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ON THE RIEMANN INTEGRABILITY OF THE n-TH LOCAL MODULUS OF CONTINUITY

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Abstract: The present note proves Riemann integrability of the n-th local modulus of continuity which is used within the definition of averaged moduli of smoothness (τ-moduli). In addition it is shown that an averaged supremum norm (δ-norm) can be calculated using the Riemann integral.

Keywords: Averaged modulus of smoothness, local modulus of continuity, Riemann integrability.

1. Introduction

This paper deals with moduli of continuity. To this end, let us first introduce some notations. $B[a,b]$ denotes the space of bounded, real valued functions on a compact interval $[a,b]$. Let $f \in B[a,b]$ and $n \in \mathbb{N}$ where $\mathbb{N} = \{1,2,...\}$ is the set of natural numbers. The n-th difference of f at a point $x \in [a,b]$ is defined as

$$\Delta_h^n f(x) := \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+jh).$$

We deal with the n-th local modulus of continuity of f at a point $x \in [a,b]$. For $\delta > 0$ it is defined as

$$\omega_n(\delta, f, x) := \sup \left\{ |\Delta_h^n f(t)| : t, t+nh \in [x-\delta, x+\delta] \cap [a,b], 0 \leqslant h \leqslant \frac{2\delta}{n} \right\}. \quad (1.1)$$

Lebesgue integrability of $\omega_n(\delta, f, \cdot)$ is well known for Lebesgue measurable, bounded functions f (cf. [5], p. 12ff). Nevertheless, because Riemann integrability of these f is equivalent to (see [5], p. 11, cf. Section 5, (5.3))

$$\lim_{\delta \to 0+} \text{L-} \int_a^b \omega_1(\delta, f, t)\, dt = 0, \quad (1.2)$$

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it seems quite natural to replace the Lebesgue integral by a Riemann integral. Therefore, apart from Sendov's and Korovkin's definition of τ-moduli via the Lebesgue integral, the following definition is established in the context of functions f belonging to the space $R[a,b]$ of Riemann integrable functions (cf. [1], [2], [3], [6], $1 \leqslant p < \infty$):

$$\tau_{n,p}(\delta, f) := \left[\overline{\int_a^b} \omega_n(\delta, f, t)^p \, dt \right]^{\frac{1}{p}}. \tag{1.3}$$

The upper Riemann integral in (1.3) can be replaced by the Riemann integral in case $n = 1$. In this connection it would be desirable to use the Riemann integral for all n. Unfortunately, the short proof for $n = 1$ (see [3], pp. 276–277) strongly depends on the structure of the first difference, which is devided into two first differences. Then one of the differences can be estimated using continuity following from Riemann integrability of f. This gives continuity of $\omega_1(\delta, f, \cdot)$ a.e. and therefore Riemann integrability. However, the argument does not work for higher differences.

In what follows Riemann integrability of $\omega_n(\delta, f, x)$ is proved for arbitrary n (see Theorem 4.1) using a different approach. First we discuss the monotonicity of slightly modified local moduli for bounded functions f. Here we do not only get Riemann integrability of these moduli but also of a local supremum of f. The δ-norm, which is often defined by an upper Riemann integral of this local supremum (cf. [1], [6]), can therefore generally be rewritten using the Riemann integral. The second step of the proof utilizes Riemann integrability of f in order to extend the first result to the original local modulus (1.1). This is similar to the proof of Lebesgue measurability in [5], p. 12ff, where also two steps are performed and measurability of f only is used in the final step. The paper concludes with the well known characterization (1.2) of Riemann integrability which now can be written using Riemann integrals only.

2. Using monotonicity

To show integrability of local moduli of continuity for arbitrary differences we first establish a theorem by using properties of a local supremum. Here monotonicity leads to continuity thus proving Riemann integrability. In this section Riemann integrability of f and the structure of the n-th difference are not used.

