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FLECK'S CONGRUENCE, ASSOCIATED MAGIC SQUARES AND A ZETA IDENTITY

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Abstract: Let the *Fleck numbers*, $C_n(t,q)$, be defined such that

$$C_n(t,q) = \sum_{k \equiv q \pmod{n}} (-1)^k \binom{t}{k}.$$

For prime p, Fleck obtained the result $C_p(t,q) \equiv 0 (\bmod\ p^{\lfloor (t-1)/(p-1) \rfloor})$, where $\lfloor . \rfloor$ denotes the usual floor function. This congruence was extended 64 years later by Weisman, in 1977, to include the case $n = p^\alpha$.

In this paper we show that the Fleck numbers occur naturally when one considers a symmetric $n \times n$ matrix, M, and its inverse under matrix multiplication. More specifically, we take M to be a symmetrically constructed $n \times n$ associated magic square of odd order, and then consider the reduced coefficients of the linear expansions of the entries of M^t with $t \in \mathbb{Z}$. We also show that for any odd integer, $n = 2m+1$, $n \geqslant 3$, there exist geometric polynomials in m that are linked to the Fleck numbers via matrix algebra and p-adic interaction. These polynomials generate numbers that obey a reciprocal type of congruence to the one discovered by Fleck.

As a by-product of our investigations we observe a new identity between values of the Zeta functions at even integers. Namely

$$\zeta(2j) = (-1)^{j+1} \left(\frac{j\pi^{2j}}{(2j+1)!} + \sum_{k=1}^{j-1} \frac{(-1)^k \pi^{2j-2k}}{(2j-2k+1)!} \zeta(2k) \right).$$

We conclude with examples of combinatorial congruences, Vandermonde type determinants and Number Walls that further highlight the symmetric relations that exist between the Fleck numbers and the geometric polynomials.

Keywords: combinatorial identities, combinatorial functions, matrices, determinants, p-adic theory and binomial coefficients.

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1. Introduction

We begin by recalling Fleck's congruence [9]. Let p be a prime and q be an integer. In 1913 A. Fleck discovered that

$$\sum_{k \equiv q \pmod{p}} (-1)^k \binom{t}{k} \equiv 0 \left(\bmod\, p^{\left\lfloor \frac{t-1}{p-1} \right\rfloor} \right) \tag{1.1}$$

for all positive integers $t > 0$. In 1977 C. S. Weisman [28] extended Fleck's congruence to obtain

$$\sum_{k \equiv q \pmod{p^\alpha}} (-1)^k \binom{t}{k} \equiv 0 \left(\bmod\, p^{\left\lfloor \frac{t-p^{\alpha-1}}{\phi(p^\alpha)} \right\rfloor} \right), \tag{1.2}$$

where α, t are positive integers $\geqslant 0$, $t \geqslant p^{\alpha-1}$, ϕ denotes the Euler totient function and $\lfloor . \rfloor$ is the well-known floor function. When $\alpha = 1$ it is clear that (1.2) reduces to (1.1). Much research is current in this area [8], [6], [24].

We define the *Fleck numbers*, $C_n(t,q)$, to be the numbers generated by the generalised sum in (1.1) and (1.2), such that

$$C_n(t,q) = \sum_{k \equiv q \pmod{n}} (-1)^k \binom{t}{k}. \tag{1.3}$$

These sums have many well known properties [25] such as

$$nC_n(t,q) = \sum_{k=0}^{t} (-1)^k \binom{t}{k} \sum_{\gamma^n=1} \gamma^{k-q} = \sum_{\gamma^n=1} \gamma^{-q}(1-\gamma)^t, \tag{1.4}$$

from which we can deduce the recurrence relation

$$C_n(t+1,q) = C_n(t,q) - C_n(t,q-1). \tag{1.5}$$

By the modular definition of the sum in (1.3) we can also deduce that

$$C_n(t,q) = C_n(t,q+n). \tag{1.6}$$

Hence when considering the Fleck numbers for any fixed value of n, we can restrict to a two-dimensional array, $T(n)$ of width n, constructed from the Fleck numbers, $C_n(t,q)$, with t increasing as we move down the page. We note that for values of t, with $t < n$, the Fleck numbers are simply the binomial coefficients $\pm{}^t\mathrm{C}_q$.

In this paper we show that the Fleck numbers occur naturally when one considers a special kind of inverse magic square [13] under matrix multiplication. Moreover, we also show that for any odd integer, $n \geqslant 3$, there exist geometric polynomials that are linked to the Fleck numbers through matrix algebra. The two-dimensional arrays, $T(n)$, constructed from the Fleck numbers and obeying the relation (1.5), can be extended upwards to include the values generated by these geometric polynomials. These numbers appear to be new and for now we shall refer to them simply as the *geometric numbers*. As with the Fleck numbers, the geometric numbers also exhibit interesting p-adic properties [16] and the p-adic interaction between the two sets of numbers is worthy of note. Further symmetric relations between the geometric polynomials and the Fleck numbers can be observed when one considers Vandermonde type determinants [14] constructed from them. As a by-product of our investigations we observe a new identity between values of the Zeta functions at even integers. Namely

$$\zeta(2j) = (-1)^{j+1}\left(\frac{j\pi^{2j}}{(2j+1)!} + \sum_{k=1}^{j-1}\frac{(-1)^k\pi^{2j-2k}}{(2j-2k+1)!}\zeta(2k)\right). \tag{1.7}$$

Fundamentally our results stem from studying the preservation of symmetry for square matrices under matrix multiplication. Hence, in order that we may state all of our results fully, we now set up and develop some notation.

2. Some Definitions

The magic square [1]

8	1	6
3	5	7
4	9	2

was used in China as a hopscotch to symbolise harmony and the balance of natural forces [20], has the property that the rows, columns and main diagonals add up to 15, and any pair of associated elements adds up to 10; two such positions within the matrix are called associated if the centre of the line adjoining them is also the centre of the square. More formally, an associated magic square satisfies the three symmetry conditions ($s1$), ($s2$) and ($s3$), defined below.

Symmetry Conditions. Let $A = (a_{i,j})$ be an $n \times n$ square matrix and c a constant rational number. We define three symmetry conditions on A as follows:

(s1) Row and column symmetry

$$\sum_{\substack{j=1\\1\leqslant i\leqslant n}}^{n} a_{i,j} = c, \qquad \sum_{\substack{i=1\\1\leqslant j\leqslant n}}^{n} a_{i,j} = c.$$

(s2) Principal diagonals symmetry

$$\sum_{i=1}^{n} a_{i,i} = c, \qquad \sum_{i=1}^{n} a_{i,(n+1-i)} = c.$$

(s3) Associated symmetry

$$a_{i,j} + a_{(n+1-i),(n+1-j)} = \frac{2c}{n},$$

for all (i, j).

A square in which only the rows and columns sum to c (condition ($s1$)) is called *semi-magic*. The extra condition that the two main or principal diagonals also sum to c (conditions ($s1$) and ($s2$)) defines the standard *magic square* and if the square also has the associated property (condition ($s3$)) then we have an *associated magic square* with associated sum $2c/n$. We note that condition ($s3$) implies condition ($s2$).

A *Latin square* of order n is an $n \times n$ array of n different symbols, each used n times, arranged in such a way, that each row or column of the array contains each symbol exactly once. In our language, a Latin square is semi-magic under formal addition of the symbols. We call a Latin square magic when both the principal diagonals also contain each symbol exactly once (conditions ($s1$) and ($s2$)). Traditionally the n symbols are identified with the numbers $0, 1, \ldots, n-1$.

Two Latin squares $B = (b_{i,j})$ and $C = (c_{i,j})$ are said to be *orthogonal* when the n^2 ordered pairs $(b_{i,j}, c_{i,j})$ are all different, so that every possible pair of symbols actually occurs as a pair $(b_{i,j}, c_{i,j})$. Euler observed [21] in 1779 that if B and C are an orthogonal pair of traditional magic Latin squares of order n, then $A = nB + C$ is a magic square with entries $0, 1, 2, \ldots, n^2 - 1$. In this construction the auxiliary magic Latin squares B and C are called the *radix* and the *unit* respectively. The use of symmetry to construct the auxiliary squares motivates our results.

We now translate the properties of associated magic squares into matrix algebra.

Definition. *First we define the $n \times n$ permutation matrices. Let $\underline{e}_1, \cdots, \underline{e}_n$ be the unit vectors $(1, 0, \cdots), (0, 1, 0, \cdots), \cdots, (0, \cdots, 0, 1)$ written as rows. A permutation of the n rows $\underline{m}_1, \cdots, \underline{m}_n$ of an $n \times n$ matrix M can be accomplished by the product $P_\sigma M$, where P_σ is the matrix with rows $\underline{e}_{\sigma 1}, \cdots, \underline{e}_{\sigma n}$. Similarly MP_σ has columns $\underline{k}_{\tau 1}, \cdots, \underline{k}_{\tau n}$ where $\underline{k}_1, \cdots, \underline{k}_n$ are the columns of M, and τ is the permutation inverse to σ. In particular let J be the matrix with rows $\underline{e}_n, \underline{e}_{n-1}, \cdots, \underline{e}_1$, and let K be the matrix with rows $\underline{e}_2, \cdots, \underline{e}_n, \underline{e}_1$. In the 3×3 case they are*

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The matrices J and K under multiplication generate the dihedral group D_{2n}. The product JMJ^{-1} has the original entry in row $n+1-i$, column $n+1-j$, where $J^{-1} = J$. Let E be defined so that E is the $n \times n$ matrix with every entry 1. We define any matrix that can be expressed as a linear combination of products of powers of J, K and E to be diagonally expressible.

We call an $n \times n$ matrix M semi-magic of weight w *if M and its transpose M^T satisfy*

$$ME = nwE = M^T E. \tag{2.1}$$

Traditionally a magic square contains the integers $0, 1, 2, \ldots, n^2-1$ or $1, 2, 3, \ldots, n^2$ and so has weight w of $(n^2-1)/2$ or $(n^2+1)/2$ respectively.

The condition (2.1) says that the rows and columns sum to nw. The permutation matrices P_σ are semi-magic of weight $1/n$. The associated magic squares of the title (type A for short) are the matrices M which satisfy (2.1) and also

$$M + JMJ = 2wE, \tag{2.2}$$

which says that the sum of the two associated elements is always $2w$, so the main diagonals also sum to nw. If M satisfies (2.1) and

$$JMJ = M, \tag{2.3}$$

then we say that M is a balanced semi-magic square (type B for short). These conditions are linear, so the type A squares form a vector space $\mathcal{V}$, which contains the transpose M^T for every M in $\mathcal{V}$ [11]. The matrix E is a basis matrix of $\mathcal{V}$, and $M - wE$ is a matrix in $\mathcal{V}$ with weight zero. Similarly the type B squares form a vector space $\mathcal{W}$, and for $n = 3$ and $n = 5$ we give the non-trivial type B examples

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 9 & 13 & 8 & 2 & 3 \\ 12 & 5 & 6 & 11 & 1 \\ 10 & 4 & 7 & 4 & 10 \\ 1 & 11 & 6 & 5 & 12 \\ 3 & 2 & 8 & 13 & 9 \end{bmatrix}.$$

We note that although a type B square can never be a traditional magic square, the 5×5 example shown above satisfies conditions ($s1$) and ($s2$).

In the next section we examine the behavior of $n \times n$ type A and B squares under matrix multiplication [26].

3. Multiplication and Constructions

Lemma 3.1.

(1) *If M and N are semi-magic with weights z and w, then MN is semi-magic with weight nzw.*
(2) *If M and N are both type A, then MN is type B.*
(3) *If M is type A and N is type B, then MN and NM are type A.*
(4) *If M is type B and N is type B, then MN and NM are type B.*
(5) *If M is semi-magic and invertible, then M^{-1} is semi-magic with weight $1/n^2z$.*
(6) *If M is type B and invertible, then M^{-1} is also type B.*
(7) *If M is type A and invertible, then M^{-1} is type A with weight $1/n^2z$.*

Corollary.

(i) *If M is type A then M^t is type A for all positive odd t and type B for all positive even t. If M is also non-singular then the positive condition can be removed from the above statement.*

(ii) *If M is type B then M^t is type B for all positive t. If M is also non-singular then the result holds for all t.*

Proof. If M and N are both semi-magic with weights z and w, then

$$MNE = MnwE = n^2zwE = N^TM^TE = (MN)^TE,$$

so MN is semi-magic with weight nzw.

If M and N are both type A, then

$$\begin{aligned} JMNJ &= (JMJ)(JNJ) = (2zE - M)(2wE - N) \\ &= 4zwE^2 - 2zEN - 2wME + MN \\ &= 4nzwE - 2znwE - 2wnzE + MN = MN, \end{aligned}$$

so MN is type B.

If M is type A and N is type B, then

$$JMNJ = (JMJ)(JNJ) = (2zE - M)N = 2zEN - MN = 2nzwE - MN,$$

so MN is type A.

If M is type B and N is type B, then

$$JMNJ = (JM(JJ)NJ) = (JMJ)(JNJ) = MN,$$

so MN is type B.

