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# SOLVING EXPLICITLY $F(x,y) = G(x,y)$ OVER FUNCTION FIELDS

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**Abstract:** Consider binary forms $F(x,y)$, $G(x,y)$ with coefficients in $\mathbb{Q}[t]$, assume that $F$ is irreducible. We give effective upper bounds for the heights of the solutions and an efficient algorithm to solve

$$w \cdot F(x,y) = z \cdot G(x,y)$$

$$\text{in} \quad x,y \in \mathbb{Q}[t], \quad w,z \in \mathbb{Q}[t] \cap U_S, \quad \gcd(x,y)=1, \quad \gcd(w,z)=1,$$

where $U_S$ denotes a group of $S$-units in $\mathbb{Q}(t)$. We derive that there are only finitely many solutions up to constant factors. We also show that this is not true for global function fields. This is a generalization of the well known Thue equations. Effective upper bounds for the solutions of this general equation were given over number fields but it was not yet considered over function fields. We illustrate our method with a detailed numerical example.

**Keywords:** Thue equations; function fields.

## 1. Introduction

Let $F \in \mathbb{Z}[x,y]$ be an irreducible binary form of degree $\geqslant 3$ and $m$ a non-zero integer. Following the classical result of Thue [17] several authors considered equations of type

$$F(x,y) = m \qquad \text{in } \ x,y \in \mathbb{Z}$$

as well as its extensions.

A typical generalization of Thue's equation is the Diophantine equation of type

$$F(x,y) = G(x,y) \qquad \text{in } \ x,y \in \mathbb{Z}$$

where $G$ is also a polynomial or a binary form with coefficients in $\mathbb{Z}$. In case $G$ is a form with $\deg G < \deg F$ Shorey and Tijdeman [15] gave upper bounds for the solutions. For various generalizations see Evertse, Győry, Shorey and Tijdeman [4]. Efficient algorithms for calculating explicitly "small" solutions of this equation

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were given by I.Gaál over $\mathbb{Z}$ [6], over imaginary quadratic fields [7] and over number fields [8] under certain conditions.

Shorey and Tijdeman [15] also gave a far reaching generalization of this problem. They considered equations of type

$$w \cdot F(x,y) = z \cdot G(x,y) \qquad \text{in} \;\; x,y,w,z \in \mathbb{Z}, \;\gcd(x,y)=1, \;\gcd(w,z)=1$$

assuming that $G$ is also a form and the variables $w, z$ are only divisible by certain fixed primes (they are $S$ units).

The purpose of the present paper is to give a function field analogue of this result over $\mathbb{Q}(t)$. In Section 5 we derive an effective upper bound for the heights of the solutions. Moreover, in Section 6 we describe an efficient algorithm for solving the equation explicitly. A detailed example is given in Section 7.

Note that in the function field case most results on Diophantine equations are obtained over algebraically closed constant fields, cf. Schmidt [13], Mason [12]. Recently Gaál and Pohst [9], [10], [11] obtained results on Diophantine equations over global function fields, i.e. a finite extension of $\mathbb{F}(t)$, for finite fields $\mathbb{F}$.

## 2. The function field

Let $K$ be a finite extension of $\mathbb{Q}(t)$. The degree and the genus of the function field $K$ will be denoted by $d$ and $g$, respectively. The integral closure of $\mathbb{Q}[t]$ in $K$ is denoted by $O_K$. The set of all (exponential) *valuations* of $K$ (which are trivial on $\mathbb{Q}$) is denoted by $V$, the subset of infinite valuations by $V_\infty$. For a non-zero element $f \in K$ we denote by $v(f)$ the valuation of $f$ at $v$. For the *normalized valuations* $v_N(f) = v(f) \cdot \deg v$ the *product formula*

$$\sum_{v \in V} v_N(f) = 0, \qquad \forall f \in K \setminus \{0\}$$

holds. The *height* of a non-zero element $f$ of $K$ is defined as usual

$$H(f) \;:=\; \sum_{v \in V} \max\{0, v_N(f)\} = \;-\sum_{v \in V} \min\{0, v_N(f)\} \;\;.$$

In the following all valuations $v$ will mean normalized valuations without subscript $N$.