Theorem 2.1. *Let $f \in B[a,b]$ and*

$$\{F_h^t : B[t, t+h] \to \mathbb{R} \, : \, h \geqslant 0, t, t+h \in [a,b]\}$$

be a family of bounded functionals with $|F_h^t g| \leqslant C \|g\|_{B[t,t+h]}$ for a constant $C < \infty$ independent of t, h and g, where $\|g\|_{B[c,d]} := \sup\{|g(s)| : s \in [c,d]\}$. For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $x \in [a,b]$ and $\delta > 0$ let

$$\omega_F(\delta, f, x)_k := \sup \left\{ |F_h^t f| : t, t+h \in [x-\delta, x+\delta] \cap [a,b], 0 \leqslant h \leqslant 2\delta \frac{k}{k+1} \right\}$$

be a generalized local modulus of continuity. Then there holds true:

$$\omega_F(\delta, f, \cdot)_k \in R[a, b],$$

and on $[a, b]$ *the function* $\omega_F(\delta, f, \cdot)_k$ *has not more than a countable set of points of discontinuity.*

Proof. We have to show that, apart from a countable set of points, $\omega_F(\delta, f, \cdot)_k$ is continuous on $[a, b]$. Then Riemann integrability follows because of the boundedness $\|\omega_F(\delta, f, \cdot)_k\|_{B[a,b]} \leqslant C\|f\|_{B[a,b]}$. On $[a, \min\{a+\delta, b\}]$ the function $\omega_F(\delta, f, \cdot)_k$ is increasing, on $[\max\{a, b-\delta\}, b]$ it is decreasing. Therefore, on these intervals it is discontinuous only for a countable set of points. If one has $b - \delta \leqslant a + \delta$ the theorem is proven. Therefore, let $a + \delta < b - \delta$ and $x_0 \in (a + \delta, b - \delta)$. For convenience we set $h_0 := 2\delta\frac{k}{k+1}$. Consider the following condition: There exists $\nu > 0$ with $x_0 + \nu \leqslant b - \delta$ such that for all $t \in [x_0, x_0 + \nu]$ there holds true

$$\omega_F(\delta, f, t)_k \leqslant \omega_F(\delta, f, x_0)_k. \tag{2.1}$$

Case 1: Condition (2.1) holds true. Then we distinguish between two further cases:

Case 1a: There exists $\nu_{x_0} > 0$, $\nu_{x_0} \leqslant \nu$ such that for all $t \in [x_0, x_0 + \nu_{x_0}]$:

$$\omega_F(\delta, f, t)_k = \omega_F(\delta, f, x_0)_k.$$

Case 1b: If Case 1a does not hold true, then there exists

$$x_\delta \in \left(x_0, x_0 + \min\left\{\nu, \frac{\delta}{k+1}\right\}\right) \tag{2.2}$$

such that

$$\omega_F(\delta, f, x_\delta)_k < \omega_F(\delta, f, x_0)_k. \tag{2.3}$$

Because of (2.2) one gets

$$\left[x_0 - \delta + \frac{\delta}{k+1}, x_0 + \delta\right] \subset [x_\delta - \delta, x_\delta + \delta] \cap [x_0 - \delta, x_0 + \delta]$$

and

$$\sup\left\{|F_h^t f| : t, t+h \in \left[x_0 - \delta + \frac{\delta}{k+1}, x_0 + \delta\right], 0 \leqslant h \leqslant h_0\right\} \leqslant$$
$$\leqslant \omega_F(\delta, f, x_\delta)_k. \tag{2.4}$$

With (2.3) and (2.4) it follows that

$$\sup\left\{|F_h^t f| : t \in \left[x_0 - \delta, x_0 - \delta + \frac{\delta}{k+1}\right], 0 \leqslant h \leqslant h_0\right\} = \omega_F(\delta, f, x_0)_k. \tag{2.5}$$