If M is semi-magic and invertible, then from (2.1)

$$E = nzM^{-1}E,$$

and

$$E = nz(M^T)^{-1}E = nz(M^{-1})^TE,$$

so M^{-1} is semi-magic with weight $1/n^2z$.

If M is type B, then by (2.3)

$$M^{-1} = J^{-1}M^{-1}J^{-1} = JM^{-1}J,$$

and so M^{-1} is also type B.

If M is type A, then we calculate

$$\begin{aligned}(2zE - JMJ)\left(\frac{2E}{n^2z} - JM^{-1}J\right) &= \frac{4E^2}{n^2} - 2zEJM^{-1}J - \frac{2}{n^2z}JMJE \\ &\quad + (JMJ)(JM^{-1}J) \\ &= \frac{4E}{n} - 2zEM^{-1}J - \frac{2}{n^2z}JME + JMM^{-1}J \\ &= \frac{4E}{n} - 2z\frac{n}{n^2z}EJ - \frac{2}{n^2z}JnzE + J^2,\end{aligned}$$

so

$$\frac{4E}{n} - \frac{2E}{n} - \frac{2E}{n} + I = I.$$

The first factor $2zE - JMJ$ is just M by (2.2), so

$$M^{-1} = \frac{2E}{n^2z} - JM^{-1}J,$$

and hence

$$M^{-1} \in \mathcal{V}.$$

If M is type A then by statements (2) and (3) of the Lemma, M^2 is type B, M^3 is type A. We inductively assume that M^t is type A for $t = 2k+1$, multiply by M^2, and apply statement (3) of the Lemma to complete the proof for positive odd powers t.

If M is type A then by statement (3) of the Lemma, M^2 is type B and by statement (4) of the Lemma $M^2M^2 = M^4$ is also type B. We inductively assume that M^t is type B for $t = 2k$, multiply by M^2, and again apply statement (4) of the Lemma to complete the proof for positive even powers t.

The proofs in the non-singular cases are similar and the second statement in the Corollary also follows from statement (4) of the Lemma. ■

Remark. The identity matrix I_n is of type B and the $n \times n$ matrix with zero entries 0_n is simultaneously of types A and B. Hence the Corollary implies that the set of all $n \times n$ type A and type B squares is closed under multiplication and addition and so forms a ring, $\mathcal{R}(A, B)$, containing the subring $\mathcal{R}(B)$, of all $n \times n$ type B squares. This raises interesting questions such as "if M is type B then does the solution to the matrix equation

$$M = N^2$$

exist, and if so must N be of type A?" If this is the case then we can think of the ring $\mathcal{R}(A, B)$ as being a "quadratic extension" to the ring $\mathcal{R}(B)$.

From a group theory perspective, the Corollary implies that the set of all non-singular type A and B squares over a field $\mathbb{F}$ forms a group, $G(A,B)$, under multiplication, containing the subgroup, $G(B)$, of all $n \times n$ non-singular type B squares. Both groups are subgroups of $GL(n, \mathbb{F})$.

How do we construct squares of types A and B? If M is semi-magic, then so is $N = M - JMJ$ and

$$N + JNJ = M - JMJ + JMJ - M,$$

so N satisfies (2.2) with $z = 0$. The permutation matrices are semi-magic, and

$$JK^r J = K^{-r}. \tag{3.1}$$

so

$$K^r - JK^r J = K^r - K^{-r} \tag{3.2}$$

is a matrix in $\mathcal{V}$ of weight zero, and

$$K^r + JK^r J = K^r + K^{-r} \tag{3.3}$$

is a matrix in $\mathcal{W}$ of weight $2/n$.

We now define the basis matrices, that along with J and E, span the vector spaces of diagonally expressible type A and B squares.

Definition. *For $n = 2m + 1$ and $r \in \mathbb{Z}$ let K^r be the permutation matrices of order n and let*

$$A_r = K^{2r-1} - K^{-(2r-1)}, \tag{3.4}$$

and

$$B_r = K^{2r} + K^{-2r}. \tag{3.5}$$

Then

$$JA_rJ = -A_r \in \mathcal{V} \tag{3.6}$$

with weight zero and

$$JB_rJ = B_r \in \mathcal{W} \tag{3.7}$$

with weight $2/n$.

We note that $A_rE = EA_r = 0_n$ and $EB_r = B_rE = 2E$.

Under the above definition, the following identities hold.

$$A_{-r} = -A_{r+1}, \qquad A_{m+r} = -A_{m+2-r}, \qquad A_{m+1} = 0_n, \tag{3.8}$$

$$A_rA_s = B_{r+s-1} - B_{r-s}, \qquad A_rB_s = A_{r+s} + A_{r-s}, \qquad B_rB_s = B_{r+s} + B_{r-s}, \tag{3.9}$$

$$B_{-r} = B_r, \qquad B_{m+r} = B_{m+1-r}, \qquad B_0 = 2I. \tag{3.10}$$

The following two Lemmas concerning vector space dimensions and enumeration properties are stated here for completeness rather than necessity. For proofs of these and other related results see [17].

Lemma 3.2. *For natural number n, the dimension, N, of the vector space $\mathcal{V}$ of $n \times n$ type A squares, satisfies either*

$$N \leqslant \frac{n^2-2n+3}{2}, \qquad \text{or} \qquad N \leqslant \frac{n^2-2n+2}{2},$$

depending on whether n is odd or even respectively.

For natural number m, let $n = 2m+1$ be odd. Let $\mathcal{V}' \subseteqq \mathcal{V}$ and $\mathcal{W}' \subseteqq \mathcal{W}$ be the vector subspaces of $n \times n$ diagonally expressible type A and B squares respectively. Then $\mathcal{V}'$ and $\mathcal{W}'$ are spanned by $E, A_1, \ldots, A_m, JA_1, \ldots, JA_m$ and $E, B_1, \ldots, B_m, JB_1, \ldots, JB_m$ respectively. Hence the vector subspaces $\mathcal{V}'$ and $\mathcal{W}'$ each have dimension n.

Lemma 3.3. *For $m \in \mathbb{N}$, let D_n be the total number of diagonally expressible traditional type A squares of odd order $n = 2m+1$. Then*

$$D_n = (2^{m-1} m!)^2. \tag{3.11}$$

4. Two and Three Parameter Families

Lemma 4.1. *For natural number m, let $n = 2m+1$, and let the three parameter family of $n \times n$ type A squares be defined such that*

$$M(z,y,x) = (zI - yJ)\sum_{r=1}^{m}(m+1-r)A_r + (m(z+y)+x)E. \tag{4.1}$$

Then the inverse matrix is given by

$$M^{-1}(z,y,x) = \frac{(zI-yJ)}{n(z^2-y^2)}A_0 + \frac{E}{n^2(m(z+y)+x)}.$$

Corollary. *For some positive integer $t \geqslant 1$ we have*

$$M^{-t}(z,y,x) \;=\; \frac{(zI-yJ)^{t(mod 2)}(z^2-y^2)^{\left[\frac{t}{2}\right]}}{n^t(z^2-y^2)^t}A_0^t + \frac{E}{n^{t+1}(m(z+y)+x)^t}, \tag{4.2}$$

and

$$\begin{aligned} M^t(z,y,x) = (zI-yJ)^{t(mod 2)}(z^2-y^2)^{\left[\frac{t}{2}\right]}\left(\sum_{r=1}^{m}(m+1-r)A_r\right)^t \\ + n^{t-1}(m(z+y)+x)^t E. \end{aligned} \tag{4.3}$$

For the inverse matrix, with $0 \leqslant t \leqslant m$, *we note the simple binomial relations*

$$A_0^{2t} = (-1)^t \binom{2t}{t} I + \sum_{r=1}^{t} (-1)^{t+r} \binom{2t}{t+r} B_r, \tag{4.4}$$

and

$$A_0^{2t+1} = \sum_{r=1}^{t+1} (-1)^{t+r} \binom{2t+1}{t+r} A_r. \tag{4.5}$$

Proof. $A_r E$ vanishes so that

$$MM^{-1} = \frac{1}{n} \sum_{r=1}^{m} (m+1-r) A_r A_0 + \frac{E}{n} = \frac{1}{n} \left(\sum_{r=1}^{m} (m+1-r)(B_{r-1} - B_r) + E \right)$$

by (3.9)

$$= \frac{1}{n} \left(mB_0 - \sum_{r=1}^{m} B_r + E \right) = \frac{1}{n} \left(2mI - (E-I) + E\right) = \frac{(2m+1)}{n} I = I.$$

To see the Corollary, we have

$$(zI - yJ) A_r (zI - yJ) = (z^2 - y^2) A_r,$$

from which the identity (4.2) follows. Multiplying (4.2) by (4.3) then gives

$$\begin{aligned} M^t M^{-t} &= \frac{1}{n^t} \left(\left(\sum_{r=1}^{m} (m+1-r) A_r A_0 \right)^t + n^{t-1} E \right) \\ &= \frac{1}{n^t} \left((nI - E)^t + n^{t-1} E \right) = \frac{1}{n^t} \left(n^{t-1}(nI - E) + n^{t-1} E \right) = I, \end{aligned}$$

as required.

This highlights the natural representation of $M^0(z, y, x)$ as

$$M^0(z, y, x) = \left(I - \frac{1}{n} E \right) + \frac{1}{n} E. \tag{4.6}$$

Hence $M^0(z, y, x)$ can be thought of as the sum of two auxiliary Latin type B squares, one of which has weight zero and the other 1/n.

To obtain the identities (4.4) and (4.5) we make repeated use of the identities in (3.9) and then collect terms. ∎

For simplicity we now restrict our calculations to the two parameter family $M(z, y)$ and its inverse matrix.

As a worked example, if $n = 5$, $z = 5$ and $y = 1$ then we have

$$M(5,1) = (5I - J)(2A_1 + A_2) + 12E$$

$$= 5 \times \begin{array}{|c|c|c|c|c|}\hline 0 & 2 & -1 & 1 & -2 \\ \hline -2 & 0 & 2 & -1 & 1 \\ \hline 1 & -2 & 0 & 2 & -1 \\ \hline -1 & 1 & -2 & 0 & 2 \\ \hline 2 & -1 & 1 & -2 & 0 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|}\hline -2 & 1 & -1 & 2 & 0 \\ \hline 1 & -1 & 2 & 0 & -2 \\ \hline -1 & 2 & 0 & -2 & 1 \\ \hline 2 & 0 & -2 & 1 & -1 \\ \hline 0 & -2 & 1 & -1 & 2 \\ \hline \end{array} + 12E_5$$

$$= \begin{array}{|c|c|c|c|c|}\hline 10 & 23 & 6 & 19 & 2 \\ \hline 3 & 11 & 24 & 7 & 15 \\ \hline 16 & 4 & 12 & 20 & 8 \\ \hline 9 & 17 & 0 & 13 & 21 \\ \hline 22 & 5 & 18 & 1 & 14 \\ \hline \end{array},$$

and

$$M^{-1}(5,1) = \frac{1}{5 \times 24}(5I - J)A_0 + \frac{1}{25 \times 12}E$$

$$= \frac{1}{120}\left(5 \times \begin{array}{|c|c|c|c|c|}\hline 0 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & -1 & 0 & 0 \\ \hline 0 & 1 & 0 & -1 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 \\ \hline -1 & 0 & 0 & 1 & 0 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|}\hline 1 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 \\ \hline 0 & -1 & 0 & 1 & 0 \\ \hline -1 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & -1 \\ \hline \end{array} \right) + \frac{1}{300}E_5$$

$$= \frac{1}{2.5.60} \begin{array}{|c|c|c|c|c|}\hline 7 & -23 & 2 & -3 & 27 \\ \hline 27 & 2 & -28 & 2 & 7 \\ \hline 2 & 22 & 2 & -18 & 2 \\ \hline -3 & 2 & 32 & 2 & -23 \\ \hline -23 & 7 & 2 & 27 & -3 \\ \hline \end{array}.$$

The simplicity of the inverse square is quite striking.

