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SOME PERMANENCE RESULTS OF THE DUNFORD–PETTIS AND GROTHENDIECK PROPERTIES IN lcHs

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Dedicated to the memory of Susanne Dierolf

Abstract: We prove some permanence results with respect to quotient spaces and to projective and injective tensor products of the Dunford–Pettis and Grothendieck properties in the setting of locally convex Hausdorff spaces.

Keywords: Dunford–Petty property, Grothendieck property, Schur property, quotient space, tensor product.

1. Introduction

Grothendieck spaces with the Dunford–Pettis property (briefly, GDP) play a prominent role in the theory of Banach spaces and vector measures, see [15, Ch. VI]. In the last years, these properties have been also investigated in the more general setting of locally convex Hausdorff spaces, mainly in connection with the study of mean ergodic operators in Fréchet spaces (see [1, 2, 3, 11]). In the papers [1, 11] many properties and examples of non normable GDP spaces have been pointed out. The aim of the present note is to study further permanence results of the Dunford–Pettis and Grothendieck properties in the setting of locally convex Hausdorff spaces and then to combine them in order to obtain the proper analogous results for GDP-spaces.

After recalling some definitions, the first part of the paper is mainly devoted to establish permanence results with respect to quotient spaces of the Dunford–Pettis and Grothendieck properties. As a consequence, it is shown, e.g., that the quotient space $\frac{\lambda_\infty(A)}{\lambda_0(A)}$ is a GDP space whenever the Köthe matrix A is regularly decreasing. Then, both properties are investigated in the setting of Köthe (LF)-sequence spaces. Finally, the last section is devoted to prove some permanence results under taking projective and injective tensor products of Fréchet spaces.

Notations. Let E be a locally convex Hausdorff space (briefly, lcHs) and Γ_E a system of continuous seminorms determining the topology of E. Denote by $\mathcal{B}(E)$

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the collection of all bounded subsets of E. If Γ_E is countable and E is complete, then E is called a Fréchet space. The identity operator on a lcHs E is denoted by I.

By E_σ we denote E equipped with its weak topology $\sigma(E, E')$, where E' is the topological dual space of E. The strong topology in E (resp. E') is denoted by $\beta(E, E')$ (resp. $\beta(E', E)$) and we write E_β (resp. E'_β). The strong dual space $(E'_\beta)'_\beta$ of E'_β is denoted simply by E''. By E'_σ we denote E' equipped with its weak-star topology $\sigma(E', E)$.

Let F be another lcHs. Denote by $\mathcal{L}(E, F)$ the space of all continuous linear operators from E in F. The strong operator topology τ_s in the space $\mathcal{L}(E, F)$ is determined by the family of seminorms

$$q_x(S) := q(Sx), \qquad S \in \mathcal{L}(E, F),$$

for each $x \in E$ and $q \in \Gamma_F$ (in which case we write $\mathcal{L}_s(E, F)$). The topology τ_b of the uniform convergence on bounded sets is defined in $\mathcal{L}(E, F)$ by the seminorms

$$q_B(S) := \sup_{x \in B} q(Sx), \qquad S \in \mathcal{L}(E, F),$$

for each $B \in \mathcal{B}(E)$ and $q \in \Gamma_F$ (in which case we write $\mathcal{L}_b(E, F)$).

Finally, we refer to [28] for definitions and notions about projective and injective tensor products of locally convex spaces.

2. Dunford–Pettis and Grothendieck properties

A lcHs E is said to have the *Dunford–Pettis property* (briefly, DP) if every element of $T \in \mathcal{L}(E, F)$, for F any quasicomplete lcHs, which transforms elements of $\mathcal{B}(E)$ into relatively $\sigma(F, F')$-compact subsets of F, also transforms $\sigma(E, E')$-compact subsets of E into relatively compact subsets of F, [25, p.633–634]. Actually, it suffices that F runs through the class of Banach spaces, [11, p.79]. A reflexive lcHs satisfies the DP-property if and only if it is Montel, [25, p.634]. A lcHs E is called a *Grothendieck space* if every sequence in E' which is convergent in E'_σ is also convergent for $\sigma(E', E'')$. Clearly, every reflexive lcHs is a Grothendieck space. Grothendick spaces with the Dunford Pettis property are called briefly GDP spaces. Every Montel lcHs is a GDP-space, [11, Remark 2.2], [1, Corollary 3.8]. Other examples of non-normable GDP spaces are given in [1], [11].

Equivalent and useful reformulations of the DP-property for some classes of lcHs's are the following.

Proposition 2.1. *Let E be any Fréchet space or any complete (LF)-space or any complete (DF)-space. Then the following properties are equivalent.*

(i) *E has the DP-property.*
(ii) *For every $\sigma(E, E')$-null sequence $\{x_k\}_{k=1}^\infty \subseteq E$ and every $\sigma(E', E'')$-null sequence $\{x'_k\}_{k=1}^\infty \subseteq E'$ we have $\lim_{k\to\infty}\langle x_k, x'_k\rangle = 0$.*
(iii) *For every $\sigma(E, E')$-Cauchy sequence $\{x_k\}_{k=1}^\infty \subseteq E$ and every $\sigma(E', E'')$-null sequence $\{x'_k\}_{k=1}^\infty \subseteq E'$ we have $\lim_{k\to\infty}\langle x_k, x'_k\rangle = 0$.*

Proof. (i)$\Leftrightarrow$(ii). See [1, Corollary 3.4].