Because of

$$\left\{t-h : t \in \left[x_0+\delta-\frac{\delta}{k+1}, x_0+\delta\right], 0 \leqslant h \leqslant h_0\right\} =$$
$$= \left[x_0+\delta-\frac{\delta}{k+1}-h_0, x_0+\delta\right] = \left[x_0-\delta+\frac{\delta}{k+1}, x_0+\delta\right]$$

and (2.4) we conclude

$$\omega_F(\delta, f, x_\delta)_k \geqslant \sup\left\{|F_h^{t-h} f| : t \in \left[x_0+\delta-\frac{\delta}{k+1}, x_0+\delta\right], 0 \leqslant h \leqslant h_0\right\}. \quad (2.6)$$

Putting together (2.5), (2.3) and (2.6) one gets

$$\sup\left\{|F_h^t f| : t \in \left[x_0-\delta, x_0-\delta+\frac{\delta}{k+1}\right], 0 \leqslant h \leqslant h_0\right\} >$$
$$> \sup\left\{|F_h^{t-h} f| : t \in \left[x_0+\delta-\frac{\delta}{k+1}, x_0+\delta\right], 0 \leqslant h \leqslant h_0\right\}. \quad (2.7)$$

Case 2: Suppose condition (2.1) is not valid. We show that $\omega_F(\delta, f, \cdot)_k$ is increasing on $\left[x_0, x_0+\frac{2\delta}{k+1}\right]$. To this end, let $x_1, x_2 \in \left(x_0, x_0+\frac{2\delta}{k+1}\right]$ with $x_1 < x_2$. Because (2.1) does not hold true, there exists $x_\nu \in (x_0, x_1)$ such that

$$\omega_F(\delta, f, x_\nu)_k > \omega_F(\delta, f, x_0)_k. \quad (2.8)$$

This gives

$$\sup\left\{|F_h^t f| : t \in \left[x_\nu-\delta, x_0-\delta+\frac{2\delta}{k+1}\right], 0 \leqslant h \leqslant h_0\right\} \leqslant$$
$$\leqslant \sup\left\{|F_h^t f| : t \in \left[x_0-\delta, x_0-\delta+\frac{2\delta}{k+1}\right], 0 \leqslant h \leqslant h_0\right\} \leqslant$$
$$\leqslant \omega_F(\delta, f, x_0)_k < \omega_F(\delta, f, x_\nu)_k. \quad (2.9)$$

Note that, if one would replace condition $0 \leqslant h \leqslant h_0$ by $0 \leqslant h \leqslant 2\delta$ in (2.9), the estimation by $\omega_F(\delta, f, x_0)_k$ would not be possible. Obviously,

$$\omega_F(\delta, f, x_1)_k = \sup\left\{|F_h^t f| : t, t+h \in [x_1-\delta, x_1+\delta] \cap [a,b], 0 \leqslant h \leqslant h_0\right\} \leqslant$$
$$\leqslant \sup\left\{|F_h^t f| : t, t+h \in [x_\nu-\delta, x_1+\delta] \cap [a,b], 0 \leqslant h \leqslant h_0\right\},$$

and therefore by (2.9)

$$\omega_F(\delta, f, x_1)_k \leqslant$$
$$\leqslant \sup\left\{|F_h^t f| : t, t+h \in [x_\nu-\delta, x_1+\delta] \cap [a,b], 0 \leqslant h \leqslant h_0\right\} =$$
$$= \sup\left\{|F_h^t f| : t, t+h \in \left[x_0-\delta+\frac{2\delta}{k+1}, x_1+\delta\right] \cap [a,b], 0 \leqslant h \leqslant h_0\right\} \leqslant$$
$$\leqslant \sup\left\{|F_h^t f| : t, t+h \in [x_2-\delta, x_2+\delta] \cap [a,b], 0 \leqslant h \leqslant h_0\right\} =$$
$$= \omega_F(\delta, f, x_2)_k.$$

Thus, $\omega_F(\delta, f, \cdot)_k$ is increasing on $\left(x_0, x_0 + \frac{2\delta}{k+1}\right]$. Together with (2.8) one has

$$\omega_F(\delta, f, x_0)_k < \omega_F(\delta, f, x_\nu)_k \leqslant \omega_F(\delta, f, x_1)_k,$$

and $\omega_F(\delta, f, \cdot)_k$ is increasing on $\left[x_0, x_0 + \frac{2\delta}{k+1}\right]$.

