^{A}(f)(x)-\mu_\lambda^{\tilde{A}}(f_2)(x_0)\right|dx$$

$$\leqslant \frac{1}{|Q|^{1+\beta/n}}\int_Q I(x)dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q II(x)dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q III(x)dx$$

$$:=I+II+III.$$

Now, let us estimate I, II and III, respectively. First, for $x\in Q$ and $z\in\tilde{Q}$, using Lemma 2 and Lemma 5, we get

$$|R_m(\tilde{A};x,z)|\leqslant C|x-z|^m\sum_{|\alpha|=m}\sup_{x\in\tilde{Q}}|D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}}|$$

$$\leqslant C|x-z|^m|Q|^{\beta/n}\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta},$$

thus, taking r,s such that $1\leqslant r<p$ and $1/s=1/r-\delta/n$, by (L^r,L^s) boundedness of μ_λ (see Lemma 7) and the Hölder inequality, we obtain

$$I\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\frac{1}{|Q|}\int_Q|\mu_\lambda(f_1)(x)|dx$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}||\mu_\lambda(f_1)||_{L^s}|Q|^{-1/s}$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}||f_1||_{L^r}|Q|^{-1/s}$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\left(\frac{1}{|\tilde{Q}|^{1-r\delta/n}}\int_{\tilde{Q}}|f(y)|^r dy\right)^{1/r}$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,r}(f)(\tilde{x}).$$

Secondly, using the following inequality(see [13])

$$||(D^\alpha A-(D^\alpha A)_{\tilde{Q}})f\chi_{\tilde{Q}}||_{L^r}\leqslant C|Q|^{1/s+\beta/n}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,r}(f),$$

we gain similarly to the proof of I, that

$$II\leqslant\frac{C}{|Q|^{1+\beta/n}}\sum_{|\alpha|=m}||\mu_\lambda((D^\alpha A-(D^\alpha A)_{\tilde{Q}})f_1)||_{L^s}|Q|^{1-1/s}$$

$$\leqslant C|Q|^{-\beta/n-1/r}\sum_{|\alpha|=m}||(D^\alpha A-(D^\alpha A)_{\tilde{Q}})f_1||_{L^r}$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,r}(f)(\tilde{x}).$$

For III, we write

$$\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x,y) - \left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2} F_t^{\tilde{A}}(f_2)(x_0,y)$$

$$= \int_{|y-z|\leqslant t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \left[\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m}\right] \frac{\Omega(y-z)R_m(\tilde{A};x,z)f_2(z)}{|y-z|^{n-\delta-1}}dz$$

$$+ \int_{|y-z|\leqslant t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \frac{\Omega(y-z)f_2(z)}{|y-z|^{n-\delta-1}|x_0-z|^m}[R_m(\tilde{A};x,z) - R_m(\tilde{A};x_0,z)]dz$$

$$+ \int_{|y-z|\leqslant t} \left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} - \left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\right] \frac{\Omega(y-z)R_m(\tilde{A};x_0,z)f_2(z)}{|y-z|^{n-\delta-1}|x_0-z|^m}dz$$

$$- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{|y-z|\leqslant t} \left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \frac{(x-z)^\alpha}{|x-z|^m} \left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2} \frac{(x_0-z)^\alpha}{|x_0-z|^m}\right]$$

$$\times \frac{\Omega(y-z)D^\alpha\tilde{A}(z)f_2(z)}{|y-z|^{n-\delta-1}}dz$$

$$:= III_1 + III_2 + III_3 + III_4.$$

Note that $|x-z| \sim |x_0-z|$ for $x \in \tilde{Q}$ and $z \in R^n \setminus \tilde{Q}$. By the condition of Ω and similar to the proof of Lemma 7, we obtain