## 3. Unit equations in two variables

Let $S$ be a finite subset of $V$, containing the infinite valuations. Then the non-zero elements $\gamma \in K$ satisfying $v(\gamma) = 0$ for all $v \notin S$ form a multiplicative group $U_S$ in $K$. These elements are called *S-units*. (For $S = V_\infty$ the $S$-units are just the units of the ring $O_K$.) We consider the *unit equation*

$$x + y = 1 \qquad \text{in} \;\; x, y \in U_S. \tag{3.1}$$

**Lemma 3.1.** *For all solutions $x, y \in U_S$ of equation (3.1) we have*

$$\max(H(x), H(y)) \leqslant 2g - 2 + \sum_{v \in S} \deg v \ . \tag{3.2}$$

Since $x, y$ are $S$-units, this implies the finiteness of the number of solutions of equation (3.1). The proof of this Lemma is analogous to the proof of Lemma 3.1 of [9]. The necessary premises are fulfilled since $\mathbb{Q}$ is perfect, cf. Stichenoth [16], Artin and Whaples [1].

## 4. Formulating the equation

Let $F(x,y), G(x,y)$ be binary forms with coefficients in $\mathbb{Q}[t]$. We assume that $F$ is irreducible and that these forms split in $K$ into linear factors

$$F(x,y) = (x - \alpha_1 y) \dots (x - \alpha_n y)$$

$$G(x,y) = (x - \beta_1 y) \dots (x - \beta_m y)$$

where $\alpha_i$ $(1 \leqslant i \leqslant n)$ and $\beta_j$ $(1 \leqslant j \leqslant m)$ are elements of $O_K$. Since $F$ is irreducible, the conjugates $\alpha_1, \dots, \alpha_n$ are obviously distinct. We also assume that $\alpha_i \neq \beta_j$ (common factors can be eliminated).

Let $V_0$ be the set of all valuations of $\mathbb{Q}(t)$ which are trivial on $\mathbb{Q}$. Let $S_0$ be a finite subset of $V_0$ containing the infinite (degree) valuation and denote by $U_{S_0}$ the group of $S_0$ units of $\mathbb{Q}(t)$. Our purpose is to consider the solutions of

$$w \cdot F(x,y) = z \cdot G(x,y) \tag{4.1}$$

in $\quad x, y \in \mathbb{Q}[t], \quad w, z \in \mathbb{Q}[t] \cap U_{S_0}, \quad \gcd(x,y) = 1, \quad \gcd(w,z) = 1.$

Without loss of generality we can assume $xy \neq 0$. (The case $x = 0$ or $y = 0$ is trivial.) This means that in addition to the binary forms the factors $w, z$ appear on both sides, divisible only by polynomials belonging to the valuations in the finite set $S_0$. The conditions $\gcd(x,y) = 1, \gcd(w,z) = 1$ are clearly necessary to ensure the finiteness of the number of solutions up to rational factors.

## 5. Upper bounds for the heights of the solutions

In this section we give explicit upper bounds for the heights of the solutions $x, y$. This implies upper bounds for the heights of $w, z$.

Let $A$ be an upper bound for the heights (in $K$) of $\alpha_i$ $(1 \leqslant i \leqslant n)$. Let $S_1$ be the set of extensions of valuations in $S_0$ to $K$. Denote by $W$ the finite set of valuations of $K$ containing $S_1$ and those finite valuations $v$ for which any of $v(\alpha_i - \beta_j)$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$ or $v(\alpha_i - \alpha_j)$ $(1 \leqslant i < j \leqslant n)$ is positive. Denote by $h(.)$ the height in $\mathbb{Q}[t]$.

**Theorem 5.1.** *Equation (4.1) has only finitely many solutions up to common constant factors. For all solutions $x, y, w, z$ of equation (4.1) we have*

$$\max(h(x), h(y)) \leqslant \frac{1}{d}\left(14g - 14 + 7\sum_{v\in W}\deg v + 36A\right).$$

**Remark 5.2.** The assertion is not valid if the constant field has prime characteristic. In that case in [10] we showed an example for a Thue equation (that is $G(x, y) = 1$) with infinitely many solutions.

**Remark 5.3.** Some ideas of Section 6 would enable to make the bound in (5.1) somewhat smaller but would also make the formulation of the Theorem more complicated.

**Proof of Theorem 5.1.** Assume that $x, y, w, z$ is a solution of equation (4.1). We apply some arguments used in [15] in the number field case.

