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ON THE SUM OF A PRIME AND A k-FREE NUMBER

ALESSANDRO LANGUASCO

Abstract: We prove a refined asymptotic formula for the number of representations of sufficiently large integer as a sum of a prime and a k-free number, $k \geqslant 2$.
Keywords: prime numbers, k-free numbers.

1. Introduction

The problem of counting the number of representations of an integer as a sum of a prime and a square-free integer was first considered by Estermann [3] in 1931. He obtained an asymptotic formula that was subsequently refined by Page [11] and then by Walfisz [13] in 1936. In 1949 Mirsky [10] generalized such results to the case of the sum of a prime and a k-free number, where $k \geqslant 2$ is a fixed integer. He obtained, for every $A > 0$, that

$$r_k(n) = \sum_{p \leqslant n} \mu_k(n-p) = \mathfrak{S}_k(n)\mathrm{li}(n) + O\left(\frac{n}{\log^A n}\right) \quad \text{as } n \to +\infty, \tag{1}$$

where $\mu_k(n) = \sum_{a^k \mid m} \mu(a)$ is the characteristic function of the k-free numbers, $\mu(n)$ is the Möbius function, $\mathrm{li}(n) = \int_2^n \frac{dt}{\log t}$ and

$$\mathfrak{S}_k(n) = \prod_{p \nmid n} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \tag{2}$$

is the singular series of this problem.

The aim of this paper is to prove a refinement of Walfisz-Mirsky asymptotic formula (1). This refinement depends on inserting a new term connected with the existence of the Siegel zero of Dirichlet L-functions (see Lemmas 1-2 below) and by sharping the error term in the asymptotic formula.

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Denoting by $\Lambda(n)$ the von Mangoldt function, we define

$$R_k(n) = \sum_{m \leqslant n} \Lambda(m)\mu_k(n-m)$$

to be the weighted number of representations of an integer n as a sum of a prime and a k-free number. As usual R_k is easily related with r_k. We have the following

Theorem. *Let $k \geqslant 2$ be a fixed integer. Then there exists a constant $c = c(k) > 0$ such that, for every sufficiently large $n \in \mathbb{N}$, we have*

$$R_k(n) = \left(n - \delta_{\widetilde{\beta}}\widetilde{\chi}(n)\frac{n^{\widetilde{\beta}}}{\widetilde{\beta}}\right)\mathfrak{S}_k(n) + O_k(nG\exp(-c\sqrt{\log n})),$$

where $\widetilde{\beta}$ is the Siegel zero, $\widetilde{\chi}$ is the Siegel character, $\widetilde{r}$ is the Siegel modulus associated with the set of Dirichlet L-functions with modulus $q \leqslant \exp(c'\sqrt{\log n})$, where $c' = c'(k) > 0$ is a suitable constant,

$$G = \begin{cases} (1-\widetilde{\beta})\sqrt{\log n} & \text{if } \widetilde{\beta} \text{ exists} \\ 1 & \text{if } \widetilde{\beta} \text{ does not exist,} \end{cases} \qquad \delta_{\widetilde{\beta}} = \begin{cases} 1 & \text{if } \widetilde{\beta} \text{ exists} \\ 0 & \text{if } \widetilde{\beta} \text{ does not exist} \end{cases}$$

(see also Lemmas 1-2 below).

An analogous result, but with a weaker error term, can also be obtained via the circle method using some recent results on exponential sums over k-free numbers proved by Brüdern-Granville-Perelli-Vaughan-Wooley [1].

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2. Lemmas

We recall now some analytic results on the zero-free region of Dirichlet L-functions.

Lemma 1. [Davenport [2], §13-14] *Assume $T' \geqslant 0$. There exists a constant $c_1 > 0$ such that $L(\sigma + it, \chi) \neq 0$ whenever*

$$\sigma \geqslant 1 - \frac{c_1}{\log T'}, \quad |t| \leqslant T'$$

for all the Dirichlet characters χ modulo $q \leqslant T'$, with the possible exception of at most one primitive character $\widetilde{\chi} \pmod{\widetilde{r}}$, $\widetilde{r} \leqslant T'$. If it exists, the character $\widetilde{\chi}$ is real and the exceptional zero $\widetilde{\beta}$ of $L(s, \widetilde{\chi})$ is unique, real, simple and there exists a constant $c_2 > 0$ such that

$$\frac{c_2}{\widetilde{r}^{1/2}\log^2 \widetilde{r}} \leqslant 1 - \widetilde{\beta} \leqslant \frac{c_1}{\log T'}, \quad |t| \leqslant T'.$$