To summarize the results of Case 1 and Case 2 we define

$$A := \bigcup_{x_0 \in (a+\delta, b-\delta):\ \text{for } x_0 \text{ Case 1a holds true}} [x_0, x_0 + \nu_{x_0}]$$

$$\cup \bigcup_{x_0 \in (a+\delta, b-\delta):\ \text{for } x_0 \text{ Case 2 holds true}} \left[x_0, x_0 + \frac{2\delta}{k+1}\right].$$

The components of connectedness of A are intervals, which are not degenerated to points. Therefore, one can write A as a countable union of disjunct intervals such that $\omega_F(\delta, f, \cdot)_k$ is increasing on each single interval, i.e., $\omega_F(\delta, f, \cdot)_k$ has not more than a countable set of points of discontinuity on each interval. Apart from a countable set of points each point in A is an inner point of A. This means that $\omega_F(\delta, f, \cdot)_k$ is continuous with respect to $[a, b]$ at all but a countable set of points of A.

One gets similar results if one discusses the following condition instead of (2.1): There exists $\nu > 0$ with $x_0 - \nu \geqslant a + \delta$ such that for all $t \in [x_0 - \nu, x_0]$ there holds true

$$\omega_F(\delta, f, t)_k \leqslant \omega_F(\delta, f, x_0)_k. \tag{2.10}$$

If this condition holds true for some x_0, either $\omega_F(\delta, f, \cdot)_k$ is constant on an interval $[x_0 - \nu_{x_0}, x_0]$ or one has

$$\begin{aligned} &\sup\left\{|F_h^{t-h} f| : t \in \left[x_0 + \delta - \frac{\delta}{k+1}, x_0 + \delta\right], 0 \leqslant h \leqslant h_0\right\} > \\ &> \sup\left\{|F_h^t f| : t \in \left[x_0 - \delta, x_0 - \delta + \frac{\delta}{k+1}\right], 0 \leqslant h \leqslant h_0\right\}. \end{aligned} \tag{2.11}$$

If condition (2.10) is false for x_0, it follows that $\omega_F(\delta, f, \cdot)_k$ is decreasing on $\left[x_0 - \frac{2\delta}{k+1}, x_0\right]$. Let B be the union of all these intervals and all intervals $[x_0 - \nu_{x_0}, x_0]$ where $\omega_F(\delta, f, \cdot)_k$ is constant. B is a countable union of intervals such that $\omega_F(\delta, f, \cdot)_k$ is decreasing on each of those intervals. Therefore, $\omega_F(\delta, f, \cdot)_k$ is continuous on B with respect to $[a, b]$ at all but a countable set of points.

Because inequalities (2.7) and (2.11) contradict each other at a specific x_0, it follows:

$$(a + \delta, b - \delta) = (A \cup B) \cap (a + \delta, b - \delta).$$

This shows that the set of points, at which the function $\omega_F(\delta, f, \cdot)_k$ is not continuous, is countable. ■

3. Riemann integrability of modified local moduli of continuity and of the δ-norm

Here we apply the result of the previous section to n-th differences. Let $f \in B[a,b]$, $x \in [a,b]$, $n, k \in \mathbb{N}$ and $\delta > 0$. We define local moduli of continuity by (cf. (1.1))

$$\omega_n(\delta, f, x)_k :=$$
$$:= \sup\left\{|\Delta_h^n f(t)| : t, t+nh \in [x-\delta, x+\delta] \cap [a,b], 0 \leqslant h \leqslant \frac{2\delta}{n}\frac{k}{k+1}\right\}.$$

Furthermore, let (cf. [4])

$$M(\delta, f, x) := \|f\|_{B[\max\{a, x-\delta\}, \min\{x+\delta, b\}]} = \sup\{|f(t)| : t \in [x-\delta, x+\delta] \cap [a,b]\}.$$