We have

$$w\cdot(x-\alpha_1 y)\dots(x-\alpha_n y) = z\cdot(x-\beta_1 y)\dots(x-\beta_m y). \tag{5.1}$$

Assume that $v$ is a finite valuation of $K$ such that $v(x-\alpha_i y) > 0$ for some $i$. Then either $v \in S_1$ or there must be a $j$ such that $v(x-\beta_j y) > 0$. In the later case

$$\begin{aligned} v(y) + v(\alpha_i - \beta_j) &= v((\alpha_i-\beta_j)y) = v((x-\beta_j y) - (x-\alpha_i y)) \\ &\geqslant \min(v(x-\alpha_i y), v(x-\beta_j y)) > 0. \end{aligned}$$

We also have

$$\begin{aligned} v(x) + v(\alpha_i - \beta_j) &= v((\alpha_i-\beta_j)x) = v(\alpha_i(x-\beta_j y) - \beta_j(x-\alpha_i y)) \\ &\geqslant \min(v(\alpha_i(x-\beta_j y), v(\beta_j(x-\alpha_i y)) \\ &= \min(v(\alpha_i) + v(x-\beta_j y), v(\beta_j) + v(x-\alpha_i y)) \\ &\geqslant \min(v(x-\beta_j y), v(x-\alpha_i y)) > 0, \end{aligned}$$

where we used $v(\alpha_i) \geqslant 0, v(\beta_j) \geqslant 0$ which follows from $\alpha_i, \beta_j \in O_K$. Because of $\gcd(x, y) = 1$ the values of $x, y$ at $v$ can not both be positive, hence the two inequalities above together imply $v(\alpha_i - \beta_j) > 0$.

Let

$$S_2 = S_1 \cup \{v \in V \mid v(\alpha_i - \beta_j) > 0 \text{ for some } i, j\},$$

then

$$\mu = (x-\alpha_1 y)\dots(x-\alpha_n y) \tag{5.2}$$

is an $S_2$-unit.

Next we use unit equations in two variables for solving equation (5.2), obviously a Thue equation.

Set $\gamma_i = x - \alpha_i y$ for $1 \leqslant i \leqslant n$. For distinct $i,j$ $(1 \leqslant i,j \leqslant n-1)$ Siegel's identity implies

$$(\alpha_i - \alpha_j)\gamma_n + (\alpha_j - \alpha_n)\gamma_i + (\alpha_n - \alpha_i)\gamma_j = 0,$$

whence

$$\frac{(\alpha_j - \alpha_n)\gamma_i}{(\alpha_j - \alpha_i)\gamma_n} + \frac{(\alpha_n - \alpha_i)\gamma_j}{(\alpha_j - \alpha_i)\gamma_n} = 1. \tag{5.3}$$

Setting

$$W = S_2 \cup \{v \in V \mid v(\alpha_i - \alpha_j) > 0 \text{ for some } i,j\}$$

both terms on the left hand side of equation (5.3) are $W$-units. Lemma 3.1 implies that

$$(\alpha_j - \alpha_n)\gamma_i = \nu_i(\alpha_j - \alpha_i)\gamma_n$$

where $\nu_i$ is a $W$-unit of height

$$H(\nu_i) \leqslant 2g - 2 + \sum_{v \in W} \deg v =: C. \tag{5.4}$$

Therefore we obtain

$$\gamma_i = \mu_i \cdot \gamma_n \quad (1 \leqslant i \leqslant n-1) \tag{5.5}$$

with a $W$-unit

$$\mu_i = \nu_i \cdot \frac{\alpha_j - \alpha_i}{\alpha_j - \alpha_n} \tag{5.6}$$

with

$$H(\mu_i) \leqslant C + 4A. \tag{5.7}$$

(We recall that $H(\alpha_i) \leqslant A$.) Let again $v$ be a finite valuation and consider the values of $\gamma_n = x - \alpha_n y$. We have $v(\gamma_n) \geqslant 0$ by $\gamma_n \in O_K$ and positive values occur at most for $v \in S_2$ (since $\gamma_n$ is an $S_2$-unit). Assume that $k_v = v(\gamma_n) > 0$. For an $i < n$ we have, firstly,