(ii)$\Rightarrow$(iii). Let $\{x_k\}_{k=1}^{\infty} \subseteq E$ be a $\sigma(E, E')$-Cauchy sequence and $\{x'_k\}_{k=1}^{\infty} \subseteq E'$ be a $\sigma(E', E'')$-null sequence. Suppose that $\langle x_k, x'_k\rangle \not\to 0$ as $k \to \infty$. Then, there exists $\varepsilon_0 > 0$ such that

$$|\langle x_k, x'_k\rangle| \geqslant \varepsilon_0 \tag{2.1}$$

for all $k \in \mathbb{N}$ (eventually, by passing to a subsequence). On the other hand, for each $k \in \mathbb{N}$ there exists $n_k (> n_{k-1})$ such that

$$|\langle x_k, x'_{n_k}\rangle| < \varepsilon_0/2 \tag{2.2}$$

as $\{x'_k\}_{k=1}^{\infty} \subseteq E'$ is a $\sigma(E', E'')$-null sequence and hence, $\langle x_k, x'_n\rangle \to 0$ as $n \to \infty$.

Now, we observe that

$$\langle x_{n_k}, x'_{n_k}\rangle = \langle x_k, x'_{n_k}\rangle + \langle x_{n_k} - x_k, x'_{n_k}\rangle, \qquad k \in \mathbb{N},$$

where $\{x_{n_k} - x_k\}_{k=1}^{\infty}$ is a $\sigma(E, E')$-null sequence. So, from (ii) it follows that $\langle x_{n_k} - x_k, x'_{n_k}\rangle \to 0$ as $k \to \infty$ and hence,

$$|\langle x_{n_k} - x_k, x'_{n_k}\rangle| < \frac{\varepsilon_0}{4}, \qquad k \geqslant k_0, \tag{2.3}$$

for some $k_0 \in \mathbb{N}$. Combining (2.1), (2.2) and (2.3) we obtain

$$\varepsilon_0 \leqslant |\langle x_{n_k}, x'_{n_k}\rangle| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{4} = \frac{3\varepsilon_0}{4}, \qquad k \geqslant k_0.$$

This is a contradiction.

(iii)$\Rightarrow$(ii). It is obvious. ∎

A sequence $\{x_n\}_{n=1}^{\infty}$ in a lcHs E is called a *basis* if, for every $x \in E$, there is a unique sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ such that the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges to $x \in E$. By setting $f_n(x) := \alpha_n$ we obtain a linear form $f_n \colon E \to \mathbb{C}$. If $\{f_n\}_{n=1}^{\infty} \subseteq E'$, then $\{x_n\}_{n=1}^{\infty}$ is called a *Schauder basis* for E. A sequence $\{x_n\}_{n=1}^{\infty}$ in a lcHs E is called a *basic sequence* if it is a Schauder basis for the closed linear hull $[\{x_n\}_{n=1}^{\infty}] := \overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ in E. Moreover, a basic sequence $\{x_n\}_{n=1}^{\infty}$ in a lcHs E is called *equivalent* to the canonical unit basis vectors of ℓ^1 if there exists an isomorphism $J \colon \ell^1 \to [\{x_n\}_{n=1}^{\infty}]$ with $J(e_n) = x_n$ for all $n \in \mathbb{N}$, i.e., if there exist $q_0 \in \Gamma_E$ and $M_0 > 0$ such that

$$\|x\|_1 \leqslant M_0 q_0(Jx), \qquad x \in \ell^1, \tag{2.4}$$

and for each $q \in \Gamma_E$ there exists $M_q > 0$ satisfying

$$q(Jx) \leqslant M_q \|x\|_1, \qquad x \in \ell^1. \tag{2.5}$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in E and $\inf_{n\in\mathbb{N}} q_0(x_n) > 0$.

Lemma 2.2. *Let E be a Fréchet space and $F \subseteq E$ be a closed subspace. Let $\frac{E}{F}$ be the quotient space endowed with the lc-topology defined via the quotient map $Q\colon E \to \frac{E}{F}$. If $\{x_n\}_{n=1}^{\infty} \subseteq E$ is a basic sequence in E which is equivalent to the canonical unit basis vectors of ℓ^1 such that $\lim_{n\to\infty} Qx_n = 0$ in $\frac{E}{F}$, then F contains an isomorphic copy of ℓ^1.*

Proof. Let $\Gamma_E = (q_k)_{k\geqslant 0}$ be increasing. Since $\{x_n\}_{n=1}^{\infty}$ is equivalent to the canonical unit basis vectors of ℓ^1, by (2.4) and (2.5) we may suppose (eventually by relabelling), for every $x = \sum_{n=1}^{\infty} \alpha_n x_n \in [\{x_n\}_{n=1}^{\infty}]$, that

$$\frac{1}{M_0} \sum_{n=1}^{\infty} |\alpha_n| \leqslant q_0 \left(\sum_{n=1}^{\infty} \alpha_n x_n \right) \leqslant M_0' \sum_{n=1}^{\infty} |\alpha_n|, \tag{2.6}$$

where $M_0' := M_{q_0}$, and that

$$q_k \left(\sum_{n=1}^{\infty} \alpha_n x_n \right) \leqslant M_k \sum_{n=1}^{\infty} |\alpha_n|, \qquad k \geqslant 1, \tag{2.7}$$

where $M_k := M_{q_k}$. On the other hand, the lc-topology of $\frac{E}{F}$ is generated by the increasing sequence of seminorms $(\hat{q}_k)_{k\geqslant 0}$ given by

$$\hat{q}_k(Qx) := \inf\{q_k(x-y) : y \in F\}, \qquad k \geqslant 0,\ x \in E,$$

and hence, $\hat{q}_k(Qx_n) \to 0$ as $n \to \infty$ for all $k \geqslant 0$.

Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive real numbers with $\varepsilon := \sum_{k=1}^{\infty} \varepsilon_k < \frac{1}{2M_0}$. Then we can find an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that

$$\hat{q}_k(Qx_{n_k}) < \varepsilon_k, \qquad k \geqslant 0,$$

and hence, a sequence $\{y_k\}_{k=0}^{\infty} \subseteq F$ satisfying

$$q_k(x_{n_k} - y_k) < \varepsilon_k, \qquad k \geqslant 0. \tag{2.8}$$

These inequalities imply that $\{y_k\}_{k=0}^{\infty}$ is a basic sequence in F which is equivalent to the canonical unit basis vectors of ℓ^1 and hence, F contains an isomorphic copy of ℓ^1. Indeed, from (2.6) and (2.8) it follows, for every $h \in \mathbb{N}$ and every choice of scalars $\alpha_0, \alpha_1 \ldots, \alpha_h$, that

$$\begin{aligned} q_0 \left(\sum_{k=0}^{h} \alpha_k y_k \right) &\leqslant q_0 \left(\sum_{k=0}^{h} \alpha_k (y_k - x_{n_k}) \right) + q_0 \left(\sum_{k=0}^{h} \alpha_k x_{n_k} \right) \\ &\leqslant \sum_{k=0}^{h} |\alpha_k| \varepsilon_k + M_0' \sum_{k=0}^{h} |\alpha_k| \leqslant (\varepsilon + M_0') \sum_{k=0}^{h} |\alpha_k| \end{aligned}$$

and that

$$q_0\left(\sum_{k=0}^{h}\alpha_k y_k\right) \geqslant q_0\left(\sum_{k=0}^{h}\alpha_k x_{n_k}\right) - q_0\left(\sum_{k=0}^{h}\alpha_k (y_k - x_{n_k})\right)$$
$$\geqslant \frac{1}{M_0}\sum_{k=0}^{h}|\alpha_k| - \varepsilon\sum_{k=0}^{h}|\alpha_k| \geqslant \frac{1}{2M_0}\sum_{k=0}^{h}|\alpha_k|.$$

Therefore, we obtain, for every $h \in \mathbb{N}$ and every choice of scalars $\alpha_0, \ldots, \alpha_h$, that

$$\frac{1}{2M_0}\sum_{k=0}^{h}|\alpha_k| \leqslant q_0\left(\sum_{k=0}^{h}\alpha_k y_k\right) \leqslant (\varepsilon + M_0')\sum_{k=0}^{h}|\alpha_k|. \tag{2.9}$$

Moreover, from (2.7) and (2.8) it follows, for every $l \geqslant 1$, $h \geqslant l$ and every choice of scalars $\alpha_0, \alpha_1 \ldots, \alpha_h$, that

$$\begin{aligned} q_l\left(\sum_{k=0}^{h}\alpha_k y_k\right) &\leqslant q_l\left(\sum_{k=l}^{h}\alpha_k y_k\right) + q_l\left(\sum_{k=0}^{l-1}\alpha_k y_k\right) \\ &\leqslant q_l\left(\sum_{k=l}^{h}\alpha_k (y_k - x_{n_k})\right) + q_l\left(\sum_{k=l}^{h}\alpha_k x_{n_k}\right) + \sum_{k=0}^{l-1}|\alpha_k| q_l(y_k) \\ &\leqslant \sum_{k=l}^{h}|\alpha_k|\varepsilon_k + M_l\sum_{k=l}^{h}|\alpha_k| + N_l\sum_{k=0}^{l-1}|\alpha_k| \qquad (2.10) \\ &\leqslant (\varepsilon + M_l + N_l)\sum_{k=0}^{h}|\alpha_k| \end{aligned}$$

with $N_l := \max_{0\leqslant k\leqslant l-1} q_l(y_k) < \infty$. By the arbitrariness of h and of $\alpha_0, \alpha_1 \ldots, \alpha_h$, (2.9) and (2.10) imply that $\{y_k\}_{k=0}^{\infty}$ is a basic sequence in F which is equivalent to the canonical unit basis vectors of ℓ^1. ■

Let E be a lcHs and let $Q\colon E \to \frac{E}{F}$ be a quotient map with F closed subspace of E. We say that Q *lifts bounded sets* (*lifts bounded sets with closure*, resp.) if for every bounded set B in $\frac{E}{F}$ there exists a bounded set A in E such that $Q(A) \supseteq B$ ($\overline{Q(A)} \supseteq B$, resp.). If E is a metrizable lcHs, we know by [9, Theorem] that Q lifts bounded sets with closure if and only if Q lifts bounded sets. Moreover, in case E is a normable lcHs, it is obvious that every quotient map lifts bounded sets.

For Banach spaces, the following useful result is due to Lohman [30].

Lemma 2.3. *Let E be a Fréchet space and $F \subseteq E$ be a closed subspace. Let $\frac{E}{F}$ be the quotient space endowed with the lc-topology defined via the quotient map $Q\colon E \to \frac{E}{F}$. If F does not contain any isomorphic copy of ℓ^1 and the quotient map Q lifts bounded sets, then every $\sigma\left(\frac{E}{F}, \left(\frac{E}{F}\right)'\right)$-Cauchy sequence in $\frac{E}{F}$ admits a subsequence which is the image of a $\sigma(E, E')$-Cauchy sequence in E under the map Q.*

Proof. Suppose that $\{\hat{x}_n\}_{n=1}^\infty \subseteq \frac{E}{F}$ is a $\sigma\left(\frac{E}{F}, \left(\frac{E}{F}\right)'\right)$-Cauchy sequence which does not admits any subsequence coming from a $\sigma(E, E')$-Cauchy sequence in E under the map Q. Since $\{\hat{x}_n\}_{n=1}^\infty$ is bounded in $\frac{E}{F}$ and Q lifts bounded sets, there exists $\{x_n\}_{n=1}^\infty \subseteq \mathcal{B}(E)$ such that $\hat{x}_n = Qx_n$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}_{n=1}^\infty$ does not contain $\sigma(E, E')$-Cauchy subsequences and hence, by [16, Lemma 3, p.172], it admits a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which is equivalent to the canonical unit basis vectors of ℓ^1. For the sake of simplicity, we denote such a subsequence again by $\{x_n\}_{n=1}^\infty$.