Note that, in contrast to $\omega_n(\delta, f, \cdot)$, where $h \leqslant 2\delta/n$ (cf. (1.1)), the definition of $\omega_n(\delta, f, \cdot)_k$ indeed requires $h \leqslant \frac{2\delta}{n}\frac{k}{k+1}$. This modification gives the piecewise monotonicity used to prove Riemann integrability in Section 2. In spite of the different range of h, both definitions are equivalent in the following manner:

$$\omega_n(\delta, f, x)_k \leqslant \omega_n(\delta, f, x) \leqslant 2^n \omega_n(\delta, f, x)_k. \tag{3.1}$$

This inequality follows from a well known feature of the global modulus of continuity

$$\omega_n(\delta, f, [c,d]) := \sup\{|\Delta_h^n f(t)| : t, t+nh \in [c,d], 0 \leqslant h \leqslant \delta\}$$

with $[c,d] \subset [a,b]$: For each $j \in \mathbb{N}$ one has $\omega_n(j\delta, f, [c,d]) \leqslant j^n \omega_n(\delta, f, [c,d])$ (cf. [5], p. 2). Choosing $j = 2$ and $[c,d] := [x-\delta, x+\delta] \cap [a,b]$ one gets (3.1):

$$\omega_n(\delta, f, x) = \omega_n(2\delta/n, f, [c,d]) \leqslant$$
$$\leqslant 2^n \omega_n(\delta/n, f, [c,d]) = 2^n \omega_n(\delta, f, x)_1 \leqslant 2^n \omega_n(\delta, f, x)_k.$$

For $f \in R[a,b]$ it is well known that $M(\delta, f, \cdot) \in R[a,b]$ (see [4]). Theorem 2.1 shows that the precondition $f \in R[a,b]$ is not necessary:

Corollary 3.1. *Let $f \in B[a,b]$, $n, k \in \mathbb{N}$ and $\delta > 0$. Then one has*

$$\omega_n(\delta, f, \cdot)_k \in R[a,b], \quad M(\delta, f, \cdot) \in R[a,b],$$

and these functions do not have more than a countable set of points of discontinuity in $[a,b]$. The δ-norm $\|f\|_\delta := \overline{\int_a^b} M(\delta, f, t)\, dt$ (cf. [1], [6]) can be calculated via the Riemann integral:

$$\|f\|_\delta = \int_a^b M(\delta, f, t)\, dt.$$

Proof. The results immediately follow from Theorem 2.1 with the particular choice $F^t_{nh}f := \Delta^n_h f(t)$ and $F^t_h f := f(t)$, respectively. ∎

Using the Riemann integral we can now define the following slightly modified τ-moduli for bounded functions $f \in B[a,b]$ $(1 \leqslant p < \infty)$:

$$\tau_{n,p}(\delta,f)_k := \left[\int_a^b \omega_n(\delta,f,t)^p_k \, dt\right]^{\frac{1}{p}}.$$

Because of (3.1) this new modulus behaves like $\tau_{n,p}(\delta,f)$ (cf. (1.3)):

$$\tau_{n,p}(\delta,f)_k \leqslant \tau_{n,p}(\delta,f) \leqslant 2^n \tau_{n,p}(\delta,f)_k.$$

The sequence $(\omega_n(\delta,f,x)_k)_{k=1}^{\infty}$ is pointwise (with respect to x) convergent to

$$\hat{\omega}_n(\delta,f,x) := \sup\{|\Delta^n_h f(t)| : t, t+nh \in [x-\delta, x+\delta] \cap [a,b], 0 \leqslant h < 2\delta/n\}.$$

With Beppo-Levi's theorem Lebesgue integrability of the limit function follows and

$$\lim_{k\to\infty} \tau_{n,p}(\delta,f)_k = \left[\text{L-}\int_a^b \hat{\omega}_n(\delta,f,t)^p \, dt\right]^{\frac{1}{p}}.$$

Since $\omega_n(\delta,f,x) = \hat{\omega}_n(\delta,f,x)$ for $x \in [a, \min\{a+\delta, b\}) \cup (\max\{a, b-\delta\}, b]$ and

$$\omega_n(\delta,f,x) = \max\left\{\hat{\omega}_n(\delta,f,x), \left|\Delta^n_{\frac{2\delta}{n}} f(x-\delta)\right|\right\}$$

for $x \in [a+\delta, \infty) \cap (-\infty, b-\delta]$ one gets Lebesgue measurability of $\omega_n(\delta,f,\cdot)$ for each measurable and bounded f (see [5], p. 12ff, for a similar proof).