$$\begin{aligned} v(y) + v(\alpha_i - \alpha_n) &= v((\alpha_i - \alpha_n)y) = v((x - \alpha_n y) - (x - \alpha_i y)) \\ &\geqslant \min(v(x - \alpha_i y), v(x - \alpha_n y)) \\ &= \min(v(\mu_i) + k_v, k_v), \end{aligned}$$

and secondly

$$\begin{aligned} v(x) + v(\alpha_i - \alpha_n) &= v((\alpha_i - \alpha_n)x) = v(\alpha_i(x - \alpha_n y) - \alpha_n(x - \alpha_i y)) \\ &\geqslant \min(v(\alpha_i(x - \alpha_n y), v(\alpha_n(x - \alpha_i y)) \\ &= \min(v(\alpha_i) + v(x - \alpha_n y), v(\alpha_n) + v(x - \alpha_i y)) \\ &\geqslant \min(v(x - \alpha_n y), v(x - \alpha_i y)) = \min(k_v, v(\mu_i) + k_v). \end{aligned}$$

It follows again from $\gcd(x,y) = 1$ that $v(x), v(y)$ can not both be positive, hence

$$\min(k_v, v(\mu_i) + k_v) \leqslant v(\alpha_i - \alpha_n) \tag{5.8}$$

that is

$$k_v \leqslant v(\alpha_i - \alpha_n) + \max(0, -v(\mu_i)) = v(\alpha_i - \alpha_n) + \max(0, v(1/\mu_i)). \tag{5.9}$$

By

$$\gamma_1 \dots \gamma_{n-1}\gamma_n = \mu \tag{5.10}$$

we have from (5.5) and the above estimate

$$\begin{aligned} v(\mu) &= v(\gamma_1) + \ldots + v(\gamma_{n-1}) + v(\gamma_n) \\ &= v(\mu_1) + \ldots + v(\mu_{n-1}) + n \cdot v(\gamma_n) \\ &\leqslant v(\mu_1) + \ldots + v(\mu_{n-1}) + n \cdot v(\alpha_i - \alpha_n) + n \cdot \max(0, v(1/\mu_i)). \end{aligned} \tag{5.11}$$

Observe that $\mu \in \mathbb{Q}[t]$, that is the values of $\mu$ are non-negative at all finite valuations. Moreover all infinite valuations of $K$ are extensions of the degree valuation of $\mathbb{Q}[t]$ with equal values at $\mu$, therefore $\mu$ has non-positive values at all infinite valuations of $K$. Hence by (5.7) and (5.11) we conclude

$$H(\mu) \leqslant (n-1)(C+4A) + n \cdot 2A + n(C+4A) \leqslant (2n-1)C + (10n-4)A. \tag{5.12}$$

Further, (5.10) and (5.5) imply

$$\gamma_n^n = \frac{\mu}{\mu_1 \dots \mu_{n-1}},$$

hence by (5.12) and (5.7) we get

$$H(\gamma_n) \leqslant \frac{3n-2}{n} C + \frac{14n-8}{n} A \leqslant 3C + 14A. \tag{5.13}$$

From this we infer

$$H(\gamma_1) \leqslant H(\mu_1) + H(\gamma_n) \leqslant 4C + 18A.$$

Finally, by

$$x = \frac{\alpha_1(x - \alpha_n y) - \alpha_n(x - \alpha_1 y)}{\alpha_1 - \alpha_n}, \qquad y = \frac{(x - \alpha_n y) - (x - \alpha_1 y)}{\alpha_1 - \alpha_n}$$

we get

$$\max(H(x), H(y)) \leqslant A + H(\gamma_n) + A + H(\gamma_1) + 2A \leqslant 7C + 36A.$$

From this the assertion of Theorem 5.1 for the heights of the solutions is immediate. Note that there are only finitely many $\gamma_n, \gamma_i$ up to common constant factors, therefore there are also only finitely many $x, y$ up to common constant factors. ∎

## 6. An efficient algorithm for solving the equation explicitly

The simplest way to attack our equation is to enumerate all $S_2$ units $\gamma_n$ satisfying (5.13). These elements can be calculated up to a rational factor. Then use the automorphism $\sigma$ for which $\sigma(\alpha_n) = \alpha_1$ to calculate $\gamma_1 = \sigma(\gamma_n)$ and then solve the system of equations

$$\gamma_1 = x - \alpha_1 y$$
$$\gamma_n = x - \alpha_n y$$

to obtain $x, y$ up to a rational factor.