The sequence $\{\hat{x}_{2n} - \hat{x}_{2n-1}\}_{n=1}^\infty$ is a $\sigma\left(\frac{E}{F}, \left(\frac{E}{F}\right)'\right)$-null sequence in $\frac{E}{F}$ and hence, there exist an increasing sequence $\{k_n\}_{n=1}^\infty$ of positive integers and a sequence $\{\hat{y}_n\}_{n=1}^\infty$ in $\frac{E}{F}$ with $\hat{y}_n$ a convex combination of elements of $\{\hat{x}_{2j} - \hat{x}_{2j-1} : k_n \leqslant j < k_{n+1}\}$ (i.e., $\{\hat{y}_n\}_{n=1}^\infty \subseteq \mathrm{co}\{\hat{x}_{2n} - \hat{x}_{2n-1}\}_{n=1}^\infty$) such that $\lim_{n\to\infty} \hat{y}_n = 0$ in $\frac{E}{F}$. If $\{y_n\}_{n=1}^\infty$ is the corresponding sequence of convex combinations in E of elements of $\{x_{2j} - x_{2j-1} : k_n \leqslant j < k_{n+1}\}$, then the sequence $\{y_n\}_{n=1}^\infty$ is also equivalent to the canonical unit basis vectors of ℓ^1. But, $Q(y_n) = \hat{y}_n \to 0$ in $\frac{E}{F}$ as $n \to \infty$. So, by Lemma 2.2 we can conclude that F contains an isomorphic copy of ℓ^1, which is a contradiction. ■

For Banach spaces, the following result is given in [14, Theorem 9].

Theorem 2.4. *Let E be a Fréchet space with the DP-property and let $F \subseteq E$ be a closed subspace. If F does not contain any isomorphic copy of ℓ^1 and the quotient map $Q\colon E \to \frac{E}{F}$ lifts bounded sets, then $\frac{E}{F}$ has the DP-property.*

Proof. Let $\{\hat{x}_n\}_{n=1}^\infty$ and $\{f_n\}_{n=1}^\infty$ be null sequences in $\sigma\left(\frac{E}{F}, \left(\frac{E}{F}\right)'\right)$ and in $\sigma\left(\left(\frac{E}{F}\right)', \left(\frac{E}{F}\right)''\right)$, respectively. Let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of positive integers. Then $\{\hat{x}_{n_k}\}_{k=1}^\infty$ is a $\sigma\left(\frac{E}{F}, \left(\frac{E}{F}\right)'\right)$-Cauchy sequence in $\frac{E}{F}$ and hence, by Lemma 2.3 there exist a $\sigma(E, E')$-Cauchy sequence $\{y_k\}_{k=1}^\infty$ in E and a subsequence $\{j_k\}_{k=1}^\infty$ of $\{n_k\}_{k=1}^\infty$ such that $Q(y_k) = \hat{x}_{j_k}$ for all $k \in \mathbb{N}$. Since $\{f_{j_k}\}_{k=1}^\infty$ is a $\sigma\left(\left(\frac{E}{F}\right)', \left(\frac{E}{F}\right)''\right)$-null sequence in $\left(\frac{E}{F}\right)'$ and hence, $\{Q' f_{j_k}\}_{k=1}^\infty$ is a $\sigma(E', E'')$-null sequence in E', it follows that

$$\langle \hat{x} j_k, f_{j_k} \rangle = \langle Q y_k, f_{j_k} \rangle = \langle y_k, Q' f_{j_k} \rangle \to 0 \qquad \text{as} \quad k \to \infty,$$

because E has the DP-property, see Proposition 2.1(iii).

The arbitrariness of the sequence $\{n_k\}_{k=1}^\infty$ implies that $\langle \hat{x}_n, f_n \rangle \to 0$ as $n \to \infty$. So, applying again Proposition 2.1(iii), we can conclude that $\frac{E}{F}$ has the DP-property. ■

We recall for a subspace Y of a lcHs X (of X', resp.) that its *annihilator* $Y^\perp := \{x' \in X' : \langle y, x' \rangle = 0 \ \forall y \in Y\}$ (${}^\perp Y := \{x \in X : \langle x, y' \rangle = 0 \ \forall y' \in Y\}$, resp.) coincides with its polar $Y^\circ := \{x' \in X' : |\langle y, x' \rangle| \leqslant 1 \ \forall y \in Y\}$ (${}^\circ Y := \{x \in X : |\langle x, y' \rangle| \leqslant 1 \ \forall y' \in Y\}$, resp.).

A consequence of Theorem 2.4 is the following result.

Corollary 2.5. *Let E be a complete (DF)-space and $F \subseteq E$ be a closed subspace which is quasibarrelled. If E'_β has the DP-property and $F^\perp$ does not contain any isomorphic copy of ℓ^1, then F'_β has the DP-property and hence, F has the DP-property.*

Proof. Denote by $J\colon F \to E$ the canonical inclusion. Then J is an isomorphism into and hence, the dual operator $Q := J'\colon E'_\beta \to F'_\beta$ is a homomorphism onto with $\ker Q = F^\perp$, i.e., Q coincides with the canonical quotient map from E'_β onto the quotient space $\frac{E'_\beta}{F^\perp}$. So, $F'_\beta = \frac{E'_\beta}{F^\perp}$ algebraically and topologically via the map Q. Moreover, the quasibarrelledness of F ensures that Q lifts bounded sets. Therefore, by Theorem 2.4 F'_β has the DP-property.

Since F is quasibarrelled and its strong dual F'_β possesses the DP-property, from a result of Grothendieck (see [25, §9.4.3(e), p.637]) we can conclude that F has the DP-property. ■

The next result is a dual proper analogue to Theorem 2.4 and is inspired by [24, Theorem 2].

Theorem 2.6. *Let E be a complete (DF)-space with the DP-property and F be a $\sigma(E', E)$-closed subspace of E' which does not contain any isomorphic copy of ℓ^1. If ${}^\perp F$ is quasibarrelled, then ${}^\perp F$ has the DP-property.*

Proof. Denote by $J\colon {}^\perp F \to E$ the canonical inclusion. As already observed in the proof of Corollary 2.5, the dual operator $Q := J'\colon E'_\beta \to ({}^\perp F)'_\beta$ coincides with the canonical quotient map from E'_β onto the quotient space $\frac{E'_\beta}{F}$ (as $({}^\perp F)^\perp = F$ by the assumption) and lifts bounded sets. So, $({}^\perp F')_\beta = \frac{E'_\beta}{F}$ algebraically and topologically via the map Q. Moreover, ${}^\perp F$ is a complete (DF)-space as a closed and quasibarrelled subspace of a complete (DF)-space.