In order to show Riemann integrability of $\omega_n(\delta,f,\cdot)$ we now give a slightly different argument using Corollary 3.1 and Riemann integrability of f.

4. Riemann integrability of the n-th local modulus of continuity

Since $\tau_{n,p}(\delta,\cdot)$ typically is used in connection with Riemann integrable functions, the following limitation to $f \in R[a,b]$ is of no practical impact.

Theorem 4.1. *Let $n \in \mathbb{N}$ and $\delta > 0$. For each $f \in R[a,b]$ one has $\omega_n(\delta,f,\cdot) \in$ $\in R[a,b]$.*

Proof. Since $f \in R[a,b]$, the set A of points of discontinuity of f on $[a,b]$ has Lebesgue measure zero. Because of A the Lebesgue measure of

$$B_1 := \bigcup_{j=0}^{n} \left\{x \in (-\infty, b-\delta] \cap [a+\delta, \infty) : f \text{ is not continuous at } x - \delta + j\frac{2\delta}{n}\right\} =$$

$$= \bigcup_{j=0}^{n} \left[\left(\delta - j\frac{2\delta}{n}\right) + A\right] \cap [(-\infty, b-\delta] \cap [a+\delta, \infty)]$$

is zero, too. The set

$$B_2 := \bigcup_{j=1}^{\infty} \{x \in (a,b) : \omega_n(\delta, f, \cdot)_j \text{ is not continuous at } x\}$$

is a countable union of countable sets (see Corollary 3.1). Therefore the Lebesgue measure of $B := B_1 \cup B_2$ is zero.

We now show the continuity of $\omega_n(\delta, f, \cdot)$ for each $x_0 \in [a,b] \setminus B$. Since $\omega_n(\delta, f, \cdot)$ is bounded, Riemann integrability then follows.

Case 1: Here we deal with the situation that $d := \min\{b, x_0 + \delta\} - \max\{a, x_0 - \delta\} < 2\delta$, i.e., x_0 is near the boundary of $[a,b]$. We choose $k \in \mathbb{N}$ such that $2\delta\frac{k}{k+1} > d$ and set $\delta_d := 2\delta\frac{k}{k+1} - d$. Then $\omega_n(\delta, f, t) = \omega_n(\delta, f, t)_k$ for all $t \in (x_0 - \delta_d, x_0 + \delta_d) \cap [a,b]$. Since $\omega_n(\delta, f, \cdot)_k$ is continuous at x_0, this also holds true for $\omega_n(\delta, f, \cdot)$.

Alternatively, one immediately can show the continuity of $\omega_n(\delta, f, \cdot)$ a.e. on $[a, \min\{a+\delta, b\}]$ and $[\max\{a, b-\delta\}, b]$ by utilizing the monotonicity of the function (cf. proof of Theorem 2.1).

Case 2: Here we discuss $[x_0 - \delta, x_0 + \delta] \subset [a,b]$. Since $x_0 \notin B$, the function f is continuous at $t_j := x_0 - \delta + j\frac{2\delta}{n}$, $0 \leqslant j \leqslant n$. For each $\varepsilon > 0$ there exists $0 < \delta_\varepsilon < 4\delta$ such that

$$|f(x_1) - f(x_2)| < \frac{\varepsilon}{2^{n+2}} \tag{4.1}$$

for all $x_1, x_2 \in (t_j - \delta_\varepsilon, t_j + \delta_\varepsilon) \cap [a,b]$, $0 \leqslant j \leqslant n$. Now we select $k \in \mathbb{N}$ such that

$$h_0 := \frac{2\delta}{n}\frac{k}{k+1} > \frac{2}{n}\left(\delta - \frac{\delta_\varepsilon}{4}\right). \tag{4.2}$$