However, as we shall explain in the following this procedure can be made much more efficient by having a closer look at the equation.

The element $\mu$ in (5.2) is an $S_2$-unit. Some of the valuations of $S_2$ can very often be eliminated. Consider a finite valuation $v \in S_2$ which is the only extension to $K$ of a valuation of $\mathbb{Q}(t)$. Then $v(x - \alpha_i y)$ is the same for all conjugates. By $\gcd(x, y) = 1$ the arguments leading to (5.8) show that

$$\min(v(x - \alpha_n y), v(x - \alpha_i y)) \leqslant v(\alpha_i - \alpha_n).$$

If $v(\alpha_i - \alpha_n) < 1$ for some $i$, then this implies that the values of $x - \alpha_i y$ are all zero, we have $v(\mu) = 0$ in (5.2). Otherwise, if $v(\alpha_i - \alpha_n) \geqslant 1$ for all $i$, the value $v(\mu)$ can be restricted by $n \cdot \min_{1 \leqslant i \leqslant n-1} v(\alpha_i - \alpha_n)$, (see (5.2)).

Denote by $S_2^*$ the result of reducing the set $S_2$ as described above. Then extend $S_2^*$ with the valuations occuring in any of the $\alpha_i - \alpha_j, \alpha_i - \alpha_n, \alpha_j - \alpha_n$, denote the resulting set by $W^*$.

Then determine all solutions $\nu_i$ of the $W^*$-unit equation (5.3). This is done by enumerating all $W^*$-units $\nu_i$ of bounded height (smaller or equal than the bound in (5.4)). The possible elements $\nu_i$ are determined up to constant factors. By (5.6) we obtain the $\mu_i$ of (5.5).

Denote by $S_0^*$ the set of valuations of $\mathbb{Q}(t)$ such that $S_2^*$ is just the set of extensions of the valuations of $S_0^*$ to $K$. (As we have possibly reduced the set $S_2$, we get here a set with possibly fewer valuations than in $S_0$.) We obtain

$$\gamma_n^n = \frac{\mu}{\mu_1 \ldots \mu_{n-1}}$$

where $\mu_1, \ldots, \mu_{n-1}$ are known and $\mu \in \mathbb{Q}[t]$ is an $S_0^*$-unit. Assume that the finite valuations of $S_0^*$ correspond to irreducible polynomials $P_1, \ldots, P_s \in \mathbb{Q}[t]$. Up to a rational factor we have

$$\mu = \prod_{i=1}^{s} P_i(t)^{k_i}$$

($k_i \in \mathbb{Z}^{\geqslant 0}$). By division with remainder we get $k_i = q_i n + r_i$ subject to $0 \leqslant r_i < n$ $(1 \leqslant i \leqslant s)$. Setting

$$P(t) := \prod_{i=1}^{s} P_i(t)^{q_i}$$

we get

$$\mu = P(t)^n \prod_{i=1}^{s} P_i(t)^{r_i}$$

and therefore

$$\gamma_n^n = \frac{P_1(t)^{r_1} \dots P_s(t)^{r_s}}{\mu_1 \dots \mu_{n-1}} \cdot P(t)^n.$$

Hence

$$\gamma_n = \delta_n \cdot P(t)$$

with an $S_2^*$-unit $\delta_n$. Taking a conjugate of $\gamma_n$ to obtain $\gamma_1$ we can see, that the factor $P(t) \in \mathbb{Q}[t]$ occurs in $\gamma_1$, as well, and this is the case also with

$$x = \frac{\alpha_1\gamma_n - \alpha_n\gamma_1}{\alpha_1 - \alpha_n}, \qquad y = \frac{\gamma_n - \gamma_1}{\alpha_1 - \alpha_n}.$$

Therefore to obtain coprime solutions $x, y$ we only have to calculate $\gamma_n$ from

$$\gamma_n^n = \frac{P_1(t)^{r_1} \dots P_s(t)^{r_s}}{\mu_1 \dots \mu_{n-1}}$$

for all $0 \leqslant r_1, \dots, r_s < n$. Then we calculate the corresponding $\gamma_1$ by taking conjugates and $x, y$ from the above formulas. We have to test if these possible values $x, y$ are indeed solutions.