Definition. *For natural number $t \geqslant 1$, let*

$$V_0^t = \left(\sum_{q=1}^{m} (m + 1 - q) A_q \right)^t, \qquad V_0^{-t} = \frac{1}{n^t} A_0^t,$$

and

$$V_0^0 = V_0^t V_0^{-t} = I_n - \frac{1}{n}E,$$

so that V_0^{-t} can be thought of as the pseudo *inverse matrix of V_0^t. Then we have*

$$M^t(z,y) = (zI - yJ)^{t(mod 2)}(z^2 - y^2)^{\left[\frac{t}{2}\right]} V_0^t + n^{t-1}(m(z+y))^t E, \tag{4.7}$$

and

$$M^{-t}(z,y) = \frac{(zI - yJ)^{t(mod 2)}(z^2 - y^2)^{\left[\frac{t}{2}\right]}}{(z^2 - y^2)^t} V_0^{-t} + \frac{E}{n^{t+1}(m(z+y))^t}, \tag{4.8}$$

We call V_0^t *and* V_0^{-t} *the* fundamental *matrices of* $M^t(z,y)$ *and* $M^{-t}(z,y)$ *respectively.*

When $t=2k+1$ *in* (4.3)*, then using* (3.9) *and* (3.8) *we have*

$$V_0^t = \left(\sum_{r=1}^{m}(m+1-q)A_q\right)^{2k+1} = V_0^{2k+1} = \sum_{q=1}^{m} a_q^{(2k+1)}A_q, \tag{4.9}$$

and when $t=2k$ *we use* (3.9) *and* (3.10) *to obtain*

$$V_0^t = \left(\sum_{q=1}^{m}(m+1-q)A_q\right)^{2k} = V_0^{2k} = \sum_{q=0}^{m} a_q^{(2k)}B_q, \tag{4.10}$$

so that $a_q^{(2k+1)}$ *is the coefficient of the diagonal type A matrix* A_q *in the expression* (4.9) *for* V_0^{2k+1} *and* $a_q^{(2k)}$ *is the coefficient of the diagonal type B matrix* B_q *in the expression* (4.10) *for* V_0^{2k}*.*

Hence, when t *is odd, the fundamental matrix of* $M^t(z,y)$ *can be written as a linear combination of the diagonal type A matrices,* $A_1, A_2, \ldots, A_m$*, each of weight zero, and when* t *is even, the fundamental matrix of* $M^t(z,y)$ *can be written as a linear combination of the diagonal type B matrices,* $B_0, B_1, B_2, \ldots, B_m$*, each of weight* $2/n$*. We note that both fundamental matrices,* V_0^t *and* V_0^{-t}*, have weight zero.*

For the fundamental matrix of the inverse square $M^{-t}(z,y)$ *we have*

$$V_0^{-t} = \frac{1}{n^{2k+1}}A_0^{2k+1} = \sum_{q=1}^{m} a_q^{-(2k+1)}A_q = \frac{1}{n^{2k+1}}\sum_{q=1}^{m} b_q^{-(2k+1)}A_q, \tag{4.11}$$

when $t=2k+1$ *is odd, and*

$$V_0^{-t} = \frac{1}{n^{2k}}A_0^{2k} = \sum_{q=0}^{m} a_q^{-(2k)}B_q = \frac{1}{n^{2k}}\sum_{q=0}^{m} b_q^{-(2k)}B_q, \tag{4.12}$$

when $t=2k$ *is even, and we call* $b_q^{-(t)}$ *the* reduced coefficient *of* A_q *in the linear expansion of* V_0^{-t}*.*

Lemma 4.2. *For natural number* $t \geqslant 1$*, the reduced coefficients* $b_q^{-(t)}$ *of* A_q *in the linear expansion of* $n \times n$ V_0^{-t} *are the Fleck numbers* $C_n(t, [\frac{t}{2}]+q)$*.*

Proof. Using the identities (3.8), (3.9) and (3.10) it can be proven inductively that

$$b_q^{(-(2k+1))} = (-1)^{k+q}\ {}^{2k+1}\mathrm{C}_{k+q} + \sum_{a=1}^{\infty}(-1)^{k+q+a}\left({}^{2k+1}\mathrm{C}_{k+q-an} + {}^{2k+1}\mathrm{C}_{k+q+an}\right), \tag{4.13}$$

and

$$b_q^{(-2k)} = (-1)^{k+q}\ {}^{2k}\mathrm{C}_{k+q} + \sum_{a=1}^{\infty}(-1)^{k+q+a}\left({}^{2k}\mathrm{C}_{k+q-an} + {}^{2k}\mathrm{C}_{k+q+an}\right), \qquad (4.14)$$

where we have employed ${}^t\mathrm{C}_q$ notation to highlight the symmetries in the expression.

The right hand sides of (4.13) and (4.14) are just rearrangements of the left hand sides of the sum in (1.1) and (1.2). Therefore, the reduced coefficients $b_q^{(-(2k+1))}$ and $b_q^{(-2k)}$ correspond exactly to the Fleck numbers $C_n(t, [\frac{t}{2}] + q)$. That is, the alternating lower index summations of the binomial coefficients over the residue class $k + q \pmod n$ with upper indices $2k + 1$ and $2k$ respectively.

From Fleck's and Weisman's congruences it follows that for $n \times n$ A_0, with $n = 2m + 1 = p^\alpha$, a prime power, we have

$$b_q^{(-t)} \equiv 0 \left(\bmod p^{\left\lfloor \frac{t-p^{\alpha-1}}{\phi(p^\alpha)} \right\rfloor}\right), \qquad (4.15)$$

which, in order notation, can be written as

$$\mathrm{Ord}_p\ b_q^{(-t)} \geqslant \left\lfloor \frac{t - p^{\alpha-1}}{\phi(p^\alpha)} \right\rfloor = F(p^\alpha, t), \qquad (4.16)$$

say. When $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ is not a prime power, these congruences do not in general seem to hold and it often appears to be the case that $\mathrm{Ord}_{p_i}\ b_q^{(-t)} = 0$. For simplicity we define $F(n, t) = 0$ when n is not a prime power. ∎

5. Polynomials related to the Fleck numbers

Having established that the reduced coefficients of the fundamental matrix of $M^{-t}(z, y)$ correspond to the Fleck numbers, we now investigate the fundamental matrix V_0^t of $M^t(z, y)$. We begin with some lemmas.

Lemma 5.1. *Let m be a positive integer and let*

$$f_k = \frac{1}{2k+1}\binom{m+k}{2k}, \qquad k \geqslant 0. \qquad (5.1)$$

Then $(2m + 1)f_k$ takes integer values.

Proof. We put $t = m + k$, $r = 2k + 1$ in the binomial identity

$$\binom{t}{r} + \binom{t+1}{r} = \frac{2t+2-r}{r}\binom{t}{r-1} = \frac{2m+1}{2k+1}\binom{m+k}{2k}. \qquad (5.2)$$

The identity is easily verified by cancellation of factorial terms on both sides. ∎

Remark. If f_k is an integer for all k in $1 \leqslant k \leqslant m - 1$, then $2m + 1$ must be prime. It can be shown that the converse statement also holds [10].

We now state a result from [12].

Proposition 5.2. *Let $\lambda, \mu, \nu, \epsilon$ be integers such that $\lambda, \mu \geqslant 0$ and $\nu \geqslant \epsilon \geqslant 0$. Then the following binomial identities hold.*

$$\sum_{k=0}^{\lambda} \binom{\lambda-k}{\mu}\binom{\epsilon+k}{\nu} = \binom{\lambda+\epsilon+1}{\mu+\nu+1}, \tag{5.3}$$

"diagonals $\times$ reversed diagonals".

$$\sum_{k=0}^{\lambda} \binom{k}{\mu} = \binom{\lambda+1}{\mu+1}, \tag{5.4}$$

"summation on the upper index".

Lemma 5.3. *The following two binomial relations hold.*

$$\sum_{k=1}^{m-r} (2k-1)\binom{m+r+1-k}{2r+1} = \frac{2m+1}{2r+3}\binom{m+r+1}{2r+2}. \tag{5.5}$$

and

$$\sum_{k=2r+1}^{m+r-q} \binom{m+r+1-k-q}{1}\binom{k}{2r+1} = \binom{m+r+2-q}{2r+3}. \tag{5.6}$$

Proof. Using (5.3) with $\lambda = m+r+1, \mu = 2r+1, \nu = 1$ and $\epsilon = 0$ gives

$$\sum_{k=1}^{m-r} \binom{k}{1}\binom{m+r+1-k}{2r+1} = \binom{m+r+2}{2r+3}, \tag{5.7}$$

and by (5.4)

$$\sum_{k=1}^{m-r} \binom{m+r+1-k}{2r+1} = \binom{m+r+1}{2r+2}. \tag{5.8}$$

Combining the two terms (5.7) and (5.8) we deduce the result (5.5)

To establish (5.6) we again use (5.3), but this time with $\lambda = m+r+1-q, \mu = 1, \nu = 2r+1$ and $\epsilon = 0$. ∎

Lemma 5.4. *For $r \geqslant 0$, let*

$$V_r = \sum_{q=1}^{m-r} \binom{m+r+1-q}{2r+1} A_q, \qquad W_r = \sum_{q=1}^{m-r} \binom{m+r-q}{2r} B_q. \tag{5.9}$$

Then we have

$$A_r V_0^2 = n^2 \sum_{q=1}^{r-1} (r-q)A_q - n\sum_{q=1}^{m} (2r-1)(m+1-q)A_q, \tag{5.10}$$

$$V_0^2 = n\left(-3f_1 I + f_1 E - W_1)\right), \tag{5.11}$$

$$V_0 V_r = n\left(-(r+1)f_{r+1}B_0 + f_{r+1}W_0 - W_{r+1}\right), \tag{5.12}$$

$$V_0 W_r = \left(n - (2r+1)\right) f_r V_0 - nV_r, \tag{5.13}$$

and

$$V_0^2 V_r = n^2\left(V_{r+1} - f_{r+1}V_0\right). \tag{5.14}$$

Proof. Equation (5.10) follows with some straightforward manipulation using (3.8) and (3.9). We give the proofs for (5.11) and (5.14). The proofs for (5.12) and (5.13) are similar although it is easy to show that together they satisfy (5.14). By (3.9)

$$V_0^2 = \sum_{r=1}^{m}\sum_{s=1}^{m} A_r B_s = \sum_{r=1}^{m}\sum_{s=1}^{m}(B_{r+s-1} - B_{r-s}).$$

Now

$$\begin{aligned}\sum_{s=1}^{m}(m+1-s)B_{r+s-1} &= \sum_{s=r}^{m}(m+r-s)B_s \\ &\quad + \sum_{s=m+2-r}^{m}(r+s-m-1)B_s,\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}-\sum_{s=1}^{m}(m+1-s)B_{s-r} &= -2(m+1-r)I - \sum_{s=1}^{r-1}(m+1+s-r)B_s \\ &\quad - \sum_{s=1}^{m-r+1}(m+1-s-r)B_s,\end{aligned} \tag{5.16}$$

where we have used the identities in (3.10). The s multiples of B_s in (5.15) and (5.16) cancel out completely, and collecting together the remaining terms we have

$$\begin{aligned}V_0^2 &= \sum_{r=1}^{m}(m+1-r)\left(-2(m+1-r)I + (2r-1)(E-I) - (2m+1)\sum_{s=1}^{r-1}B_s\right) \\ &= \sum_{r=1}^{m}(m+1-r)\left(-nI + (2r-1)E - n\sum_{s=1}^{r-1}B_s\right).\end{aligned}$$

The coefficient of B_k in

$$\sum_{r=2}^{m}\sum_{s=1}^{r-1}(m+1-r)B_s$$

is

$$\sum_{j=1}^{m-k} j = \binom{m+1-k}{2}, \qquad 1 \leqslant k \leqslant m-1,$$

so that

$$\sum_{r=2}^{m}\sum_{s=1}^{r-1}(m+1-r)B_s = \sum_{k=1}^{m-1}\binom{m+1-k}{2} = W_1.$$

Hence we have

$$V_0^2 = n\left(-3f_1 I + f_1 E - W_1\right)\right),$$

which is (5.11).

To obtain (5.10) we use

$$\begin{aligned}
V_r V_0^2 &= \sum_{k=1}^{m-r}\binom{m+r+1-k}{2r+1}A_k V_0^2 \\
&= \sum_{k=1}^{m-r}\binom{m+r+1-k}{2r+1}\left(n^2\sum_{q=1}^{k-1}(k-q)A_q - n\sum_{q=1}^{m}(2k-1)(m+1-q)A_q\right) \\
&= n^2\sum_{k=2}^{m-r}\binom{m+r+1-k}{2r+1}\sum_{q=1}^{k-1}(k-q)A_q - n\sum_{k=1}^{m-r}\binom{m+r+1-k}{2r+1}(2k-1)V_0.
\end{aligned}$$

For fixed s, the coefficient of A_s in

$$\sum_{k=2}^{m-r}\binom{m+r+1-k}{2r+1}\sum_{q=1}^{k-1}(k-q)A_q$$

is given by (5.6). Therefore using both parts of Lemma 5.2 we have

$$V_r V_0^2 = n^2\left(V_{r+1} - f_{r+1}V_0\right),$$

and hence the result. ∎

An immediate consequence of (5.14) is that

$$V_0^3 + n^2 f_1 V_0 = n^2 V_1, \tag{5.17}$$

and repeated use of this identity yields the result

$$\sum_{k=0}^{r} n^{2(r-k)} f_{r-k} V_0^{2k+1} = n^{2r} V_r. \tag{5.18}$$

Applying (5.12), multiplying through by n and rearranging gives

$$\sum_{k=0}^{r} n^{2(r-k)+1} f_{r-k} V_0^{2k} = n^{2r}\left(2\binom{m+r}{2r+1}I - W_r\right), \tag{5.19}$$

where as usual we have taken

$$V_0^0 = I - \frac{1}{n}E. \tag{5.20}$$

Together (5.18) and (5.19) imply that the diagonal coefficients $a_q^{(t)}$ of V_0^t can be written in the form $n^{t-1}b_q^{(t)}$, and we call $b_q^{(t)}$ the *reduced coefficients* of V_0^t. Hence the equations in (4.9) and (4.10) for the fundamental matrix of $M^t(z,y)$ become

$$\left(\sum_{q=1}^{m}(m+1-q)A_q\right)^{2k+1} = V_0^{2k+1} = \sum_{q=1}^{m} n^{2k} b_q^{(2k+1)} A_q, \tag{5.21}$$

when $t = 2k+1$ is odd, and

$$\left(\sum_{q=1}^{m}(m+1-q)A_q\right)^{2k} = V_0^{2k} = \sum_{q=0}^{m} n^{2k-1} b_q^{(2k)} B_q, \tag{5.22}$$

when $t = 2k$ is even. We note that by (5.19), $b_0^{(0)} = m$, and that (5.18) and (5.19) can also be used to obtain the characteristic polynomial of $M(z,y)$. For a detailed account see [17].