Fix now $T_1 > 0$ such that $\log T_1 \asymp \sqrt{\log n}$. According to Lemma 1, applied with $T' = T_1$, we denote by $\widetilde{\beta}$ the Siegel zero, $\widetilde{\chi}$ the Siegel character and by $\widetilde{r}$ its modulus. Let now

$$T_2 = \begin{cases} T_1 & \text{if } \widetilde{r} \leqslant T_1^{1/4} \\ T_1^{1/4} & \text{otherwise.} \end{cases}$$

Now Lemma 1 remains true for $T' = T_2$, with a suitable change in the constant c_1. In the following we will continue to call c_1 this modified constant. Hence $\widetilde{r} \leqslant T_2^{1/4}$, if it exists. From now on we set $T = T_2$.

Moreover we need also the following form of Deuring-Heilbronn phenomenon whose proof can be found in Knapowski [9], see also §4 of Gallagher [5].

Lemma 2. *Under the same hypotheses of Lemma 1 applied with $T' = T$, if $\widetilde{\beta}$ exists, then for all the Dirichlet characters χ modulo $q \leqslant T$, there exists a constant $c_3 > 0$ such that $L(\sigma + it, \chi) \neq 0$ whenever*

$$\sigma \geqslant 1 - \frac{c_3}{\log T} \log\left(\frac{ec_1}{(1-\widetilde{\beta})\log T}\right), \quad |t| \leqslant T$$

and $\widetilde{\beta}$ is still the only exception.

The next Lemma is the explicit formula for $\psi(x, \chi)$.

Lemma 3. [Davenport [2], §19] *Let χ a Dirichlet character to the modulus q and $2 \leqslant T \leqslant x$. Then*

$$\sum_{m \leqslant x} \Lambda(m)\chi(m) = \delta_\chi x - \delta_{\chi,\widetilde{\chi}} \frac{x^{\widetilde{\beta}}}{\widetilde{\beta}} - \sideset{}{'}\sum_{|\rho| \leqslant T} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}\log^2 qx + x^{1/4}\log x\right),$$

where $\delta_\chi = 1$ if χ is the principal character, $\delta_\chi = 0$ otherwise, $\delta_{\chi,\widetilde{\chi}} = 1$ if $\chi = \widetilde{\chi}$ and $\delta_{\chi,\widetilde{\chi}} = 0$ otherwise and $\sum'$ means that the sum runs over the non-exceptional zeros.

We will need also a zero-density result for Dirichlet's L-functions.

Lemma 4. [Huxley [7] and Ramachandra [12]] *Let χ be a Dirichlet character (mod q) and $N(\sigma, T, \chi) = |\{\rho = \beta + i\gamma : L(\rho, \chi) = 0, \beta \geqslant \sigma$ and $|\gamma| \leqslant T\}|$. Then, for $\sigma \in [1/2, 1]$, there exists a positive absolute constant c_4 such that*

$$\sum_\chi N(\sigma, T, \chi) \ll (qT)^{12/5(1-\sigma)} (\log qT)^{c_4}. \tag{3}$$

3. Proof of the theorem

Following Walfisz [13] and Mirsky [10], we have

$$R_k(n) = \sum_{m \leqslant n} \Lambda(m) \sum_{d^k | (n-m)} \mu(d) = \sum_{m \leqslant n} \Lambda(m) \Bigg[\sum_{\substack{d^k | (n-m) \\ d \leqslant D}} \mu(d) + \sum_{\substack{d^k | (n-m) \\ d > D}} \mu(d) \Bigg] =$$

$$= \sum_{d \leqslant D} \mu(d) \sum_{\substack{m \leqslant n \\ d^k | (n-m)}} \Lambda(m) + \sum_{d > D} \mu(d) \sum_{\substack{m \leqslant n \\ d^k | (n-m)}} \Lambda(m) = \tag{4}$$

$$= \sum_{d \leqslant D} \mu(d) \psi(n; d^k, n) + \sum_{d > D} \mu(d) \psi(n; d^k, n) = A + B,$$

say, where $\psi(x; q, a) = \sum_{\substack{m \leqslant x \\ m \equiv a \,(\mathrm{mod}\, q)}} \Lambda(m)$ and $1 \leqslant D \leqslant n^{1/k}$ will be chosen later in (12).