Let $\{x_n\}_{n=1}^\infty \subseteq {}^\perp F$ be a $\sigma({}^\perp F, ({}^\perp F)')$-null sequence and $\{\hat{x}'_n\}_{n=1}^\infty \subseteq ({}^\perp F)'$ be a $\sigma(({}^\perp F)', ({}^\perp F)'')$-Cauchy sequence satisfying $\inf_{n\in\mathbb{N}} |\langle x_n, \hat{x}'_n\rangle| \geqslant c > 0$. By Lemma 2.3 there exists then a $\sigma(E', E)$-Cauchy sequence $\{x'_{n_k}\}_{k=1}^\infty \subseteq E'$ such that $Qx'_{n_k} = \hat{x}'_{n_k}$ for all $k \in \mathbb{N}$. It follows that

$$\langle Jx_{n_k}, x'_{n_k}\rangle = \langle x_{n_k}, Qx'_{n_k}\rangle = \langle x_{n_k}, \hat{x}'_{n_k}\rangle \geqslant c > 0, \qquad k \in \mathbb{N}.$$

On the other hand, by Proposition 2.1(iii) the DP-property of E implies that

$$\langle Jx_{n_k}, x'_{n_k}\rangle \to 0 \qquad \text{as} \quad k \to \infty.$$

So, we have a contradition. Then the proof is complete. ■

We recall that a lcHs E has the *Schur property* (briefly, E is a *Schur space*) if every $\sigma(E, E')$-convergent sequence is also convergent in E. If E is any Fréchet space or any complete (DF)-space or any complete (LF)-space with the Schur property, then, by Proposition 2.1, E has the DP-property.

Theorem 2.7. *Let E be a Fréchet space. If E has the DP-property and does not contain any isomorphic copy of ℓ^1, then E'_β is a Schur space and hence, E'_β has the DP-property.*

Proof. Suppose that E'_β is not a Schur space. So, there exists a sequence $\{x'_n\}_{n=1}^\infty \subseteq E'$ such that $x'_n \to 0$ in $(E', \sigma(E', E''))$ as $n \to \infty$ but, $\{x'_n\}_{n=1}^\infty$ does not converge to 0 in E'_β. Therefore, there exists $B \in \mathcal{B}(E)$ such that

$$\sup_{x \in B} |\langle x, x'_n \rangle| \geqslant c > 0, \qquad n \in \mathbb{N},$$

(eventually by passing to a subsequence). So, we can find a sequence $\{x_n\}_{n=1}^\infty \subseteq B$ satisfying

$$|\langle x_n, x'_n \rangle| \geqslant \frac{1}{2} c, \qquad n \in \mathbb{N}. \tag{2.11}$$

By [16, Lemma 3, p.172] the bounded sequence $\{x_n\}_{n=1}^\infty \subseteq B$ has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which is either $\sigma(E, E')$-Cauchy or is equivalent to the canonical unit basis vectors of ℓ^1. But E does not contain any isomorphic copy of ℓ^1 and hence, $\{x_{n_k}\}_{k=1}^\infty$ must be a $\sigma(E, E')$-Cauchy sequence. Since E has the DP-property, by Proposition 2.1(iii) it follows that

$$\langle x_{n_k}, x_{n_k} \rangle \to 0 \qquad as \;\; k \to \infty.$$

This is in contradiction with (2.11). Then the proof is complete. ■

We end this section with a result concerning the Grothendieck property.

Theorem 2.8. *Let E be a lcHs and $F \subseteq E$ be a closed subspace. If E is a Grothendieck space and the quotient map $Q\colon E \to \frac{E}{F}$ lifts bounded sets with closure, then the quotient space $\frac{E}{F}$ is also a Grothendieck space.*

Proof. The assumption on the quotient map Q ensures that its dual operator $Q'\colon \left(\frac{E}{F}\right)'_\beta \to E'_\beta$ is a topological isomorphism into and hence, it is also a topological isomorphism with respect the weak topologies $\sigma\left(\left(\frac{E}{F}\right)', \left(\frac{E}{F}\right)''\right)$ and $\sigma(E', E'')$, [28, Proposition 9.6.1]. Moreover, $\operatorname{Im} Q' = F^\perp$.

Let $\{x'_n\}_{n=1}^\infty \subseteq \left(\frac{E}{F}\right)'$ be a $\sigma\left(\left(\frac{E}{F}\right)', \frac{E}{F}\right)$-convergent sequence. Since Q' also belongs to $\mathcal{L}\left(\left(\frac{E}{F}\right)'_\sigma, E'_\sigma\right)$, it follows that $\{Q'x'_n\}_{n=1}^\infty$ is a $\sigma(E', E)$-convergent sequence in E' and hence, it is also a $\sigma(E', E'')$-convergent sequence in E' because E is a Grothendieck space. But $\{Q'x'_n\}_{n=1}^\infty \subseteq F^\perp$ and $\sigma(E', E'')|_{F^\perp} = \sigma(F^\perp, (F^\perp)')$, [28, Corollary 8.7.3]. Therefore, $\{Q'x'_n\}_{n=1}^\infty$ is a $\sigma(F^\perp, (F^\perp)')$-convergent sequence in $F^\perp$. Via the weakly isomorphism Q' we obtain that $\{x'_n\}_{n=1}^\infty$ is a convergent sequence with respect to $\sigma\left(\left(\frac{E}{F}\right)', \left(\frac{E}{F}\right)''\right)$. So, the proof is complete. ■