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q ||III_1||dx$$

$$\leqslant \frac{C}{|Q|^{1+\beta/n}} \int_Q \left(\int_{R^n\setminus\tilde{Q}} \frac{|x-x_0|}{|x_0-z|^{m+n+1-\delta}} |R_m(\tilde{A};x,z)||f(z)|dz \right) dx$$

$$\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^{k+1}\tilde{Q}} \frac{|x-x_0|}{|x_0-z|^{n+1-\delta}} |f(z)|dz$$

$$\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \sum_{k=1}^{\infty} 2^{-k} \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}} \int_{2^k\tilde{Q}} |f(z)|dz$$

$$\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \sum_{k=1}^{\infty} 2^{-k} M_{\delta,1}(f)(\tilde{x})$$

$$\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} M_{\delta,1}(f)(\tilde{x}).$$

For III_2, by the formula (see [5])

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)=\sum_{|\eta|<m}\frac{1}{\eta!}R_{m-|\eta|}(D^\eta\tilde{A};x,x_0)(x-z)^\eta$$

and Lemma 5, we get

$$|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}|x-x_0||x_0-z|^{m-1}.$$

Thus

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q ||III_2||dx\\
&\leqslant C\frac{1}{|Q|^{1+\beta/n}}\int_Q\int_{R^n\setminus\tilde{Q}}\frac{|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n-\delta}}|f(z)|dzdx\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}|f(z)|dz\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(\tilde{x}).
\end{aligned}$$

For III_3, by inequality $a^{1/2}-b^{1/2}\leqslant(a-b)^{1/2}$ if $a\geqslant b>0$, we gain

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q ||III_3||dx\\
&\leqslant\frac{C}{|Q|^{1+\beta/n}}\int_Q\int_{R^n}\left(\int_{R_+^{n+1}}\left[\frac{t^{n\lambda/2}|x-x_0|^{1/2}\chi_{\Gamma(z)}(y,t)|f_2(z)|}{(t+|x-y|)^{(n\lambda+1)/2}|y-z|^{n-1-\delta}}\frac{|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\right]^2\frac{dydt}{t^{n+3}}\right)^{\frac{1}{2}}dzdx\\
&\leqslant\frac{C}{|Q|^{1+\beta/n}}\int_Q\int_{R^n}\frac{|R_m(\tilde{A};x_0,z)||f_2(z)||x-x_0|^{1/2}}{|x_0-z|^{m+n+1/2-\delta}}dzdx\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=1}^\infty 2^{-k/2}M_{\delta,1}(f)(\tilde{x})\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,1}(f)(\tilde{x}).
\end{aligned}$$

For III_4, by Lemma 4, we get

$$|D^\alpha A(z)-(D^\alpha A)_{\tilde{Q}}|\leqslant||D^\alpha A||_{\dot{\Lambda}_\beta}|x_0-z|^\beta.$$

Thus, similarly to the proof of III_1 and III_3, we obtain

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q ||III_4||dx$$

$$\leqslant \frac{C}{|Q|^{1+\beta/n}}\int_Q \sum_{|\alpha|=m}\int_{R^n}\left(\frac{|x-x_0|}{|x_0-z|^{n+1-\delta}}+\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2-\delta}}\right)|f_2(z)||D^\alpha \tilde{A}(z)|dzdx$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=1}^{\infty}(2^{k(\beta-1)}+2^{k(\beta-1/2)})M_{\delta,1}(f)(\tilde{x})$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,1}(f)(\tilde{x}).$$

Thus

$$III\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}M_{\delta,1}(f)(\tilde{x}).$$

We now put these estimates together, and taking the supremum over all Q such that $\tilde{x}\in Q$, and using Lemma 1 with Lemma 3, we obtain

$$||\mu_\lambda^A(f)||_{\dot{F}_q^{\beta,\infty}}\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. By the same argument as in the proof of Theorem 1, we have, for $1\leqslant s<p$ and $1/r=1/s-\delta/n$,

$$\frac{1}{|Q|}\int_Q|\mu_\lambda^A(f)(x)-\mu_\lambda^{\tilde{A}}(f_2)(x_0)|dx\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}(M_{\beta+\delta,r}(f)+M_{\beta+\delta,1}(f)),$$

thus

$$(\mu_\lambda^A(f))^{\#}\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}(M_{\beta+\delta,r}(f)+M_{\beta+\delta,1}(f)).$$