First of all, we estimate B. By Brun-Titchmarsh Theorem, see, *e.g.*, Friedlander-Iwaniec [4], and Theorem 328 of Hardy-Wright [6], we get

$$B \leqslant \sum_{d > D} \psi(n; d^k, n) \ll \sum_{d > D} \frac{n}{\varphi(d^k)} \ll_k n \sum_{d > D} \frac{\log \log d}{d^k} \ll_k n D^{1-k} \log \log D. \tag{5}$$

Then we remark that, if $(d, n) > 1$, we have $\psi(n; d^k, n) \ll_k \log^2(dn)$ and hence

$$A = \sum_{\substack{d \leqslant D \\ (d,n)=1}} \mu(d) \psi(n; d^k, n) + O_k(D \log^2(Dn)). \tag{6}$$

We now insert $\psi(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi}(a) \psi(x, \chi)$ in (6). Hence, by Lemma 3 and the previous remarks, we get

$$A = \sum_{\substack{d \leqslant D \\ (d,n)=1}} \frac{\mu(d)}{\varphi(d^k)} \Bigg[n - \delta_{\widetilde{\beta}} \widetilde{\chi}(n) \frac{n^{\widetilde{\beta}}}{\widetilde{\beta}} - \sum_{\substack{\chi \,(\mathrm{mod}\, d^k) \\ \chi \neq \chi_0, \widetilde{\chi}}} \overline{\chi}(n) \sideset{}{'}\sum_{|\rho| \leqslant T} \frac{n^\rho}{\rho} +$$

$$+ O\Bigg(\varphi(d^k) \Big(\frac{n}{T} \log^2(d^k n) + n^{1/4} \log n \Big) \Bigg) \Bigg] + O_k(D \log^2(Dn)) =$$

$$= \Bigg(n - \delta_{\widetilde{\beta}} \widetilde{\chi}(n) \frac{n^{\widetilde{\beta}}}{\widetilde{\beta}} \Bigg) \sum_{\substack{d \leqslant D \\ (d,n)=1}} \frac{\mu(d)}{\varphi(d^k)} - \sum_{\substack{d \leqslant D \\ (d,n)=1}} \frac{\mu(d)}{\varphi(d^k)} \sum_{\substack{\chi \,(\mathrm{mod}\, d^k) \\ \chi \neq \chi_0, \widetilde{\chi}}} \overline{\chi}(n) \sideset{}{'}\sum_{|\rho| \leqslant T} \frac{n^\rho}{\rho} +$$

$$+ O\Bigg(\sum_{\substack{d \leqslant D \\ (d,n)=1}} \Big(\frac{n}{T} \log^2(d^k n) + n^{1/4} \log n \Big) \Bigg) + O_k(D \log^2(Dn)) =$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{7}$$

say.

Evaluation of Σ_1.

To evaluate the singular series we use again Theorem 328 of Hardy-Wright [6], thus obtaining

$$\sum_{\substack{d\leqslant D\\(d,n)=1}}\frac{\mu(d)}{\varphi(d^k)}=\sum_{\substack{d=1\\(d,n)=1}}^{+\infty}\frac{\mu(d)}{\varphi(d^k)}+O\Big(\sum_{d>D}\frac{1}{\varphi(d^k)}\Big)=\mathfrak{S}_k(n)+O_k(D^{1-k}\log\log D)$$

by the Euler identity and (2). Hence we easily get

$$\Sigma_1=\left(n-\delta_{\widetilde{\beta}}\widetilde{\chi}(n)\frac{n^{\widetilde{\beta}}}{\widetilde{\beta}}\right)\mathfrak{S}_k(n)+O_k(nD^{1-k}\log\log D). \tag{8}$$

Estimation of Σ_2.