We now re-write (5.18) and (5.19) in terms of the reduced coefficients $b_q^{(2k)}$ and $b_q^{(2k+1)}$ to obtain the following Lemma.

Lemma 5.5. *For $q \geqslant 1$, $r \geqslant 0$, the reduced coefficients $b_q^{(t)}$ of $M^t(z,y)$ satisfy*

$$\sum_{k=0}^{r}\binom{m+r-k}{2(r-k)}\frac{b_q^{(2k+1)}}{2(r-k)+1} = \binom{m+r-q+1}{2r+1}, \tag{5.23}$$

and

$$\sum_{k=0}^{r}\binom{m+r-k}{2(r-k)}\frac{b_q^{(2k)}}{2(r-k)+1} = -\binom{m+r-q}{2r}, \tag{5.24}$$

which can be rearranged as

$$b_q^{(2r+1)} = \binom{m+r-q+1}{2r+1} - \sum_{k=0}^{r-1} f_{r-k} b_q^{(2k+1)}, \tag{5.25}$$

and

$$b_q^{(2r)} = -\binom{m+r-q}{2r} - \sum_{k=0}^{r-1} f_{r-k} b_q^{(2k)}, \tag{5.26}$$

respectively.

Proof. By (5.18), (5.1) and (5.9) we have

$$\begin{aligned} f_0 V_0^{2r+1} &= V_0^{2r+1} = n^{2r} V_r - \sum_{k=0}^{r-1} n^{2(r-k)} f_{r-k} V_0^{2k+1} \\ &= n^{2r}\left(\sum_{q=1}^{m-r}\binom{m+r+1-q}{2r+1}A_q - \sum_{k=0}^{r-1} n^{-2k} f_{r-k}\sum_{q=0}^{m} n^{2k} b_q^{(2k+1)} A_q\right), \end{aligned}$$

by (5.22). Hence

$$\sum_{q=0}^{m} n^{2r} b_q^{(2r+1)} A_q$$

$$= n^{2r} \left(\sum_{q=1}^{m-r} \binom{m+r+1-q}{2r+1} A_q - \sum_{k=0}^{r-1} n^{-2k} f_{r-k} \sum_{q=0}^{m} n^{2k} b_q^{(2k+1)} A_q \right),$$

and comparing coefficients of A_q we have

$$b_q^{(2r+1)} = \binom{m+r+1-q}{2r+1} - \sum_{k=0}^{r-1} f_{r-k} b_q^{(2k+1)}.$$

The proof for $b_q^{(2r)}$ is similar. ∎

We note that for even powers, the coefficient of $B_0 = 2I$ is given by $b_0^{(2r)}$, where by (5.19)

$$\sum_{k=0}^{r} f_{r-k} b_0^{(2k)} = \binom{m+r}{2r+1}. \tag{5.27}$$

Comparing (5.27) with (5.23) when $q = 1$ and taking into account that $b_0^{(0)} = b_1^{(1)} = m$, it follows that $b_0^{(2r)} = b_1^{(2r+1)}$, for $r \geqslant 0$. By (5.26) it also follows that for $q \geqslant 1$, $b_q^{(0)} = -1$. Hence

$$V_0^0 = \frac{1}{n} \sum_{k=0}^{m} b_0^{(2k)} = \frac{1}{n}(2mI - (E - I)),$$

which also satisfies (5.20).

In the case $q = m$ we have

$$\binom{m+r-q+1}{2r+1} = 0, \qquad \binom{m+r-q}{2r} = 0,$$

for $r \geqslant 1$ and $r \geqslant 0$ respectively. Hence

$$b_m^{(2r+1)} = -\sum_{k=0}^{r-1} f_{r-k} b_m^{(2k+1)}, \qquad b_m^{(2r)} = -\sum_{k=0}^{r-1} f_{r-k} b_m^{(2k)}, \tag{5.28}$$

and as $b_m^{(1)} = -b_m^{(0)} = 1$, it follows that $b_m^{(2k+1)} = -b_m^{(2k)}$.

6. Determinant Expressions

In this section we prove that the reduced coefficients $b_q^{(2r)}$, $b_q^{(2r+1)}$, in the expressions for $M^t(z, y)$, defined before Lemma 5.5, can each be expressed as a special type of Toeplitz determinant, and we show that these determinants are related to known determinant generators for the Bernoulli numbers [27].

Definition. *We define any $r \times r$ determinant of the form*

$$(-1)^r \begin{vmatrix} h_1 & 1 & 0 & 0 & \dots & 0 \\ h_2 & h_1 & 1 & 0 & \dots & 0 \\ h_3 & h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{r-1} & h_{r-2} & h_{r-3} & h_{r-4} & \dots & 1 \\ h_r & h_{r-1} & h_{r-2} & h_{r-3} & \dots & h_1 \end{vmatrix} \tag{6.1}$$

to be a minor corner layered determinant *or MCL determinant for short. The name comes from the minor of $a_{1,r+1}$ in the $(r+1)\times(r+1)$ lower triangular determinant shown below.*

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ h_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ h_2 & h_1 & 1 & 0 & \dots & 0 & 0 \\ h_3 & h_2 & h_1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{r-1} & h_{r-2} & h_{r-3} & h_{r-4} & \dots & 1 & 0 \\ h_r & h_{r-1} & h_{r-2} & h_{r-3} & \dots & h_1 & 1 \end{vmatrix}.$$

Lemma 6.1. *Let $h_1, \dots, h_r$ be given. For $k = 1, \dots, r$, let Δ_k be the $k \times k$ MCL determinant* (6.1)*. Let $\Delta_0 = 1$. Then*

$$\Delta_r = -\sum_{k=0}^{r-1} h_{r-k}\Delta_k. \tag{6.2}$$

Conversely, if $\Delta_0 = 1$ and $\Delta_1, \Delta_2, \dots, \Delta_r, h_1, \dots, h_r$ are real numbers satisfying (6.2) *then Δ_r is given in terms of $h_1, \dots, h_r$ by* (6.1)*.*

Corollary. *For $r \geqslant 1$, let g_r be defined by the recurrence relation*

$$g_r = -\sum_{k=0}^{r-1} h_{r-k} g_k, \tag{6.3}$$

where $g_0 = 1$. Then g_r is given by the MCL determinant in the statement of the lemma with $\Delta_r = g_r$.

Proof. We expand the determinant along its first column starting at the r-th row so that

$$(-1)^r \Delta_r = (-1)^{r-1} 1^{r-1} h_r + (-1)^{r-2} 1^{r-2} h_{r-1} \left| h_1 \right| +$$

$$(-1)^{r-3} 1^{r-3} h_{r-2} \begin{vmatrix} h_1 & 1 \\ h_2 & h_1 \end{vmatrix} + \dots + h_1 \begin{vmatrix} h_1 & 1 & \dots & 0 \\ h_2 & h_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{r-2} & h_{r-1} & \dots & 1 \\ h_{r-1} & h_{r-2} & \dots & h_1 \end{vmatrix}$$

and hence the result. The converse follows from re-packing the original determinant.

To see the Corollary, we only need to show that each g_r can be expressed as a determinant of the required form. By (6.3) we have

$$g_1 = -h_1 g_0 = -\left|h_1\right| g_0$$

$$g_2 = -\left(h_2 g_0 + h_1 g_1\right) = -\left(h_2 - h_1^2\right) = \begin{vmatrix} h_1 & 1 \\ h_2 & h_1 \end{vmatrix}.$$

We inductively assume true for g_r, $1 \leqslant r \leqslant n$, and we consider the case g_{n+1} in the relation (6.3), replacing the g_r with the corresponding $r \times r$ determinants. The Corollary follows by the second assertion of the Lemma. ∎

An immediate consequence of this result, is that for $r \geqslant 1$, the expressions for $b_m^{(2r+1)}$ and $b_m^{(2r)}$ in (5.25) and (5.26) can be expressed as MCL determinants. That is

$$b_m^{(2r+1)} = -b_m^{(2r)} = -\sum_{k=0}^{r-1} f_{r-k} b_m^{(2k+1)} \tag{6.4}$$

$$= (-1)^r \begin{vmatrix} f_1 & 1 & 0 & 0 & \dots & 0 \\ f_2 & f_1 & 1 & 0 & \dots & 0 \\ f_3 & f_2 & f_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{r-1} & f_{r-2} & f_{r-3} & f_{r-4} & \dots & 1 \\ f_r & f_{r-1} & f_{r-2} & f_{r-3} & \dots & f_1 \end{vmatrix}. \tag{6.5}$$

Having expressed $b_m^{(2r+1)}$ and $b_m^{(2r)}$ as $r \times r$ MCL determinants, we now show that there is a family of determinants that relate to the reduced coefficients $b_q^{(2r+1)}$ and $b_q^{(2r)}$, $1 \leqslant q \leqslant m$.

Lemma 6.2. *Let the matrices V_r and W_r be defined as at the beginning of Section 5. Then the fundamental matrix V_0^t of $M^t(z,y)$ satisfies*

$$V_0^{2r+1} = n^{2r} \sum_{k=0}^{r} b_m^{(2k+1)} V_{r-k}, \tag{6.6}$$

and

$$V_0^{2r} = n^{2r-1} \left(2I \sum_{k=0}^{r-1} (r-k) f_{r-k} b_m^{(2k)} + \sum_{k=0}^{r} b_m^{(2k)} W_{r-k} \right). \tag{6.7}$$

We deduce that the reduced coefficients in the fundamental matrix satisfy

$$b_q^{(2r+1)} = \sum_{k=0}^{\min(r,\ m-q)} \binom{m-q+k+1}{2k+1} b_m^{(2r-2k+1)}, \tag{6.8}$$

and

$$b_q^{(2r)} = \sum_{k=0}^{\min(r,\ m-q)} \binom{m-q+k}{2k} b_m^{(2r-2k)}. \tag{6.9}$$

Proof. Putting $r = 0$ and 1 in (6.6) gives

$$V_0 = V_0, \qquad V_0^3 = n^2(V_1 - f_1 V_0),$$

which agrees with (5.17).

We inductively assume true so that

$$V_0^{2r+1} V_0^2 = n^{2r} \sum_{k=0}^{r} b_m^{(2k+1)} V_{r-k} V_0^2,$$

and by (5.14) we have

$$\begin{aligned} V_0^{2r+3} &= n^{2r} \sum_{k=0}^{r} b_m^{(2k+1)} n^2 (V_{r-k+1} - f_{r-k+1} V_0) \\ &= n^{2r+2} \left(\sum_{k=0}^{r} b_m^{(2k+1)} V_{r-k+1} - V_0 \sum_{k=0}^{r} b_m^{(2k+1)} f_{r-k+1} \right). \end{aligned}$$

Using (5.28) this reduces to

$$V_0^{2r+3} = n^{2r+2} \sum_{k=0}^{r+1} b_m^{(2k+1)} V_{r+1-k},$$

and the induction is complete.