3. Köthe sequence spaces

Let I denote a fixed countable index set and $A = (a_n)_{n\in\mathbb{N}}$ be a *Köthe matrix* on I (i.e., A is an increasing sequence of strictly positive functions on I). Then the *Köthe echelon space* $\lambda_p(A)$ of order $1 \leqslant p < \infty$ is defined as the vector space

$$\lambda_p(A) := \left\{ x \in \mathbb{C}^I : \forall n \in \mathbb{N}\ q_n^p(x) := \left(\sum_{i\in I} (a_n(i)|x_n(i)|)^p \right)^{1/p} < \infty \right\},$$

and the *Köthe echelon space* $\lambda_\infty(A)$ of order ∞ ($\lambda_0(A)$ of order 0, resp.) is defined as the vector space

$$\lambda_\infty(A) = \left\{ x \in \mathbb{C}^I : \forall n \in \mathbb{N}\ q_n^\infty(x) := \sup_{i\in I} a_n(i)|x_n(i)| < \infty \right\},$$

$$\Big(\lambda_0(A) = \{ x \in \mathbb{C}^I : \forall n \in \mathbb{N}\ \lim_i a_n(i)x_n(i) = 0 \},\ \text{resp.} \Big).$$

The spaces $\lambda_p(A)$, with $1 \leqslant p \leqslant \infty$ or $p = 0$, endowed with the sequence of norms $(q_n^p)_{n=1}^\infty$, are Fréchet spaces. Moreover, if $1 < p < \infty$ then $\lambda_p(A)$ is reflexive.

For a Köthe matrix $A = (a_n)_{n\in\mathbb{N}}$, let $V = (v_n)_{n\in\mathbb{N}}$ denote the associate decreasing sequence of functions $v_n := 1/a_n$, and set

$$k_p(V) = \operatorname{ind}_n \ell^p(v_n), \quad 1 \leqslant p \leqslant \infty, \qquad \text{and} \qquad k_0(V) := \operatorname{ind}_n c_0(v_n).$$

So, $k_p(V)$ is the increasing union of the Banach spaces $\ell^p(v_n)$ ($c_0(v_n)$, resp.) endowed with the strongest lc-topology under which the natural injection of each Banach space $\ell^p(v_n)$ ($c_0(v_n)$, resp.) is continuous. The spaces $k_p(V)$ are called *co-echelon spaces* of order p. The canonical inclusion $k_0(V) \to k_\infty(V)$ is clearly continuous but it is even a topological isomorphism into.

Given any decreasing sequence $V = (v_n)_{n=1}^\infty$ of strictly positive functions on I, let

$$\overline{V} := \left\{ \overline{v} = (\overline{v}(i))_{i\in I} \in [0,\infty)^I : \forall n \in \mathbb{N}\ \sup_{i\in I} \frac{\overline{v}(i)}{v_n(i)} = \sup_{i\in I} a_n(i)\overline{v}(i) < \infty \right\}.$$

Since I is countable, the system $\overline{V}$ always contains strictly positive functions. Next, we introduce the family of spaces

$$K_p(\overline{V}) := \operatorname{proj}_{\overline{v}\in\overline{V}} \ell^p(\overline{v}), \quad p \in [1,\infty], \qquad \text{and} \qquad K_0(\overline{V}) := \operatorname{proj}_{\overline{v}\in\overline{V}} c_0(\overline{v}).$$

These spaces are equipped with the complete lc-topology given by the system of seminorms

$$q_{\overline{v}}^{(p)}(x) := \left(\sum_{i\in I} (\overline{v}(i)|x(i)|)^p \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

and

$$q_{\overline{v}}^{(\infty)}(x) := \sup_{i\in I} \overline{v}(i)|x(i)|,$$

for each $\overline{v} \in \overline{V}$. For $1 \leqslant p < \infty$, $k_p(V) = K_p(\overline{V})$ as vector spaces and also topologically. In particular, the inductive limit topology is given by the system of seminorms $\{q_{\overline{v}}^{(p)} : \overline{v} \in \overline{V}\}$ and $k_p(V)$ is always complete. Moreover, $K_0(\overline{V})$ is the completion of $k_0(V)$ and the inductive limit topology of $k_0(V)$ is given by the system of seminorms $\{q_{\overline{v}}^{(\infty)} : \overline{v} \in \overline{V}\}$. However, it can happen that $k_0(V)$ is a proper subspace of $K_0(\overline{V})$. In particular, $K_0(\overline{V})$ is a barrelled (DF)-space. Finally, $k_\infty(V)$ and $K_\infty(\overline{V})$ are equal as vector spaces and the two spaces have the same bounded sets. Moreover, $k_\infty(V)$ is the bornological space associated with $K_\infty(\overline{V})$. For $1 \leqslant p < \infty$ or $p = 0$, if $\frac{1}{p} + \frac{1}{q} = 1$ (where we take $q = \infty$ for $p = 1$ and $q = 0$ for $p = 1$), then $(\lambda_p(A))'_\beta = K_q(\overline{V})$ and $(k_p(V))'_\beta = \lambda_q(A)$. For $p = 0$ we also have $(\lambda_0(A))'_\beta = k_1(V)$ and $(K_0(\overline{V}))'_\beta = \lambda_1(A)$.

For a systematic treatment of co-echelon spaces (and echelon spaces) we refer to [6].

An old result of Grothendieck ensures that if E is a quasibarrelled lcHs such that its strong dual E'_β possesses the DP-property, then so too does E (see [25, §9.4.3(e), p.637]). If we combine this result with the recent results in [1, Theorem 4.4], [11, Proposition 3.1], we obtain that:

Remark 3.1.

(a) The space $K_\infty(\overline{V})$ is the strong dual of the echelon space $\lambda_1(A)$ and has the DP-property, see [1, Theorem 4.4]. So, by [25, §9.4.3(e), p.637] the echelon space $\lambda_1(A)$ also has the DP-property.

(b) By [25, §9.4.3(e), p.637] the barrelled (DF)-space $K_0(\overline{V})$ has the DP-property as its strong dual $(K_0(\overline{V}))'_\beta = \lambda_1(A)$.