Writing $\rho=\beta+i\gamma$ we have

$$\Sigma_2\ll\sum_{\substack{d\leqslant D\\(d,n)=1}}\frac{1}{\varphi(d^k)}\sum_{\substack{\chi\,(\mathrm{mod}\ d^k)\\ \chi\neq\chi_0,\widetilde{\chi}}}\sideset{}{'}\sum_{|\rho|\leqslant T}\frac{n^\beta}{|\rho|}\leqslant\sum_{\substack{q\leqslant D^k\\(q,n)=1}}\frac{1}{\varphi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi\neq\chi_0,\widetilde{\chi}}}\sideset{}{'}\sum_{|\rho|\leqslant T}\frac{n^\beta}{|\rho|}. \tag{9}$$

Now, to estimate Σ_2, we first split the summation over ρ according to $0<|\rho|\leqslant 1$ and $1<|\rho|\leqslant T$. Arguing as in §20 of Davenport [2] and using Lemmas 1-2, we get

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi\neq\chi_0,\widetilde{\chi}}}\sideset{}{'}\sum_{0<|\rho|\leqslant 1}\frac{n^\beta}{|\rho|}\ll n^{1-f(T)}\log^2 n, \tag{10}$$

where $f(T)=\frac{c_1}{\log T}$ if the Siegel zero does not exist or $f(T)=\frac{c_3}{\log T}\log\left(\frac{ec_1}{(1-\widetilde{\beta})\log T}\right)$ if the Siegel zero exists.

In the range $1<|\rho|\leqslant T$, we follow the line of §12 of Ivić [8]. Recalling Lemmas 1-2 and 4 and Theorem 328 of Hardy-Wright [6], we have, for $D^k\leqslant T$, that

$$\frac{1}{\varphi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\ q)\\ \chi\neq\chi_0,\widetilde{\chi}}}\sideset{}{'}\sum_{1<|\rho|\leqslant T}\frac{n^\beta}{|\rho|}\ll(\log^{c_4+3}n)\max_{1/2\leqslant\sigma\leqslant 1-f(T)}n^\sigma\max_{1\leqslant t\leqslant T}(qt)^{12/5(1-\sigma)-1} \tag{11}$$

where $f(T)$ is as in (10).

Choosing now

$$T=D^{2k}\quad\text{and}\quad T=\exp(C\sqrt{\log n}), \tag{12}$$

where $C > 0$ is an absolute constant, we split the interval over σ in two parts: the first one is for $\sigma \in [1/2, 7/12]$ and the second one is for $\sigma \in [7/12, 1 - f(T)]$. In the first case the maxima are attained at $t = T$ and $\sigma = 7/12$ and in the second case they are attained at $t = 1$ and $\sigma = 1 - f(T)$. The total contribution of (11) is then

$$\ll (n^{7/12} + n^{1-f(T)})T^{1/2} \log^E n \ll T^{1/2} n^{1-f(T)} \log^E n, \tag{13}$$

where $E > 0$ is a suitable constant, not necessarily the same at each occurrence. An analogous argument for (10) gives the same estimate. Hence, by (10) and (12)–(13), we obtain

$$\Sigma_2 \ll T^{1/2} n^{1-f(T)} \log^E n. \tag{14}$$

If the Siegel zero does not exist than we have

$$\Sigma_2 \ll_k T^{1/2} n \exp(-c_1 \frac{\log n}{\log T}) \log^E n, \tag{15}$$

while, if the Siegel zero exists, we get

$$\begin{aligned} \Sigma_2 &\ll_k T^{1/2} n \exp\Big(- c_3 \frac{\log n}{\log T} \log \big(\frac{e c_1}{(1-\widetilde{\beta}) \log T} \big) \Big) \log^E n \ll \\ &\ll T^{1/2} n \big[(1-\widetilde{\beta}) \log T \big] \exp(-c_3 \frac{\log n}{\log T}) \log^E n, \end{aligned} \tag{16}$$

and hence, combining (15)–(16) we finally have

$$\Sigma_2 \ll T^{1/2} n G \exp(-c_5 \frac{\log n}{\log T}) \log^E n, \tag{17}$$

where $c_5 = \min(c_1; c_3)$ and

$$G = \begin{cases} (1-\widetilde{\beta})\sqrt{\log n} & \textit{if } \widetilde{\beta} \textit{ exists} \\ 1 & \textit{if } \widetilde{\beta} \textit{ does not exist.} \end{cases}$$