To obtain (6.7), we simply take the above equation for V_0^{2r-1}, multiply by V_0 and apply (5.12). We then rearrange using (5.28) and replace $b_m^{(2k+1)}$ with $-b_m^{(2k)}$. The results (6.8) and (6.9) then follow by considering coefficients of A_q and B_q in V_r and W_r respectively. ■

Hence, for the odd powers of the fundamental matrix we have

$$\begin{aligned} b_{m-1}^{(2r+1)} &= 2b_m^{(2r+1)} + b_m^{(2r-1)}, \\ b_{m-2}^{(2r+1)} &= 3b_m^{(2r+1)} + 4b_m^{(2r+1)} + b_m^{(2r-1)} + b_m^{(2r-3)}, \\ &\ \vdots \qquad\qquad \vdots \\ b_2^{(2r+1)} &= \sum_{k=0}^{\min(r,\ m-2)} \binom{m-1+k}{2k+1} b_m^{(2r-2k+1)}, \\ b_1^{(2r+1)} &= \sum_{k=0}^{\min(r,\ m-1)} \binom{m+k}{2k+1} b_m^{(2r-2k+1)}. \end{aligned}$$

Translating these equations into determinant format yields

$$b_{m-1}^{(2r+1)} = (-1)^{r+1} \begin{vmatrix} 2 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & f_1 & 1 & 0 & 0 & \dots & 0 \\ 0 & f_2 & f_1 & 1 & 0 & \dots & 0 \\ 0 & f_3 & f_2 & f_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & f_{r-1} & f_{r-2} & f_{r-3} & f_{r-4} & \dots & 1 \\ 0 & f_r & f_{r-1} & f_{r-2} & f_{r-3} & \dots & f_1 \end{vmatrix},$$

$$\vdots \qquad\qquad \vdots$$

$$b_{2}^{(2r+1)} = (-1)^{r+m-2} \begin{vmatrix} \binom{m-1}{1} & 1 & 0 & 0 & 0 & \dots & 0 \\ \binom{m}{3} & f_1 & 1 & 0 & 0 & \dots & 0 \\ \binom{m+1}{5} & f_2 & f_1 & 1 & 0 & \dots & 0 \\ \binom{m+2}{7} & f_3 & f_2 & f_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{m+r-2}{2r-1} & f_{r-1} & f_{r-2} & f_{r-3} & f_{r-4} & \dots & 1 \\ \binom{m+r-1}{2r+1} & f_r & f_{r-1} & f_{r-2} & f_{r-3} & \dots & f_1 \end{vmatrix},$$

$$b_{1}^{(2r+1)} = (-1)^{r+m-1} \begin{vmatrix} \binom{m}{1} & 1 & 0 & 0 & 0 & \dots & 0 \\ \binom{m+1}{3} & f_1 & 1 & 0 & 0 & \dots & 0 \\ \binom{m+2}{5} & f_2 & f_1 & 1 & 0 & \dots & 0 \\ \binom{m+3}{7} & f_3 & f_2 & f_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{m+r-1}{2r-1} & f_{r-1} & f_{r-2} & f_{r-3} & f_{r-4} & \dots & 1 \\ \binom{m+r}{2r+1} & f_r & f_{r-1} & f_{r-2} & f_{r-3} & \dots & f_1 \end{vmatrix}.$$

Corresponding families of determinants exist for the even power reduced coefficients $b_q^{(2r)}$. We give the first few polynomials in m for $b_1^{(2r+1)}$ and $b_m^{(2r+1)}$ below. We have

$$\begin{aligned}
b_1^{(1)} &= m, \qquad b_m^{(1)} = 1, \\
b_1^{(3)} &= b_m^{(3)} = -\frac{m^2}{6} - \frac{m}{6}, \\
b_1^{(5)} &= \frac{m^4}{90} + \frac{m^3}{45} + \frac{2m^2}{45} + \frac{m}{30}, \\
b_m^{(5)} &= \frac{7m^4}{360} + \frac{7m^3}{180} + \frac{13m^2}{360} + \frac{m}{60}, \\
b_1^{(7)} &= -\frac{m^6}{945} - \frac{m^5}{315} - \frac{17m^4}{2520} - \frac{31m^3}{3780} - \frac{3m^2}{280} - \frac{m}{140}, \\
b_m^{(7)} &= -\frac{31m^6}{15120} - \frac{31m^5}{5040} - \frac{17m^4}{1680} - \frac{151m^3}{15120} - \frac{2m^2}{315} - \frac{m}{420}.
\end{aligned}$$

Remark. Geometric interpretations [2] are often of interest and it is well documented [22] that their exists a natural relationship between determinants of order $r+1$ and $\pm r!$ times the volume of an r-dimensional simplex. Hence one interpretation of the reduced coefficient polynomials in m, $b_q^{(t)}$, is as a multiple of the volume of a simplex, and as such, they are geometric polynomials that generate geometric numbers for different values of m. For example, the determinant for the coefficient $b_m^{(2r+1)}$ can be written as the $(r+1)\times(r+1)$ determinant

$$(-1)^r \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & f_1 & 1 & 0 & 0 & \dots & 0 \\ 1 & f_2 & f_1 & 1 & 0 & \dots & 0 \\ 1 & f_3 & f_2 & f_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_{r-1} & f_{r-2} & f_{r-3} & f_{r-4} & \dots & 1 \\ 1 & f_r & f_{r-1} & f_{r-2} & f_{r-3} & \dots & f_1 \end{vmatrix}.$$

Here, $|b_m^{(2r+1)}/r!|$ is equal to the r-dimensional volume of a simplex with corners

$$(0,0,0\dots,0),(f_1,1,0,\dots,0),(f_2,f_1,1,\dots,0),\dots,(f_r,f_{r-1},f_{r-2},\dots,f_1).$$

A parallel interpretation can be made for the reduced coefficients of the fundamental inverse matrix of $M^{-t}(z,y)$ as they are just binomial coefficients, and these are known [4] to represent the lattice point [18] enumerators of a simplex. The difference here being that the volume is discrete rather than continuous.

The denominator of f_r is $(2r+1)!$ and we now highlight further the link between the coefficients $b_q^{(2r+1)}$ and the Bernoulli numbers (and so the even zeta values) with some known results concerning MCL determinants of these denominators.

Proposition 6.3. *The Bernoulli numbers, $\mathcal{B}_r$, are generated by the $r\times r$ MCL determinant of factorial denominators*

$$\mathcal{B}_r = (-1)^r r! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \dots & 1 \\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \dots & \frac{1}{2!} \end{vmatrix}, \tag{6.10}$$

and the even Bernoulli numbers $\mathcal{B}_{2r}$ are generated by the $r\times r$ MCL determinant of odd factorial denominators

$$\mathcal{B}_{2r} = (-1)^{r-1}\frac{2r!}{2(2^{2r-1}-1)}\begin{vmatrix} \frac{1}{3!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{5!} & \frac{1}{3!} & 1 & 0 & \cdots & 0 \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(2r-1)!} & \frac{1}{(2r-3)!} & \frac{1}{(2r-5)!} & \frac{1}{(2r-7)!} & \cdots & 1 \\ \frac{1}{(2r+1)!} & \frac{1}{(2r-1)!} & \frac{1}{(2r-3)!} & \frac{1}{(2r-5)!} & \cdots & \frac{1}{3!} \end{vmatrix}. \tag{6.11}$$

Proof. We refer the reader to [29] and [19]. A related identity is given in [27]. ■

7. Zeta Numbers and p-adic Properties

In this section we show that a Corollary to this examination of $M(z, y)$ is a particularly simple recurrence for the Bernoulli numbers (and so for the Riemann zeta function at even integers) which does not appear to have been written down before.

Lemma 7.1. *Let $b_q^{(2r+1)}$ be the reduced coefficient of A_q in the expansion of $M^{2r+1}(z, y)$. Then for $r \geqslant 1$, we can write*

$$b_q^{(2r+1)} = \sum_{j=0}^{2r} c_{q,j}^{(2r+1)} m^j, \tag{7.1}$$

where

$$c_{q,2r}^{(2r+1)} = \frac{-r(2q-1)}{(2r+1)!} - \sum_{k=1}^{r-1}\frac{c_{q,2k}^{(2k+1)}}{(2r-2k+1)!}, \tag{7.2}$$

and

$$c_{m,2r}^{(2r+1)} = -\sum_{k=0}^{r-1}\frac{c_{m,2k}^{(2k+1)}}{(2r-2k+1)!}, \tag{7.3}$$

Proof. The coefficient of m^{2r+1} in

$$\binom{m+r+1-q}{2r+1},$$

is $1/(2r+1)!$ and this always cancels with the coefficient of

$$\frac{1}{2r+1}\binom{m+r}{2r}b_q^{(1)} = \frac{1}{2r+1}\binom{m+r}{2r}(m+1-q)$$

in (5.25). Therefore $b_q^{(3)}$ is a polynomial in m of degree at most 2 and by the recursive definition it follows that for $r \geqslant 1$, $b_q^{(2r+1)}$ is a polynomial in m of degree at most $2r$.

As defined in [12], let $s(n,k)$ denote the Stirling numbers of the first kind and let $m^{\underline{n}}$ denote the falling factorial

$$m(m-1)(m-2)\dots(m-n+1).$$

The $s(n,k)$ count the number of permutations of n elements with k disjoint cycles and are related to $m^{\underline{n}}$ by the identity

$$m^{\underline{n}} = \sum_{k=1}^{n} s(n,k)m^k.$$

Replacing m with $m+i$ yields

$$\begin{aligned}(m+i)^{\underline{n}} &= (m+i)(m+i-1)(m+i-2)\dots(m+i-n+1) && (7.4)\\ &= \sum_{k=1}^{n} s(n,k)(m+i)^k = \sum_{k=1}^{n} s(n,k)\sum_{j=0}^{k}\binom{k}{j}m^j i^{k-j}, && (7.5)\end{aligned}$$

and collecting terms we have

$$(m+i)^{\underline{n}} = \sum_{j=0}^{n} m^j \sum_{t=0}^{n-j}\binom{j+t}{j} s(n,j+t)i^t.$$

By (7.4) we can write (5.25) as

$$\sum_{j=0}^{2r} c_{q,j}^{(2r+1)} m^j = \frac{(m+r+1-q)^{\underline{2r+1}}}{(2r+1)!} - \sum_{k=0}^{r-1}\frac{(m+r-k)^{\underline{2r-2k}}}{(2r-2k+1)!}\sum_{j=0}^{2k} c_{q,j}^{(2k+1)} m^j, \quad (7.6)$$

and we use (7.5) to consider coefficients of m^{2r} in (7.6). We have

$$\begin{aligned}c_{q,2r}^{(2r+1)} &= \frac{1}{(2r+1)!}\sum_{t=0}^{1}\binom{2r+t}{2r} s(2r+1,2r+t)(r+1-q)^t\\ &\quad - \frac{1}{(2r+1)!}\sum_{t=0}^{1}\binom{2r-1+t}{2r-1} s(2r,2r-1+t)r^t - \frac{1-q}{(2r+1)!}\\ &\quad - \sum_{k=1}^{r-1}\frac{c_{q,2k}^{(2k+1)}}{(2r-2k+1)!},\end{aligned}$$

so that

$$c_{q,2r}^{(2r+1)} = \frac{-r(2q-1)}{(2r+1)!} - \sum_{k=1}^{r-1}\frac{c_{q,2k}^{(2k+1)}}{(2r-2k+1)!}.$$

The proof for (7.3) is simpler as we start with (5.28). Corresponding recurrence relations for the even power coefficients $c_{q,2r}^{(2r)}$ in the polynomial $b_q^{(2r)}$ can also be derived. ■

In order that we may prove that the coefficients in $M^{2r+1}(z,y)$ are related to the even integer Zeta numbers, we first prove the identity itself.

Lemma 7.2 (first Zeta identity lemma).

$$\zeta(2j) = (-1)^{j+1}\left(\frac{j\pi^{2j}}{(2j+1)!} + \sum_{k=1}^{j-1}\frac{(-1)^k\pi^{2j-2k}}{(2j-2k+1)!}\zeta(2k)\right).$$

Proof. Let the j-th Bernoulli number, $\mathcal{B}_j$, and the j-th Bernoulli polynomial, $\mathcal{B}_j(x)$, be defined in the usual fashion, so that

$$\mathcal{B}_j = -\frac{1}{j+1}\sum_{k=0}^{j-1}\binom{j+1}{k}\mathcal{B}_k, \qquad \mathcal{B}_j(x) = \sum_{k=0}^{j}\binom{j}{k}\mathcal{B}_{j-k}x^k,$$

where $\mathcal{B}_0 = 1$. Then for $j > 1$ we have

$$\begin{aligned}0 = (2^{-2j}-1)\mathcal{B}_{2j+1} = \mathcal{B}_{2j+1}\left(\frac{1}{2}\right) &= \sum_{k=0}^{2j+1}\binom{2j+1}{k}\left(\frac{1}{2}\right)^{2j+1-k}\mathcal{B}_k\\ = \binom{2j+1}{1}\frac{1}{2^{2j}}\mathcal{B}_1 + \sum_{k=0}^{j}\binom{2j+1}{2k}\frac{1}{2^{2(j-k)+1}}\mathcal{B}_{2k}.\end{aligned}$$

Thus we get

$$\frac{(2j+1)\mathcal{B}_{2j}}{2} = \frac{2j+1}{2^{2j+1}} - \sum_{k=0}^{j-1}\binom{2j+1}{2k}\frac{1}{2^{2(j-k)+1}}\mathcal{B}_{2k},$$

yielding

$$\begin{aligned}2^{2j-1}\mathcal{B}_{2j} &= \frac{1}{2} - \frac{1}{2j+1}\sum_{k=0}^{j-1}\binom{2j+1}{2k}2^{2k-1}\mathcal{B}_{2k}\\ &= \frac{j}{2j+1} - \frac{1}{2j+1}\sum_{k=1}^{j-1}\binom{2j+1}{2k}2^{2k-1}\mathcal{B}_{2k}.\end{aligned}$$

Hence, by the well-known Bernoulli-zeta even integer relation [5]

$$\zeta(2j) = \frac{(-1)^{j+1}2^{2j-1}\pi^{2j}\mathcal{B}_{2j}}{(2j)!},$$

we can write

$$\zeta(2j) = (-1)^{j+1}\left(\frac{j\pi^{2j}}{(2j+1)!} + \sum_{k=1}^{j-1}\frac{(-1)^k\pi^{2j-2k}}{(2j-2k+1)!}\zeta(2k)\right),$$

as required. ∎

Lemma 7.3 (coefficient lemma). *We have*

$$c_{1,1}^{(2r+1)} = rc_{m,1}^{(2r+1)}. \tag{7.7}$$

$$c_{m,1}^{(2r+1)} = \frac{(-1)^r (r!)^2}{r(2r+1)!} \tag{7.8}$$

$$c_{m,2r}^{(2r+1)} = \frac{(-1)^r (2^{2r-1}-1)}{2^{2r-2}\pi^{2r}}\zeta(2r) \tag{7.9}$$

$$c_{q,2r}^{(2r+1)} = \frac{(-1)^r (2q-1)}{\pi^{2r}}\zeta(2r) \tag{7.10}$$

$$c_{q,2r-1}^{(2r+1)} = rc_{q,2r}^{(2r+1)}. \tag{7.11}$$

Proof. Equations (7.7) and (7.8) follow directly from the recurrence relations (5.25) and (5.28). Equation (7.9) follows from the relation (7.3) and the even integer zeta identity [5]

$$\sum_{k=0}^{j} \frac{(-1)^k \pi^{2k}}{(2k+1)!}(1-2^{2k-2j+1})\zeta(2j-2k) = 0, \tag{7.12}$$