Now, using Lemma 3, we gain

$$||\mu_\lambda^A(f)||_{L^q}\leqslant C||(\mu_\lambda^A(f))^{\#}||_{L^q}$$

$$\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}(||M_{\beta+\delta,r}(f)||_{L^q}+||M_{\beta+\delta,1}(f)||_{L^q})\leqslant C||f||_{L^p}.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. First we prove the following estimate

$$|\mu_\lambda^A(f)(x)|\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}(\eta_1^{\delta+\beta}Mf(x)+\eta_1^{\delta+\beta-n/r'}||f||_{L^1}^{1/r'}(Mf(x))^{1/r}),$$

for any $\eta_1 > 0$ and $n/(n-\delta-\beta) < r$. In fact, for the fixed the cube $Q = Q(x, \lambda_1)$, similarly to the proof of Lemma 6, we have

$$|\mu_\lambda^A(f)(x)| \leqslant C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} dz$$
$$= C\left(\int_Q + \int_{Q^c}\right) \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} dz = I_1 + I_2.$$

For $k > 0$ we put

$$\tilde{A}_k(y) = A(y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2^{-k}Q} y^\alpha.$$

Then, by Lemma 5, for $z \in 2^{-k}Q$,

$$|R_{m+1}(\tilde{A}_k; x, z)| \leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} (2^{-k}\eta_1)^\beta |x-z|^m.$$

Thus, by Lemma 5 and Lemma 6

$$I_1 \leqslant C \sum_{k=0}^{\infty} \int_{2^{-k}Q \setminus 2^{-k-1}Q} \frac{|f(z)||R_{m+1}(\tilde{A}_k; x, z)|}{|x-z|^{m+n-\delta}} dz$$
$$\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \sum_{k=0}^{\infty} (2^{-k}\eta_1)^\beta \int_{2^{-k}Q \setminus 2^{-k-1}Q} \frac{|f(z)|}{|x-z|^{n-\delta}} dz$$
$$\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \sum_{k=0}^{\infty} (2^{-k}\eta_1)^{\beta+\delta-n} \int_{2^{-k}Q \setminus 2^{-k-1}Q} |f(z)| dz$$
$$\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \eta_1^{\beta+\delta} M(f)(x).$$

For I_2, taking $\varepsilon > 0$ such that $(n+\varepsilon)/(n-\delta-\beta) < r$, we write $n - \delta = (n+\varepsilon)/r + n/r' - \varepsilon/r - \delta$, then, by Hölder's inequality,

$$\begin{aligned}
I_2 &\leqslant C\left(\int_{Q^c}\frac{|f(z)|dz}{|x-z|^{n+\varepsilon}}\right)^{1/r}\left(\int_{Q^c}\frac{|f(z)|}{|x-z|^{n-(\delta+\varepsilon/r)r'}}\left(\frac{|R_{m+1}(A;x,z)|}{|x-z|^m}\right)^{r'}dz\right)^{1/r'}\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\left(\sum_{k=0}^{\infty}(2^k\lambda_1)^{-\varepsilon-n}\int_{|x-z|<2^k\lambda_1}|f(z)|dz\right)^{1/r}\\
&\quad\times\left(\sum_{k=0}^{\infty}(2^k\lambda_1)^{\beta r'}\int_{2^{-k}Q\setminus 2^{-k-1}Q}\frac{|f(z)|dz}{|x-z|^{n-(\delta+\varepsilon/r)r'}}\right)^{1/r'}\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\left(\sum_{k=0}^{\infty}2^{-k\varepsilon}\eta_1^{-\varepsilon}M(f)(x)\right)^{1/r}\eta_1^{\delta+\beta-n/r'+\varepsilon/r}\\
&\quad\times\left(\sum_{k=0}^{\infty}2^{k(\delta+\beta-n/r'+\varepsilon/r)r'}\int_{2^{-k}Q\setminus 2^{-k-1}Q}|f(z)|dz\right)^{1/r'}\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\eta_1^{-\varepsilon/r}(M(f)(x))^{1/r}\eta_1^{\delta+\beta-n/r'+\varepsilon/r}||f||_{L^1}^{1/r'}\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\eta_1^{\delta+\beta-n/r'}||f||_{L^1}^{1/r'}(M(f)(x))^{1/r}.
\end{aligned}$$