Because of (cf. (1.1))

$$\begin{aligned}\omega_n(\delta, f, x) = \max\Big\{&\sup\{|\Delta_h^n f(t)| : t, t+nh \in [x-\delta, x+\delta] \cap [a,b], 0 \leqslant h \leqslant h_0\},\\ &\sup\{|\Delta_h^n f(t)| : t, t+nh \in [x-\delta, x+\delta] \cap [a,b], h_0 < h \leqslant 2\delta/n\}\Big\} =\\ = \max\{&\omega_n(\delta, f, x)_k, S_n(\delta, h_0, f, x)\}\end{aligned}$$

and the continuity of $\omega_n(\delta, f, \cdot)_k$ at x_0 (see definition of B), it remains to prove the continuity of

$$S_n(\delta, h_0, f, x) := \sup\{|\Delta_h^n f(t)| : t, t+nh \in [x-\delta, x+\delta] \cap [a,b], h_0 < h \leqslant 2\delta/n\}$$

at x_0. For $x_1 \in \left(x_0 - \frac{\delta_\varepsilon}{2}, x_0 + \frac{\delta_\varepsilon}{2}\right) \cap [a,b]$ and $t, t+nh \in [x_1 - \delta, x_1 + \delta] \cap [a,b]$ as well as $h > h_0$ and $0 \leqslant j \leqslant n$ the choice of h_0 (see (4.2)) implies

$$t \in [x_1 - \delta, x_1 + \delta - nh_0) \subset \left[x_1 - \delta, x_1 - \delta + \frac{\delta_\varepsilon}{2}\right),$$

so that because of (4.2) one has

$$
\begin{aligned}
t+jh \in & \left[x_1-\delta+jh, x_1-\delta+jh+\frac{\delta_\varepsilon}{2}\right) \subset \\
& \subset\left(x_1-\delta+j\frac{2\delta}{n}-\frac{j}{n}\frac{\delta_\varepsilon}{2}, x_1-\delta+j\frac{2\delta}{n}+\frac{\delta_\varepsilon}{2}\right) \subset \\
& \subset\left(x_1-\delta+j\frac{2\delta}{n}-\frac{\delta_\varepsilon}{2}, x_1-\delta+j\frac{2\delta}{n}+\frac{\delta_\varepsilon}{2}\right) \subset (t_j-\delta_\varepsilon, t_j+\delta_\varepsilon). \quad (4.3)
\end{aligned}
$$

The continuity estimate (4.1) in connection with (4.3) leads to

$$
\begin{aligned}
\left||\Delta_h^n f(t)|-|\Delta_{\frac{2\delta}{n}}^n f(x_0-\delta)|\right| & \leqslant |\Delta_h^n f(t)-\Delta_{\frac{2\delta}{n}}^n f(x_0-\delta)| \leqslant \\
& \leqslant \sum_{j=0}^{n}\binom{n}{j}|f(t+jh)-f(t_j)| < 2^n\frac{\varepsilon}{2^{n+2}}.
\end{aligned}
$$

That means

$$
|S_n(\delta, h_0, f, x_1)-\Delta_{\frac{2\delta}{n}}^n f(x_0-\delta)| \leqslant \frac{\varepsilon}{4},
$$

which gives continuity at x_0 since for each pair $x_1, x_2 \in \left(x_0-\frac{\delta_\varepsilon}{2}, x_0+\frac{\delta_\varepsilon}{2}\right) \cap [a,b]$ we have

$$
|S_n(\delta, h_0, f, x_1)-S_n(\delta, h_0, f, x_2)| \leqslant \frac{\varepsilon}{2} < \varepsilon.
$$

■

5. Characterization of Riemann integrability

Without the explicit use of upper Riemann sums or Lebesgue integrals one has the following characterization (5.1) of Riemann integrability:

Theorem 5.1. *Let $f \in B[a,b]$. Then for each $k \in \mathbb{N}$ there holds true* $(1 \leqslant p < \infty)$

$$
f \in R[a,b] \iff \lim_{\delta\to 0+} \tau_{1,p}(\delta, f)_k = 0, \tag{5.1}
$$

$$
f \in R[a,b] \Longrightarrow \lim_{\delta\to 0+} \tau_{1,p}(\delta, f) = 0. \tag{5.2}
$$

If f is measurable, then one has (cf. 1.2)

$$
f \in R[a,b] \iff \lim_{\delta\to 0+} L\text{-}\int_a^b \omega_1(\delta, f, t)^p\, dt = 0. \tag{5.3}
$$

Proof. The proof is well known if one replaces Riemann by Lebesgue integrals (cf. [5], p. 11). Nevertheless, the proof is given here for the sake of completeness. Because f is bounded, one only has to deal with the continuity of f. Since for

$x_0 \in [a,b]$ continuity of f at x_0 is equivalent to $\lim_{j\to\infty} \omega_1(1/j, f, x_0)_k = 0$, one has

$$f \in R[a,b] \iff \lim_{j\to\infty} \omega_1(1/j, f, \cdot)_k = 0 \text{ a.e.} \tag{5.4}$$

We apply Beppo-Levi's theorem to the increasing sequence of functions

$$\left([2\,\|f\|_{B[a,b]}]^p - \omega_1(1/j, f, \cdot)_k^p\right)_{j=1}^{\infty}$$

and get

$$\lim_{j\to\infty} \tau_{1,p}(1/j, f)_k^p = \text{L-}\int_a^b \lim_{j\to\infty} \omega_1(1/j, f, t)_k^p\, dt. \tag{5.5}$$

For $f \in R[a,b]$ it follows from (5.4) and (5.5) that $\lim_{j\to\infty} \tau_{1,p}(1/j, f)_k = 0$. This gives the right side of (5.1). In the same way (5.2) is shown. If on the other hand the right side of (5.1) is true, one gets with (5.5)

$$\text{L-}\int_a^b \lim_{j\to\infty} \omega_1(1/j, f, t)_k^p\, dt = 0$$

and $\lim_{j\to\infty} \omega_1(1/j, f, \cdot)_k = 0$ a.e. Taking (5.4) into consideration we finally get $f \in R[a,b]$ and therefore (5.1). Characterization (5.3) follows with the same argumentation and the fact that $\omega_1(\delta, f, \cdot)$ is Lebesgue measurable (see Section 3). ■

Finally, because of $\omega_n(\delta, f, \cdot)_k \leqslant 2^{n-1}\omega_1(\delta, f, \cdot)_k$ one has for $n \in \mathbb{N}$:

$$f \in R[a,b] \Longrightarrow \lim_{\delta\to 0+} \tau_{n,p}(\delta, f)_k = \lim_{\delta\to 0+} \tau_{n,p}(\delta, f) = 0.$$

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References

[1] L.T. Dechevski, τ-Moduli and interpolation. In: *Function Spaces and Applications* (Proc. Conf. Lund 1986, eds.: M. Cwikel et al.). New York etc.: Springer 1988, pp. 177–190.

[2] H. Esser, N. Kirchhoff, G. Lüttgens and R.J. Nessel, On some properties of the τ-modulus. In: *Inequalities and Applications* (ed.: R.P. Agarwal). Singapore etc.: World Scientific Publishing 1994, pp. 219–232.

[3] H. Mevissen, R.J. Nessel and E. van Wickeren: On the Riemann convergence of positive linear operators. *Rocky Mountain J. Math.* **19** (1989), 271–280.

[4] H. Mevissen and R.J. Nessel: Approximation of Riemann integrable functions by trigonometric convolution processes. *Funct. Approx. Comment. Math.* **20** (1992), 79–97.

[5] Bl. Sendov and V.A. Popov: *The Averaged Moduli of Smoothness*. New York: Wiley 1988.

[6] E. van Wickeren, Direct and inverse theorems for Bernstein polynomials in the space of Riemann integrable functions. *Constr. Approx.* **5** (1989), 189–198.

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