Estimation of Σ_3 and the final argument.
Recalling $T = D^{2k}$ and $T = \exp(C\sqrt{\log n})$, we get from (17) that

$$\Sigma_2 \ll_k n G \exp(-c_6 \sqrt{\log n}), \tag{18}$$

with

$$C = \sqrt{c_5} \quad \text{and} \quad c_6 = \sqrt{c_5}/3. \tag{19}$$

From (8) we obtain

$$\Sigma_1 = \Big(n - \delta_{\widetilde{\beta}} \widetilde{\chi}(n) \frac{n^{\widetilde{\beta}}}{\widetilde{\beta}} \Big) \mathfrak{S}_k(n) + O_k(n \exp(-C \frac{k-1}{3k} \sqrt{\log n})). \tag{20}$$

Moreover, the error terms collected in Σ_3 can be estimated as follows:

$$\Sigma_3 \ll_k \frac{nD}{T}\log^2(D^k n) + n^{1/4}D\log n + D\log^2(Dn) \ll$$
$$\ll_k n\exp(-C\frac{2k-1}{3k}\sqrt{\log n}). \tag{21}$$

Hence, if the Siegel zero does not exist, inserting (18)–(21) into (4)–(5) and (7) we have the Theorem with $c = C\frac{k-1}{3k}$ provided that $C < \frac{3k}{k-1}c_6$ (which holds by (19)).

If the Siegel zero exists, we remark that

$$n - \widetilde{\chi}(n)\frac{n^{\widetilde{\beta}}}{\widetilde{\beta}} \geqslant n - \frac{n^{\widetilde{\beta}}}{\widetilde{\beta}} = \int_T^n (1 - t^{\widetilde{\beta}-1})dt + O(T) \gg n(1 - T^{\widetilde{\beta}-1}) + O(T) \gg$$
$$\gg Gn + O(T)$$

and, by Lemma 1, that

$$G \gg \frac{\sqrt{\log n}}{\widetilde{r}^{1/2}\log^2\widetilde{r}} \gg \exp(-C\frac{k-1}{3k}\sqrt{\log n}),$$

since $\widetilde{r} \leqslant T^{1/4} = \exp((C/4)\sqrt{\log n})$.

Provided that $C < \frac{3k}{k-1}c_6$ (which holds by (19)), the Theorem follows also in this case with $c = C\frac{k-1}{3k}$ by inserting (18)–(21) into (4)–(5) and (7).

References

[1] J. Brüdern, A. Granville, A. Perelli, R.C. Vaughan, T.D. Wooley, On the exponential sum over k-free numbers, *Phil. Trans. R. Soc. Lond. A* **356** (1998), 739–761.

[2] H. Davenport, *Multiplicative Number Theory*, 3-rd ed., Springer GTM, 2001.

[3] T. Estermann, On the representations of a number as the sum of a prime and a quadratfrei number, *Journal London Math. Soc.* **6** (1931), 219–221.

[4] J. Friedlander, H. Iwaniec, The Brun-Titchmarsh theorem, in "*Analytic Number Theory*" ed. by Y. Motohashi, *Lecture Notes of the London Math. Soc.* **247** (1997), 85–93.

[5] P.X. Gallagher, A large sieve density estimate near $\sigma = 1$, *Invent. Math.* **11** (1970), 329–339.

[6] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, fifth edition, Oxford Science Publications, 1979.

[7] M.N. Huxley, Large values of Dirichlet polynomials, III, *Acta Arith.* **26** (1975), 435–444.

[8] A. Ivić, *The Riemann Zeta-Function*, Wiley, 1985.

[9] S. Knapowski, On Linnik's theorem concerning exceptional L-zeros, *Publ. Math. Debrecen.* **9** (1962), 168–178.

[10] L. Mirsky, The number of representation of an integer as the sum of a prime and a k-free integer, *Amer. Math. Monthly* **56** (1949), 17–19.
[11] A. Page, On the number of primes in an arithmetic progression, *Proc. London Math. Soc.* **39** (1935), 116–141.
[12] K. Ramachandra, On the number of Goldbach numbers in small intervals, *J. Indian Math. Soc.* **37** (1973), 157–170.
[13] A. Walfisz, Zur Additive Zahlentheorie II, *Math. Z.* **40** (1936), 592–607.

Address: Università di Padova, Dipartimento di Matematica Pura e Applicata, Via Belzoni, 7, 35131 Padova, Italy
E-mail: `languasco@math.unipd.it`
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