When $r = 1$ and 2 in (7.2) we have

$$c_{q,2}^{(3)} = \frac{-(2q-1)}{6}, \qquad c_{q,4}^{(5)} = \frac{(2q-1)}{90},$$

which agrees with (7.10). Inductively assuming true in (7.2) gives

$$c_{q,2r}^{(2r+1)} = \frac{-r(2q-1)}{(2r+1)!} - \sum_{k=1}^{r-1} \frac{(-1)^k (2q-1)}{\pi^{2r}(2r-2k+1)!}\zeta(2r),$$

and by Lemma 7.2 this implies that

$$c_{q,2r}^{(2r+1)} = \frac{(-1)^r (2q-1)}{\pi^{2r}}\zeta(2r).$$

Equation (7.11) then follows directly from the recurrence relation (5.25). ■

We note that similar relations exist for the even power coefficients $c_{q,s}^{(2r)}$ in the polynomial $b_q^{(2r)}$. One of the most notable being that

$$c_{q,2r}^{(2r)} = \frac{2(-1)^r}{\pi^{2r}}\zeta(2r).$$

Lemma 7.4 (second Zeta identity lemma).

$$\frac{\zeta(2r)}{\pi^{2r}} = \frac{2^{2r-2}}{(2^{2r-1}-1)} \sum_{s=1}^{r} \sum_{\substack{d_i \geqslant 0 \\ d_1+d_2+\ldots+d_r=s \\ d_1+2d_2+\ldots+rd_r=r}} \binom{s}{d_1, d_2, \ldots, d_r} \frac{(-1)^{s+r}}{3!^{d_1}5!^{d_2}\ldots(2r+1)!^{d_r}}. \tag{7.13}$$

Proof. From (6.8) and (6.9) the reduced coefficients $b_m^{(2k+1)}$ and $-b_m^{(2k)}$ can be used to express the reduced coefficients $b_q^{(t)}$ of V_0^t. By (6.4) we have

$$b_m^{(2r+1)} = -b_m^{(2r)} = -\sum_{k=0}^{r-1} f_{r-k} b_m^{(2k+1)}, \tag{7.14}$$

and repeated use of (7.14) gives

$$b_m^{(2r+1)} = (-1)^r \sum_{k_1=0}^{r-1} \sum_{k_2=0}^{k_1-1} \cdots \sum_{k_w=0}^{k_{w-1}-1} f_{r-k_1} f_{k_1-k_2} \cdots f_{k_{w-1}-k_w} b_m^{(2k_w+1)},$$

with $k_w = k_{w-1} - 1 = 0$, so that $b_m^{(2k_w+1)} = b_m^{(1)} = 1$. Hence we can write

$$b_m^{(2r+1)} = (-1)^r \sum_{k_1=0}^{r-1} \sum_{k_2=0}^{k_1-1} \cdots \sum_{k_w=0}^{k_{w-1}-1} f_{r-k_1} f_{k_1-k_2} \cdots f_{k_{w-1}-k_w}, \tag{7.15}$$

which is just a sum of products of f_k, where the subscripts in each product sum to r. By considering the determinant expansion (6.5) of $b_m^{(2r+1)}$ we see that number of products is 2^{r-1}. That is, when the expressions for the sum of the products is simplified, then ignoring sign, the sum of the coefficients of the products is 2^{r-1}. For example, when $r = 5$, we have

$$b_m^{(11)} = -f_1^5 + 4f_1^3 f_2 - 3f_1 f_2^2 - 3f_1^2 f_3 + 2f_2 f_3 + 2f_1 f_4 - f_5.$$

Therefore we have established that $b_m^{(2r+1)}$ is a sum of monomials of the form

$$\pm f_1^{d_1} f_2^{d_2} \ldots f_r^{d_r}, \tag{7.16}$$

with

$$d_i \geqslant 0, \qquad d_1 + 2d_2 + \ldots + rd_r = r.$$

We note that for a given $d_1 + 2d_2 + \ldots + rd_r = r$, with $d_1 + d_2 + \ldots + d_j = s$, the coefficient of the product in (7.16) is the same (ignoring sign) as that in the multinomial expansion of

$$(f_1 + f_2 + \ldots + f_r)^s.$$

Hence we can write

$$b_m^{(2r+1)} = \sum_{s=1}^{r} \sum_{\substack{d_i \geqslant 0 \\ d_1+d_2+\ldots+d_r=s \\ d_1+2d_2+\ldots+rd_r=r}} (-1)^s \binom{s}{d_1, d_2, \ldots, d_r} f_1^{d_1} f_2^{d_2} \ldots f_r^{d_r}. \tag{7.17}$$

From (7.9) of Lemma 7.3 the leading coefficient $c_{m,2r}^{(2r+1)}$ of the polynomial expansion of $b_m^{(2r+1)}$ satisfies

$$c_{m,2r}^{(2r+1)} = \frac{(-1)^r (2^{2r-1} - 1)}{2^{2r-2} \pi^{2r}} \zeta(2r), \tag{7.18}$$

from which we deduce the identity

$$\frac{\zeta(2r)}{\pi^{2r}} = \frac{2^{2r-2}}{(2^{2r-1}-1)} \sum_{s=1}^{r} \sum_{\substack{d_i \geqslant 0 \\ d_1+d_2+\ldots+d_r=s \\ d_1+2d_2+\ldots+rd_r=r}} \binom{s}{d_1, d_2, \ldots, d_r} \frac{(-1)^{s+r}}{3!^{d_1} 5!^{d_2} \ldots (2r+1)!^{d_r}}. \quad \blacksquare$$

Similar identities can be obtained by comparing (7.17) with (6.8) and (7.10).

Lemma 7.5 (denominator lemma). *Let m and k be positive integers, with*

$$2m+1 = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_r^{a_r}$$

as a product of primes. Let $p_1, \ldots, p_s$ be the prime factors of $2m+1$ with $p \leqslant 2k+1$; only these primes can occur in the denominators of $b_m^{(2k)}$, and $b_m^{(2k+1)}$. For $i \leqslant s$, let $p_i = 2v_i + 1$, and let $k = q_i v_i + w_i$, where q_i and w_i are positive integers with $q_i \geqslant 1$, $0 \leqslant w_i \leqslant v_i - 1$. Let

$$Q = p_1^{q_1} p_2^{q_2} \ldots p_s^{q_s}.$$

Then $Qb_q^{(2k)}$ and $Qb_q^{(2k+1)}$ are integers.

Corollary 1. *The rational numbers*

$$(2m+1)^k b_q^{(2k)}, \qquad (2m+1)^k b_q^{(2k+1)} \tag{7.19}$$

are integers, $1 \leqslant q \leqslant m$.

Corollary 2. *Let $\ell = \ell(k,m)$ of $b_q^{(2k+1)}$ be the smallest ℓ for which*

$$(2m+1)^{\ell} b_q^{(2k+1)}$$

is an integer, so that $(2m+1)^{\ell} b_q^{(2k)}$ is also an integer. Then for $(2m+1)$ a prime we have

$$\ell = \left[\frac{k}{m}\right].$$

Proof. In the proof of Lemma 7.4 we established that $b_m^{(2k)}$ and $b_m^{(2k+1)}$ are both sums of monomials of the form

$$\pm f_{c_1}^{d_1} f_{c_2}^{d_2} \ldots f_{c_j}^{d_j},$$

with

$$c_1 d_1 + c_2 d_2 + \ldots + c_j d_j = k.$$

Let D_k be the minimal denominator of f_k. Then $D_k \leqslant 2k+1$. We want to establish the power of $p = 2v+1$ in

$$D_{c_1}^{d_1} D_{c_2}^{d_2} \ldots D_{c_j}^{d_j}.$$

This is at most

$$d_1\left\lceil\frac{2c_1+1}{2v+1}\right\rceil+d_2\left\lceil\frac{2c_2+1}{2v+1}\right\rceil+\ldots+d_j\left\lceil\frac{2c_j+1}{2v+1}\right\rceil.$$

We note that

$$\frac{2c_i+1}{2v+1}\leqslant\frac{c_i}{v}$$

when $2cv+v\leqslant 2cv+c$, $v\leqslant c$, and that

$$\left\lceil\frac{2c_1+1}{2v+1}\right\rceil=0\leqslant\frac{c}{v},$$

when $v>c$. So the power of p in $D_{c_1}^{d_1}D_{c_2}^{d_2}\ldots D_{c_j}^{d_j}$ is at most

$$\frac{c_1d_1+\ldots+c_jd_j}{v}\leqslant\frac{k}{v}=q+\frac{w}{v}.$$

The power is an integer, so it is at most q. We deduce the result of the Lemma.

To see the first Corollary we have

$$Qb_m^{(1)},Qb_m^{(3)},\ldots,Qb_m^{(2k+1)}\in\mathbb{N},$$

with

$$Q|(2m+1)^k.$$

The $b_q^{(2k+1)}$ are just linear integer combinations of the $b_m^{(2j+1)}$, with $0\leqslant j\leqslant k$, and the results follows. In the second Corollary, $2m+1$ is the only prime that can occur in the denominator, and so in the proof of the Lemma we have

$$\ell=q=\left\lceil\frac{k}{m}\right\rceil. \qquad \blacksquare$$

We are now in a position to compare the p-adic properties of $b_q^{(t)}$ with $b_q^{(-t)}$. That is, the Fleck numbers with the numbers generated by the determinants in Section 6.

Lemma 7.6. *For natural number m, let $n=2m+1$ be odd, let $F(n,t)$ be defined as in* (4.16) *and for p_i^α an odd prime power, $\alpha\geqslant 1$, let*

$$G(p_i^\alpha,t)=\left\lfloor\frac{t}{p_i-1}\right\rfloor,\qquad 1\leqslant i\leqslant r.$$

Then when $n=p^\alpha$ we have

$$\mathrm{Ord}_p\ b_q^{(-t)}b_q^{(t)}\geqslant\left\lfloor\frac{t-p^{\alpha-1}}{\phi(p^\alpha)}\right\rfloor-\left\lfloor\frac{t}{p-1}\right\rfloor=F(p^\alpha,t)-G(p^\alpha,t),\tag{7.20}$$

and when $n=p_1^{a_1}p_2^{a_2}\ldots p_r^{a_r}$ is not a prime power then

$$\mathrm{Ord}_{p_i}\ b_q^{(-t)}b_q^{(t)}\geqslant-\left\lfloor\frac{t}{p_i-1}\right\rfloor=-G(p_i^\alpha,t),\qquad 1\leqslant i\leqslant r.\tag{7.21}$$

Proof. From (5.21) and (5.22) we can write the diagonal coefficients of V_0^t as $n^{t-1}b_q^{(t)}$, where by Lemma 7.5, we have

$$\mathrm{Ord}_{p_i}\, b_q^{(t)} \geqslant -\left\lfloor \frac{t}{p_i-1} \right\rfloor = -G(p_i,t), \qquad 1 \leqslant i \leqslant r. \tag{7.22}$$

Combining (4.16) and (7.22) when $n = p^\alpha$ then gives (7.20). The inequality (7.21) follows immediately from the definition that $F(n,t) = 0$ when n is not a prime power. ■

Experimentally it often seems to be the case that the inequality signs in (7.20) and (7.21) can be replaced with equality signs.

An interesting relation between the $b_q^{(-t)}$ and the $b_q^{(t)}$ is given in the following lemma.

Lemma 7.7. *Let $n = 2m+1$ and $t = 2k+1$ be positive odd integers. Let $\mathbf{x}$ and $\mathbf{y}$ be the respective vectors $(b_1^{(-t)}, b_2^{(-t)}, \ldots, b_m^{(-t)})$ and $(b_1^{(t)}, b_2^{(t)}, \ldots, b_m^{(t)})$, so that $\mathbf{x}$ and $\mathbf{y}$ are both of dimension m. Then*

$$\mathbf{x}.\mathbf{y} = \sum_{r=1}^{m} b_q^{(-t)} b_q^{(t)} = -m, \tag{7.23}$$

Proof. The result follows directly from the identity

$$V_0^t V_0^{-t} = I_n - \frac{1}{n}E,$$ ■

When equality holds in (7.20) and (7.21), and taking into account the prime powers already present in the vector entries, it is interesting to note that the dot product $\mathbf{x}.\mathbf{y}$ generates the prime powers

$$p^{\left\lfloor \frac{t}{p-1} \right\rfloor - \left\lfloor \frac{t-p^{\alpha-1}}{\phi(p^\alpha)} \right\rfloor} \qquad \text{or} \qquad p_i^{\left\lfloor \frac{t}{p_i-1} \right\rfloor}, \qquad 1 \leqslant i \leqslant r,$$

depending on whether n is a prime power or not. It is also interesting to note that for large values of t and n the rational vectors $\mathbf{x}$ and $\mathbf{y}$ are almost perpendicular.