(c) The space $\lambda_\infty(A)$ is the strong dual of the co-echelon space $k_1(V) = K_1(\overline{V})$ and has the DP-property, see [11, Proposition 3.1]. So, by [25, §9.4.3(e), p.637] the co-echelon space $k_1(V)$ has the DP-property.

(d) By [25, §9.4.3(e), p.637] the echelon space $\lambda_0(A)$ has the DP-property as $(\lambda_0(A))'_\beta = k_1(V)$.

Moreover, by the results in §2 and Remark 3.1 we deduce the following result. We refer to [5] and the reference therein for the definition of regularly decreasing sequences V and for results about the lifting bounded sets property.

Proposition 3.2. *The following holds.*

(i) *The quotient space* $\frac{K_\infty(\overline{V})}{K_0(\overline{V})}$ *is a Grothendieck space.*

(ii) *If the sequence V associated to the Köthe matrix A is regularly decreasing, then the quotient space* $\frac{\lambda_\infty(A)}{\lambda_0(A)}$ *is a GDP space.*

Proof. (i). We observe that $K_0(\overline{V})$ is a closed subspace of $K_\infty(\overline{V})$ and that the quotient map $Q\colon K_\infty(\overline{V}) \to \frac{K_\infty((\overline{V}))}{K_0(\overline{V})}$ lifts bounded sets with closure as $K_\infty(\overline{V})$ is a (DF)-space. Since $K_\infty(\overline{V})$ is a Grothendieck space, the result then follows from Theorem 2.8.

(ii). We first observe that $\lambda_0(A)$ is a closed subspace of $\lambda_\infty(A)$ and that the assumption on V ensures that $\lambda_0(A)$ is quasinormable. We can then apply Merzon–Palamodov theorem to conclude that the quotient map $Q\colon \lambda_\infty(A) \to \frac{\lambda_\infty(A)}{\lambda_0(A)}$ lifts bounded sets as $\ker Q = \lambda_0(A)$. Since the space $\lambda_\infty(A)$ is a Grothendieck space, from Theorem 2.8 it follows that $\frac{\lambda_\infty(A)}{\lambda_0(A)}$ is also a Grothendieck space. On the other hand, as it is easy to prove, the space $\lambda_0(A)$ does contain any isomorphic copy of ℓ^1. So, since $\lambda_\infty(A)$ possesses also the DP-property, from Theorem 2.4 it follows that $\frac{\lambda_\infty(A)}{\lambda_0(A)}$ has the DP-property too. ■

Next, let $(a_{n,k}(j))_{j,k,n\in\mathbb{N}}$, be a matrix of strictly positive number satifying the following properties

$$a_{n,k}(j) \leqslant a_{n,k+1}(j), \qquad a_{n,k}(j) \geqslant a_{n+1,k}(j)$$

for all $j,k,n \in \mathbb{N}$. For $1 \leqslant p \leqslant \infty$ or $p = 0$ and for $n \in \mathbb{N}$ we denote by E_n^p the echelon space of order p corresponding to the Köthe matrix $A_n = (a_{n,k})_{k\in\mathbb{N}}$. Then all these spaces are Fréchet spaces and, $E_n^p \hookrightarrow E_{n+1}^p$ continuously for every $n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$ or $p = 0$. The inductive limit $E^p := \operatorname{ind}_n E_n^p$ is called a *Köthe (LF)-sequence space* of order p.

Moreover, let

$$d^\times := \{(a_j)_{j\in\mathbb{N}} \in \mathbb{R}^\mathbb{N} : \\ \forall j \in \mathbb{N}\ a_j \geqslant 0,\ \forall n\ \exists \alpha_n > 0, k(n) \in \mathbb{N}\ \forall j \in \mathbb{N}\ \ a_j \leqslant \alpha_n a_{n,k(n)}(j)\}.$$

By [34, Proposition 5.1] a fundamental system of seminorms in E^p is given by the seminorms

$$\|x\|_a := \left(\sum_{j=1}^\infty |x(j)|^p a_j^p\right)^{\frac{1}{p}}, \quad a \in d_x, \qquad \text{for} \quad 1 \leqslant p < \infty,$$

$$\|x\|_a := \sup_{j\in\mathbb{N}} |x(j)| a_j, \quad a \in d_x, \qquad \text{for} \quad p = 0.$$

By using this projective representation for the spaces E^p, it is easy to verify that the sequence of vectors $(e_i)_{i\in\mathbb{N}}$, where $e_i = (0,\dots,0,1,0\dots)$, is a Schauder basis in E^p for $1 \leqslant p < \infty$ or $p = 0$.

An (LF)-space $E = \operatorname{ind}_n E_n$ is called *boundedly retractive* if for every bounded set B in E there is $n \in \mathbb{N}$ such that B is contained in E_n and the lc-topologies of E and E_n coincide on B. The results in [34, §5] ensure that E^p is boundedly retractive if and only if the matrix $(a_{nk}(j))$, $j,k,n \in \mathbb{N}$, satisfies the following condition

$$\exists (k(\nu))_{\nu\in\mathbb{N}}\ \forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall l \in \mathbb{N}, \varepsilon > 0\ \exists (a_j)_{j\in\mathbb{N}} \in d^\times\ \forall j \in \mathbb{N} : \\ a_{ml}(j) \leqslant \max\left(\varepsilon \min_{1\leqslant\nu\leqslant n} a_{\nu k(\nu)}(j), a_j\right). \tag{3.1}$$

By Theorems 5.10, 5.14 and 5.16 in [34] we know that the Köthe (LF)-space E^p is regular if and only if it is complete. Therefore, we can obtain the following result.