## 7. Example

Let the set $S_0$ consist of the infinite valuation (deg) and the valuations corresponding to the irreducible polynomials $t, t+1, t+2$. Consider the Diophantine equation

$$w \cdot \left(x^3 - tx^2y - (t+3)xy^2 - y^3\right) = z \cdot (x+y) \tag{7.1}$$

$$\text{in} \qquad x, y \in \mathbb{Q}[t], \quad w, z \in \mathbb{Q}[t] \cap U_{S_0}, \quad \gcd(x,y) = 1, \quad \gcd(w,z) = 1.$$

Assume that $x, y$ are both nonzero. To solve this equation we consider the function field $K = \mathbb{Q}(t)(\alpha_1)$ generated by a root $\alpha_1$ of the polynomial

$$f(x) = x^3 - tx^2 - (t+3)x - 1.$$

(We call $K$ simplest cubic field in correspondence to D.Shanks [14] in the number field case.) The function field $K = \mathbb{Q}(t)(\alpha_1)$ is cyclic, its automorphism group is generated by

$$\sigma(\alpha_1) = \frac{-1}{\alpha_1 + 1}$$

and the other roots

$$\alpha_2 = \frac{-1}{\alpha_1 + 1}, \qquad \alpha_3 = \frac{-1}{\alpha_2 + 1}$$

of $f$ are also contained in $K$. The degree of $K$ is $d = 3$ and the genus of $K$ is $g = 0$. There are three infinite valuations $v_{\infty,1}, v_{\infty,2}, v_{\infty,3}$ of degree 1 and there is only one valuation $v_t, v_{t+1}, v_{t+2}$ of $K$ of degree 3, corresponding to $t, t+1, t+2$, respectively. Hence the set of extensions of valuations of $S_0$ to $K$ is

$$S_1 = \{v_{\infty,1}, v_{\infty,2}, v_{\infty,3}, v_t, v_{t+1}, v_{t+2}\}.$$

We have $n = 3,\ m = 1$ and $\beta_1 = -1$. In $\alpha_i - \beta_1 = \alpha_i + 1$ only the infinite valuations occur, hence the set $S_2$ coincides with $S_1$. Further, in $\alpha_i - \alpha_j$ apart from the infinite valuations also the valuation $v_{t^2+3t+9}$ corresponding to the polynomial $t^2 + 3t + 9$ occurs (there is only one valuation on $K$ extending the valuation corresponding to $t^2 + 3t + 9$ on $\mathbb{Q}(t)$). Therefore

$$W = \{v_{\infty,1}, v_{\infty,2}, v_{\infty,3}, v_t, v_{t^2+3t+9}\}.$$

By $v_t(\alpha_i - \alpha_j) = 0,\ v_{t+1}(\alpha_i - \alpha_j) = 0,\ v_{t+2}(\alpha_i - \alpha_j) = 0$ we can eliminate $v_t, v_{t+1}, v_{t+2}$ from $S_2$. By $v_{t^2+3t+9}(\alpha_i - \alpha_j) = 1$ the exponent of $t^2 + 3t + 9$ in $\mu$ is 0 or 1.

We have

$$S_2^* = \{v_{\infty,1}, v_{\infty,2}, v_{\infty,3}\}$$

and

$$W^* = \{v_{\infty,1}, v_{\infty,2}, v_{\infty,3}, v_{t^2+3t+9}\}.$$

We solve the $W^*$-unit equation

$$\nu_1 + \nu_2 = 1.$$

Lemma 3.1 implies that $H(\nu_i) \leqslant 3\ \ (i = 1, 2)$. There are 73 such $W^*$-units (up to constant factors). For all possible $\nu_1, \nu_2$ we calculate the corresponding $\mu_1, \mu_2$ and for each of them we calculate the possible values of $\gamma_3$ from

$$\gamma_3^3 = \frac{(t^2 + 3t + 9)^{r_1} \cdot t^{r_2} \cdot (t+1)^{r_3} \cdot (t+2)^{r_4}}{\mu_1 \mu_2}$$

for all $0 \leqslant r_1 \leqslant 1,\ 0 \leqslant r_2, r_3, r_4 < 3$. Carrying out the above calculations we find that there are only the trivial solutions with $x = 0$ or $y = 0$.

## 8. Computational aspects

All calculations to solve the equation in the example were carried out by Kash [3] which took just a few seconds on a PC.

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