We conclude this section by considering Table 1 (p. 38), the extended array formed from the $b_q^{(t)}$ and the $b_q^{(-t)}$ when $m = 5$. The array links the geometric numbers, generated by the geometric polynomials $b_q^{(t)}$, to the $b_q^{(-t)}$ (and so the Fleck numbers) through a relation of the type given in (1.5).

Let $P_{11}(t,q)$ denote the entry in row t and column q of the array, so that for $t > 0$, $P_{11}(-t,q) = b_q^{(-t)} = C_{11}(t, \lfloor \frac{t}{2} \rfloor + q)$. Then moving down the array from odd to even row values of t, we have the relation

$$P_{11}(t-1,q-1) = P_{11}(t,q) - P_{11}(t,q-1),$$

whereas for even to odd row values of t. we have

$$P_{11}(t-1,q) = P_{11}(t,q) - P_{11}(t,q-1),$$

with a factor of 11 removed between row 0 and row -1.

Further observations on congruences modulo n and relations between the $b_q^{(-t)}$ and the $b_q^{(t)}$ are given in the next section.

8. Fleck Quotients and other Determinants

For positive integer n, we define the Fleck quotients to be the n numbers

$$n^{-F(n,t)}C_n(t,q) = n^{-F(n,t)} \sum_{k\equiv q \pmod n} (-1)^k \binom{t}{k}, \qquad 0 \leqslant q \leqslant n-1, \tag{8.1}$$

so that the Fleck quotients take integer values for all integers n. When $n = p$, a prime, a result by Z. W. Sun and D. W. Wan [25] determines the Fleck quotients modulo p. Their results (not stated here) imply the following proposition.

Proposition 8.1. *Let t and c be natural numbers and p a prime with $t = c \times \phi(p) - 1$. Then the reduced coefficients $b_q^{-(t)}$ (and so the Fleck quotients) of A_q in the linear expansion of $p \times p$ V_0^{-t}, satisfy*

$$\begin{aligned} &p^{-F(p,t)}\left(b_1^{(-t)}, b_2^{(-t)}, \ldots, b_{m-1}^{(-t)}, b_m^{(-t)}, 0, -b_m^{(-t)}, -b_{m-1}^{(-t)}, \ldots, -b_2^{(-t)}, -b_1^{(-t)}\right) \\ &\qquad \equiv (-1)^c\,(m, m-1, \ldots, 2, 1, 0, -1, -2, \ldots, -(m-1), -m) \pmod p, \end{aligned} \tag{8.2}$$

and when $t = c \times \phi(p)$, then the reduced coefficients $b_q^{-(t)}$ of B_q in the linear expansion of $p \times p$ V_0^{-t}, satisfy

$$\begin{aligned} &p^{-F(p,t)}\left(b_1^{(-t)}, b_2^{(-t)}, \ldots, b_m^{(-t)}, b_0^{(-t)}, -b_m^{(-t)}, \ldots, -b_2^{(-t)}, -b_1^{(-t)}\right) \\ &\qquad\qquad \equiv (-1)^{c-1}\,(1, 1, \ldots, 1, 1, 1, \ldots, 1, 1) \pmod p. \end{aligned} \tag{8.3}$$

We note that a permutation of the expression (8.2) also holds when $t = p$.

For $n = p^\alpha$ a prime power, with $\alpha \geqslant 2$, there exist symmetries modulo p but not modulo n.

With regard to the entries in the $n \times n$ square V_0^t, generated by the polynomial $b_q^{(t)}$, we again look for symmetries modulo n that are either all different or all equal. The observations are stated below as a conjecture, and it is assumed that the value of m has been substituted in the polynomial $b_q^{(t)}$.

Conjecture 8.2. *Let m, t and c be natural numbers and $n = 2m + 1$ an odd integer. Then the geometric numbers generated by the geometric polynomials in m, $b_q^{(t)}$, satisfy*

(1) *For $n = p$ a prime, and $t = c \times \phi(p) + 1$, the reduced coefficients $b_q^{(t)}$ of A_q in the linear expansion of $p \times p$ V_0^t, satisfy the congruence given in (8.2) but with the $b_q^{(-t)}$ replaced with $b_q^{(t)}$ and $F(p,t)$ replaced with $-G(p,t)$.*

(2) *For $n = p$ a prime, and $t = c \times \phi(p)$, the reduced coefficients $b_q^{(t)}$ of B_q in the linear expansion of $p \times p$ V_0^t, satisfy the congruence given in (8.3) but with the $b_q^{(-t)}$ replaced with $b_q^{(t)}$ and $F(p,t)$ replaced with $-G(p,t)$.*

(3) *For $n = p^\alpha$ a prime power, with $\alpha \geqslant 2$, there exist symmetries for the reduced coefficients $b_q^{(t)}$ modulo p.*

(4) *For $n = p_1 p_2 \dots p_r$, a square free number, where the p_i are the odd prime factors of n, let*

$$w(n) = \phi(p_1)\phi(p_2)\dots\phi(p_r).$$

Then for $t = c \times w(n) + 1$ we have

$$\begin{aligned} &p_1^{G(p_1,t)} p_2^{G(p_2,t)} \dots p_r^{G(p_r,t)} \\ &\quad \times \left(b_1^{(t)}, b_2^{(t)}, \dots, b_{m-1}^{(t)}, b_m^{(t)}, 0, -b_m^{(t)}, -b_{m-1}^{(t)}, \dots, -b_2^{(t)}, -b_1^{(t)}\right) \\ &\quad \equiv (m, m-1, \dots, 2, 1, 0, -1, -2, \dots, -(m-1), -m) \pmod{n}, \end{aligned} \tag{8.4}$$

and when $t = c \times w(n)$, then

$$\begin{aligned} &p_1^{G(p_1,t)} p_2^{G(p_2,t)} \dots p_r^{G(p_r,t)} \\ &\quad \times \left(b_1^{(t)}, b_2^{(t)}, \dots, b_{m-1}^{(t)}, b_m^{(t)}, b_0^{(t)}, -b_m^{(t)}, -b_{m-1}^{(t)}, \dots, -b_2^{(t)}, -b_1^{(t)}\right). \\ &\quad \equiv -(1, 1, \dots, 1, 1, 1, 1, 1, \dots, 1, 1) \pmod{n}. \end{aligned} \tag{8.5}$$

(5) *Let $v(n) = \mathrm{LCM}\,(\phi(p_1), \phi(p_2), \dots \phi(p_r)) < w(n)$. Then there exist values of c with $t = c \times v(n)$ or $t = c \times v(n) + 1$, that ignoring sign, satisfy the congruence relations given in* (8.4) *and* (8.5).

From a symmetry perspective, Proposition 8.1 and Conjecture 8.2 tell us when there exist natural numbers n, t and integers k such that either $n^k V_0^{-t}$ or $n^k V_0^t \equiv \pm V_0 \pmod{n}$. When this occurs, then by Euler, we know that either $(nI - J)n^k V_0^{-t}$ or $(nI - J)n^k V_0^t$ contain all of the residue classes $\pmod{n^2}$, and so $\pmod{n^2}$, satisfies the conditions of a traditional associated magic square. Hence the cycles of the residue classes $\pmod{n}$ determine the cycles of symmetry $\pmod{n^2}$.

Further symmetric relations between the Fleck numbers, $b_q^{(-t)}$, and the geometric polynomials, $b_q^{(t)}$, can be observed when one considers Vandermonde type determinants constructed from them. Knuth defines the $r \times r$ Vandermonde matrix $V = (a_{i,j})$ by the simple relation $a_{i,j} = x_i^j$. The determinant of V is then given by the product

$$\prod_{1 \leqslant j \leqslant r} x_j \prod_{1 \leqslant i < j \leqslant r} (x_j - x_i).$$

Although $b_q^{(t)} b_q^{(s)} \neq b_q^{(t+s)}$, surprisingly symmetric products are obtained when one considers the determinant of the $r \times r$ matrix $a_{i,j} = b_i^{(j)}$, and for certain values of $s \in \mathbb{Z}$, when one considers the determinants of the $r \times r$ matrices defined by $a_{i,j} = b_i^{(2j-1+2s)}$ and $a_{i,j} = b_i^{(2j+2s)}$. Some of these observations are stated below as a conjecture in which r denotes the determinants size and s the j-shift across the $b_i^{(j)}$.

Conjecture 8.3.

(1) *For natural number r, let $D'_m(r,0)$ be the $r \times r$ determinant defined such that*

$$D'_m(r,0) = \begin{vmatrix} b_1^{(1)} & b_1^{(2)} & b_1^{(3)} & \dots & b_1^{(r)} \\ b_2^{(1)} & b_2^{(2)} & b_2^{(3)} & \dots & b_2^{(r)} \\ b_3^{(1)} & b_3^{(2)} & b_3^{(3)} & \dots & b_3^{(r)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{r-1}^{(1)} & b_{r-1}^{(2)} & b_{r-1}^{(3)} & \dots & b_{r-1}^{(r)} \\ b_r^{(1)} & b_r^{(2)} & b_r^{(3)} & \dots & b_r^{(r)} \end{vmatrix}. \tag{8.6}$$

Then, ignoring sign, $D'_m(r,0)$ is given by the expression

$$\frac{m(m-1)(m-2)\dots(m-k+1)}{k!} \cdot \frac{(2m-1)(2m-3)\dots(2m-(2k-1))}{(2k+1)!!},$$

when $r = 2k$ is even, and

$$\frac{m(m-1)(m-2)\dots(m-k)}{(k+1)!} \cdot \frac{(2m-1)(2m-3)\dots(2m-(2k-1))}{(2k+1)!!}$$

when $r = 2k+1$ is odd.

We note that the above two expressions can be represented by the more concise single expression

$$D'_m(r,0) = \frac{\binom{2m}{r}}{r+1}$$

(2) *For natural number r, let $D_m(r,0)$ be the $r \times r$ determinant defined such that*

$$D_m(r,0) = \begin{vmatrix} b_1^{(1)} & b_1^{(3)} & b_1^{(5)} & \dots & b_1^{(2r-1)} \\ b_2^{(1)} & b_2^{(3)} & b_2^{(5)} & \dots & b_2^{(2r-1)} \\ b_3^{(1)} & b_3^{(3)} & b_3^{(5)} & \dots & b_3^{(2r-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{r-1}^{(1)} & b_{r-1}^{(3)} & b_{r-1}^{(5)} & \dots & b_{r-1}^{(2r-1)} \\ b_r^{(1)} & b_r^{(3)} & b_r^{(5)} & \dots & b_r^{(2r-1)} \end{vmatrix}. \tag{8.7}$$

Then, ignoring sign, we have

$$\begin{aligned} D_m(r,0) = {} & \frac{m}{1}\frac{(m-1)^2}{2^2}\frac{(m-2)^3}{3^3}\cdots\frac{(m-k+1)^k}{k^k} \\ & \times\frac{(m-k)^k}{(k+1)^k}\frac{(m-k-1)^{k-1}}{(k+2)^{k-1}}\cdots\frac{(m-(r-2))^2}{(r-1)^2}\frac{(m-(r-1))}{r} \\ & \times\frac{(2m-1)}{3}\frac{(2m-3)^2}{5^2}\frac{(2m-5)^3}{7^3}\cdots\frac{(2m-(2k-3))^{k-1}}{(2k-1)^{k-1}} \\ & \times\frac{(2m-(2k-1))^k}{(2k+1)^k}\frac{(2m-(2k+1))^{k-1}}{(2k+3)^{k-1}}\cdots \\ & \times\frac{(2m-(2r-5))^2}{(2r-3)^2}\frac{(2m-(2r-3))}{(2r-1)}, \end{aligned} \tag{8.8}$$

when $r = 2k$ is even, and

$$\begin{aligned} D_m(r,0) = {} & \frac{m}{1}\frac{(m-1)^2}{2^2}\frac{(m-2)^3}{3^3}\cdots\frac{(m-k+1)^k}{k^k} \\ & \times\frac{(m-k)^{k+1}}{(k+1)^{k+1}}\frac{(m-k-1)^{k}}{(k+2)^{k}}\cdots\frac{(m-(r-2))^2}{(r-1)^2}\frac{(m-(r-1))}{r} \\ & \times\frac{(2m-1)}{3}\frac{(2m-3)^2}{5^2}\frac{(2m-5)^3}{7^3}\cdots\frac{(2m-(2k-3))^{k-1}}{(2k-1)^{k-1}} \\ & \times\frac{(2m-(2k-1))^k}{(2k+1)^k}\frac{(2m-(2k+1))^{k-1}}{(2k+3)^{k-1}}\cdots\frac{(2m-(2r-5))^2}{(2r-3)^2} \\ & \times\frac{(2m-(2r-3))}{(2r-1)}, \end{aligned} \tag{8.9}$$

when $r = 2k+1$ is odd. As with the previous case, the above two expressions can be represented by a more concise single expression. Namely

$$D_m(r,0) = \prod_{k=0}^{[\frac{r-1}{2}]} \frac{\binom{2m-2k}{2r-4k-1}}{\binom{2r-2k}{2r-4k-1}}. \tag{8.10}$$