Thus, the desired estimate holds. Now we can prove Theorem 3. For any $\eta>0$ and $f\in L^1(R^n)$, taking $\eta_1=(\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}||f||_{L^1}\eta^{-1})^{1/(n-\delta-\beta)}$ in above estimate, we gain, by the weak type boundedness of M,

$$\begin{aligned}
&|\{x\in R^n:\mu_\lambda^A(f)(x)>\eta\}|\\
&\leqslant\left|\left\{x\in R^n:Mf(x)>\frac{\eta}{2C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\eta_1^{\delta+\beta}}\right\}\right|\\
&\quad+\left|\left\{x\in R^n:Mf(x)>\left(\frac{\eta}{2C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\eta_1^{\delta+\beta-n/r'}||f||_{L^1}^{1/r}}\right)^{r}\right\}\right|\\
&\leqslant C\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\eta_1^{\delta+\beta}||f||_{L^1}/\eta+C(\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}\eta_1^{\delta+\beta-n/r'}||f||_{L^1}^{1/r}/\eta)^r\\
&\leqslant C(\sum_{|\alpha|=m}||D^\alpha A||_{\dot\Lambda_\beta}||f||_{L^1}/\eta)^{n/(n-\delta-\beta)}.
\end{aligned}$$

This completes the proof of Theorem 3.

References

[1] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), 7–16.

[2] W.G. Chen, *Besov estimates for a class of multilinear singular integrals*, Acta Math. Sinica, **16** (2000), 613–626.

[3] J. Cohen, *A sharp estimate for a multilinear singular integral on R^n*, Indiana Univ. Math. J., **30** (1981), 693–702.

[4] J. Cohen and J. Gosselin, *On multilinear singular integral operators on R^n*, Studia Math., **72** (1982), 199–223.

[5] J. Cohen and J. Gosselin, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math., **30** (1986), 445–465.

[6] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), 611–635.

[7] R.A. DeVore and R.C. Sharpley, *Maximal functions measuring smoothness*, Mem. Amer. Math. Soc., **47** (1984).

[8] Y. Ding and S.Z. Lu, *Weighted boundedness for a class rough multilinear operators*, Acta Math. Sinica, **17** (3) (2001), 517–526.

[9] Y. Ding, S.Z. Lu and Q. Xue, *On Marcinkiewicz integral with homogeneous kernels*, J. Math. Anal. Appl., **245** (2000), 471–488.

[10] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Math., **16** (1978), 263–270.

[11] S. Janson, M. Taibleson and G. Weiss, *Elementary characterizations of the Morrey-Campanato spaces*, Lect. Noces in Math., **992** (1983), 101–114.

[12] L.Z. Liu, *Boundedness for multilinear Marcinkiewicz Operators on certain Hardy Spaces*, Inter. J. of Math. and Math. Sci., **2** (2003), 87–96.

[13] M. Paluszynski, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J., **44** (1995), 1–17.

[14] A. Torchinsky, *The real variable methods in harmonic analysis*, Pure and Applied Math. **123**, Academic Press, New York, 1986.

[15] A. Torchinsky and S. Wang, *A note on the Marcinkiewicz integral*, Colloq. Math., **60/61** (1990), 235–243.

Address: Lanzhe Liu, College of Mathematics, Changsha University of Science and Technology, Changsha 410077, P.R. of China

E-mail: lanzheliu@263.net

Received: 1 July 2004; **revised:** 20 March 2005