The first few polynomials for $D_m(r,0)$ are given by

$$D_m(1,0) = m, \qquad D_m(2,0) = m(m-1)(2m-1)/6,$$

$$D_m(3,0) = m(m-1)^2(m-2)(2m-1)(2m-3)/180,$$

$$D_m(4,0) = m(m-1)^2(m-2)^2(m-3)(2m-1)(2m-3)^2(2m-5)/75600. \tag{8.11}$$

(3) *For natural number r, and $s \in \mathbb{Z}$, define $D_m(r,s)$ such that*

$$D_m(r,s) = \begin{vmatrix} b_1^{(1+2s)} & b_1^{(3+2s)} & b_1^{(5+2s)} & \dots & b_1^{(2r-1+2s)} \\ b_2^{(1+2s)} & b_2^{(3+2s)} & b_2^{(5+2s)} & \dots & b_2^{(2r-1+2s)} \\ b_3^{(1+2s)} & b_3^{(3+2s)} & b_3^{(5+2s)} & \dots & b_3^{(2r-1+2s)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{r-1}^{(1+2s)} & b_{r-1}^{(3+2s)} & b_{r-1}^{(5+2s)} & \dots & b_{r-1}^{(2r-1+2s)} \\ b_r^{(1+2s)} & b_r^{(3+2s)} & b_r^{(5+2s)} & \dots & b_r^{(2r-1+2s)} \end{vmatrix}. \tag{8.12}$$

Then for $r > |s|$ with s negative (ignoring determinant sign) we have

$$D_m(r,s) = D_{m+s}(r+s,0), \tag{8.13}$$

and for $r \geqslant s$ with s positive (ignoring determinant sign) we have

$$D_m(r,s) = \frac{1}{(2m+1)^s} D_{m+s}(r+s,0), \tag{8.14}$$

where $D_{m+s}(r+s,0)$ means that we consider the $(r+s)\times(r+s)$ determinant defined in (8.7)*, with m replaced with $m+s$.*

For values of r and s that do not satisfy the conditions stated in (8.13) and (8.14) the equalities do not appear to hold. We give the examples when $r=5$, $s=-2$ and $s=0$.

$$\begin{aligned} D_m(5,-2) &= \frac{(m-2)}{1}\frac{(m-3)^2}{2^2}\frac{(m-4)}{3}\frac{(2m-5)}{3}\frac{(2m-7)}{5}, \\ D_m(5,0) &= \frac{m}{1}\frac{(m-1)^2}{2^2}\frac{(m-2)^3}{3^3}\frac{(m-3)^2}{4^2}\frac{(m-4)}{5} \\ &\quad \times \frac{(2m-1)}{3}\frac{(2m-3)^2}{5^2}\frac{(2m-5)^2}{7^2}\frac{(2m-7)}{9} \end{aligned}$$

An interesting question concerns when $D(r,s) \in \mathbb{Z}$. Initial investigations suggest that this is true when $2m+1$ is a prime and $r \leqslant m-s$.

Similar results to the expression in (8.10) have been observed by Bacher in his study of determinants of matrices related to the Pascal triangle, the most relevant of which are detailed below. For proofs of these and other related results we refer the reader to [3].

Proposition 8.4. *Let $P_{s,t}(m) = (p_{i,j})$ be the $m \times m$ matrix with coefficients $p_{i,j} = \binom{i+j+s+t}{i+s}$, $0 \leqslant i,j < m$, and $Q_s(2m) = (q_{i,j})$ be the $2m \times 2m$ matrix with coefficients $q_{i,j} = \binom{2s+i+j-1}{s+j} - \binom{2s+i+j-1}{s+j-1}$, $0 \leqslant i,j < 2m$. Then for $s,t,m \geqslant 0$ we have*

$$\det(P_{s,t}(m)) = \prod_{k=0}^{s-1} \frac{\binom{m+k+t}{t}}{\binom{k+t}{t}},$$

and

$$\sqrt{\det(Q_s(2m))} = \prod_{k=1}^{s-1} \frac{\binom{2m+2k}{k}}{\binom{2k}{k}}. \tag{8.15}$$

The first polynomials

$$R_s(m) = \sqrt{\det(Q_s(2m))} = \prod_{k=1}^{s-1} \frac{\binom{2m+2k}{k}}{\binom{2k}{k}},$$

are identical to those generated by $D_m(r,0)$, where we seem to have

$$D_m(r,0) = R_{r+1}(m-r). \tag{8.16}$$

Bacher considers the sequences $\mathcal{S}(m) = ((R_s(m))_{s=0,1,2,\ldots}$ (for fixed m), and notes that they also appear in [15]. In fact $\mathcal{S}(1)$ is the sequence of Catalan numbers $(1,1,2,5,14,42,\ldots)$, and the sequence $\mathcal{S}(m)$ appears in the m-th row of the upper triangle of the Catalan Number Wall, the first few rows of which are given below.

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
1 & 1 & 2 & 5 & 14 & 42 & 132 & 429 & \ldots \\
 & 1 & 1 & 3 & 14 & 84 & 594 & 4719 & \ldots \\
 & & 1 & 1 & 4 & 30 & 330 & 4719 & \ldots \\
 & & & 1 & 1 & 5 & 55 & 1001 & \ldots \\
 & & & & 1 & 1 & 6 & 91 & \ldots \\
 & & & & & \ddots & \ddots & \ddots & \ddots
\end{array} \tag{8.17}$$

We now give a brief definition of Number Walls [7], [23] (or quotient-difference tables to give the more traditional name) as described by Michael Somos.

Definition. *Let* $\ldots, a(-2), a(-1), a(0), a(1), a(2), \ldots$ *be a sequence of numbers and define the* $r \times r$ *Hankel matrix* $H_{r,s} = (h_{i,j})$, $1 \leqslant i,j \leqslant r$, *with coefficients* $h_{i,j} = a(s-r+i+j-1)$. *Similarly define the* $r \times r$ *Toeplitz matrix* $T_{r,s} = (t_{i,j})$, $1 \leqslant i,j \leqslant r$, *with coefficients* $t_{i,j} = a(s+i-j)$, *so that the matrices* $H_{r,s}$ *and* $T_{r,s}$ *are reflections of each other.*

Denote the determinant of matrix A *by* $|A|$. *The determinants of the Hankel matrices,* $|H_{r,s}|$, *form a Hankel number wall* $\mathcal{H}$, *while the determinants of the Toeplitz matrices,* $|T_{r,s}|$, *form a Toeplitz number wall* $\mathcal{T}$. *As one consequence of Jacobi's identities on matrices, we have*

$$|H_{r,s}|^2 = |H_{r,s+1}| \times |H_{r,s-1}| - |H_{r+1,s}| \times |H_{r-1,s}|, \tag{8.18}$$

$$|T_{r,s}|^2 = |T_{r,s+1}| \times |T_{r,s-1}| + |T_{r+1,s}| \times |T_{r-1,s}| \tag{8.19}$$

among other identities.

The array in (8.17) forms half of a Hankel Number Wall and so satisfies the properties of (8.18).

With the definitions in place, let $n = 2m+1$, and we consider the array

$$\begin{array}{ccccccc}
\dots & D_m(1,-2) & D_m(1,-1) & D_m(1,0) & nD_m(1,1) & n^2D_m(1,2) & \dots \\
\dots & D_m(2,-2) & D_m(2,-1) & D_m(2,0) & nD_m(2,1) & n^2D_m(2,2) & \dots \\
\dots & D_m(3,-2) & D_m(3,-1) & D_m(3,0) & nD_m(3,1) & n^2D_m(3,2) & \dots \\
\dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\
\dots & D_m(m,-2) & D_m(m,-1) & D_m(m,0) & nD_m(m,1) & n^2D_m(m,2) & \dots
\end{array} \tag{8.20}$$

where we ignore the sign of $D_m(r,s)$ and taken all values to be positive.

For entries in the array whose values of r and s satisfy the conditions stated in (8.13) and (8.14), then taking account the relation (8.16), it is to be expected that the these entries correspond to a portion of the triangle in (8.17).

What is perhaps not quite so expected is that the array splits into two sides, where both the left and the right sides obey the Hankel Number Wall relation (8.18), except from where the sides meet in a diagonal of ones. This is most clearly illustrated with an example and we give the case for $m = 5$ in (8.20) below.

$$\begin{array}{cccccccccccccc}
\dots & 1716 & 462 & 126 & 35 & 10 & 3 & 1 & 5 & 55 & 1331 & 42592 & 1449459 & \dots \\
\dots & 28314 & 2772 & 294 & 35 & 5 & \mathbf{1} & 4 & 30 & 330 & 4719 & 86515 & 1955239 & \dots \\
\dots & 28314 & 1386 & 84 & 7 & \mathbf{1} & 3 & 14 & 84 & 594 & 4719 & 40898 & 380666 & \dots \\
\dots & 4719 & 165 & 9 & \mathbf{1} & 2 & 5 & 14 & 42 & 132 & 429 & 1430 & 4862 & \dots \\
\dots & 121 & 11 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots
\end{array} \tag{8.21}$$

When the numbers in (8.21) are replaced by those generated by the relation (8.18), then they remain unchanged, apart from the diagonal of ones (shown in bold) which are replaced by $\sqrt{11}$ in this case or $\sqrt{2m+1}$ in general. We note that this discrepancy in the Number Wall can be removed by multiplying the diagonals by the required powers of n.

Omitting the powers of n in the array (8.20), and so the powers of 11 in array (8.21), leaves the Number Wall properties unchanged, although the right-hand side of the array would then consist of rational numbers whose denominators are powers of $2m+1$. By Fleck's congruence, when n is a prime, the left-hand side of the array contains increasing powers of n, further highlighting the p-adic symmetry that exists within relations between the $b_q^{(-t)}$ and the $b_q^{(t)}$.

To conclude, Fleck's and Weisman's congruences are instances of a fundamental relationship between Fleck numbers, formed from sums of binomial coefficients, and prime numbers. It appears to be the case that the geometric polynomials in m, $b_q^{(t)}$, and the numbers that they generate, are in fact closely linked to this fundamental relationship and as such they may well be worthy of further study.

Table 1: Table of $b_q^{(t)}$ values when $m = 5$ and $-12 \leqslant t \leqslant 11$.

t	$\mathbf{b_0^{(t)}}$	$\mathbf{b_1^{(t)}}$	$\mathbf{b_2^{(t)}}$	$\mathbf{b_3^{(t)}}$	$\mathbf{b_4^{(t)}}$	$\mathbf{b_5^{(t)}}$	$\mathbf{b_6^{(t)}}$	$\mathbf{b_7^{(t)}}$	$\mathbf{b_8^{(t)}}$	$\mathbf{b_9^{(t)}}$	$\mathbf{b_{10}^{(t)}}$	$\mathbf{b_{11}^{(t)}}$
11		$-310\frac{5}{11}$	$-832\frac{4}{11}$	$-1089\frac{3}{11}$	$-1000\frac{2}{11}$	$-594\frac{1}{11}$	0	$594\frac{1}{11}$	$1000\frac{2}{11}$	$1089\frac{3}{11}$	$832\frac{4}{11}$	$310\frac{5}{11}$
10	$-620\frac{10}{11}$	$-521\frac{10}{11}$	$-256\frac{10}{11}$	$89\frac{1}{11}$	$406\frac{1}{11}$	$594\frac{1}{11}$	$594\frac{1}{11}$	$406\frac{1}{11}$	$89\frac{2}{11}$	$-256\frac{10}{11}$	$-521\frac{10}{11}$	
9		99	265	346	317	188	0	-188	-317	-346	-265	-99
8	198	166	81	-29	-129	-188	-188	-129	-29	81	166	
7		-32	-85	-110	-100	-59	0	59	100	110	85	32
6	-64	-53	-25	10	41	59	59	41	10	-25	-53	
5		11	28	35	31	18	0	-18	-31	-35	-28	-11
4	22	17	7	-4	-13	-18	-18	-13	-4	7	17	
3		-5	-10	-11	-9	-5	0	5	9	11	10	5
2	-10	-5	-1	2	4	5	5	4	2	-1	-5	
1		5	4	3	2	1	0	-1	-2	-3	-4	-5
0	10	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	
-1		-1	0	0	0	0	0	0	0	0	0	-1
-2	-2	1	0	0	0	0	0	0	0	0	1	
-3		3	-1	0	0	0	0	0	0	0	1	-3
-4	6	-4	1	0	0	0	0	0	0	1	-4	
-5		-10	5	-1	0	0	0	0	0	1	-5	10
-6	-20	15	-6	1	0	0	0	0	1	-6	15	
-7		35	-21	7	-1	0	0	0	1	-7	21	-35
-8	70	-56	28	-8	1	0	0	1	-8	28	-56	
-9		-126	84	-36	9	-1	0	1	-9	36	-84	126
-10	-252	210	-120	45	-10	1	1	-10	45	-120	210	
-11		462	-330	165	-55	11	0	-11	55	-165	330	-462
-12	924	-792	495	-220	66	-11	-11	66	-220	495	-792	

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