Proposition 3.3. *Let $1 \leqslant p \leqslant \infty$ or $p = 0$ and E^p be regular. Then the following holds.*

(i) *If $1 < p \leqslant \infty$, then E^p is a Grothendieck space.*
(ii) *If $p = 0$, then E^p has the DP-property. If $p = 1$ and the matrix $(a_{n,k}(j))_{j,k,n\in\mathbb{N}}$, satisfies (3.1), then E^p has the DP-property.*
(iii) *If $1 \leqslant p < +\infty$ or $p = 0$, then E^p is a GDP space if and only if E^p is Montel.*
(iv) *If the matrix $(a_{n,k}(j))_{j,k,n\in\mathbb{N}}$, satisfies (3.1), then E^∞ is a GDP space.*

Proof. (i). If $1 < p < \infty$ then each E^p_n is a reflexive Fréchet space and hence, E^p is a Grothendieck space. If $p = \infty$ then each E^∞_n is a Grothendieck space, see [17, Proposition 5]. Since E^p is regular by assumption, the result then follows from [1, Proposition 3.10].

(ii). For $p = 1$ or $p = 0$ each space E^p_n has the DP-property, see Remark 3.1(a),(d). On the other hand, [34, Theorem 5.16] ensures that E^0 satisfies the assumption in [1, Proposition 3.11]. In the case $p = 1$, [34, Theorem 5.6] ensures that E^1 satisfies the assumption in [1, Proposition 3.11] (see [4]). So, the result follows by applying [1, Proposition 3.11] to E^p.

(iii). As already observed, if $1 \leqslant p < +\infty$ or $p = 0$ then the space E^p has a Schauder basis. Then the result follows from [1, Corollary 3.8].

(iv). For each $n \in \mathbb{N}$ the space E^∞_n is a GDP space, see [11, Proposition 3.1]. On the other hand, (3.1) implies that E^∞ satisfies the assumption in [1, Proposition 3.11] (see [4]). So, we can apply [1, Proposition 3.11] to conclude that E^∞ has also the DP-property. ∎

4. Projective tensor products of GDP-Fréchet spaces

It is well known that the DP-property is not preserved by projective and injective tensor products. Famous examples of Talagrand [32] show the existence of Banach spaces E with the DP-property such that $L^1(\Omega, E) = L^1(\Omega)\widehat{\otimes}_\pi E$ and $C(\Omega, E) = C(\Omega)\widehat{\otimes}_\varepsilon E$ do not have the DP-property. More results in the Banach spaces setting can be found in [7, 27, 22, 23, 31].

Schur property has a better behaviour, at least with respect to the injective tensor product. Extending a results of Ryan [31], Botelho and Rueda [12] proved that the injective tensor product of quasibarrelled spaces with the Schur property enjoys the Schur property. Moreover, they proved that if E'_β and F have the Schur property, then the space $\mathcal{L}_b(E, F)$ has the Schur property. Following an idea of Ryan [31] and using this result, we can obtain a sufficient condition for the DP-property with respect to the projective tensor product. We recall that a pair of Fréchet spaces (E, F) has *the property (BB) of Grothendieck* if every bounded set of $E\widehat{\otimes}_\pi F$ is contained in the closure of the absolutely convex hull of the tensor product of a bounded set of E and a bounded set of F. The so-called *problem of topologies* of Grothendieck asked if every pair of Fréchet spaces has the property (BB). It was solved negatively by Taskinen [33] in 1986. Since then several authors (among them Bonet, Defant, Díaz, Galbis, Peris) studied the problem finding classes of spaces satisfying the property (BB).

Proposition 4.1. *Let E and F be Fréchet spaces. Then the following holds.*

(i) *If E and F have the Schur property, then $E\widehat{\otimes}_\varepsilon F$ has the Schur property and hence, it has the DP-property.*
(ii) *If (E,F) has the property (BB) of Grothendieck and both E'_β and F'_β have the Schur property, then $E\widehat{\otimes}_\pi F$ has the DP-property.*

Proof. (i). It is proved in [12, Proposition 4.1].

(ii). By [12, Proposition 4.3], $\mathcal{L}_b(E,F'_\beta)$ has the Schur property and hence, the DP-property. Since $(E\widehat{\otimes}_\pi F)'_\beta = \mathcal{L}_b(E,F'_\beta)$, from [25, 9.4.3(e), p.637] it follows that $E\widehat{\otimes}_\pi F$ has the DP-property. ∎

Concerning Grothendieck spaces and injective tensor products, we recall that Freniche [26] characterized when spaces of vector valued continuous functions are Grothendieck spaces. On the other hand, Domanski, Lindström and Schlüchtermann proved in [20, Theorem 3.6] that if E is a Fréchet space and F is a Fréchet Montel space, and F or E''_β has the approximation property, then $E\widehat{\otimes}_\varepsilon F$ is a Grothendieck space if and only if E is a Grothendieck space. Moreover, they observed that the assumption about the approximation property can be removed if E is a Banach space and F is a Fréchet space, see [21, Corollary 2.7]. By combining these results with Proposition 4.1 we obtain that

Proposition 4.2. *Let E and F be Fréchet spaces. Then the following holds.*

(i) *If E has the Schur property, F is a Montel space and F or E''_β has the approximation property, then $E\widehat{\otimes}_\varepsilon F$ is a GDP space.*
(ii) *If E is a Banach space with the Schur property and F is a Schwartz space, then $E\widehat{\otimes}_\varepsilon F$ is a GDP space.*

In order to obtain sufficient conditions for the GDP property (and also for the Schur property) with respect to projective tensor products, we recall the following definition which was introduced in [10] (see also [8]) to give a partial positive answer to the problem of *topologies of Grothendieck*.

Definition 4.3. *A Fréchet space E is said to be a decomposable (FG)-space if there is an increasing fundamental sequence of seminorms $\{|\cdot|_n\}_{n\in\mathbb{N}}$ such that for every sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ of scalars with $0<\alpha_k\leqslant 1$ for every $k\in\mathbb{N}$, there exists a sequence of continuous linear operators $\{P_n\}_{n\in\mathbb{N}}\subseteq\mathcal{L}(E)$ such that*

(FG1) $x=\sum_{j=1}^{\infty}P_jx$ *for every $x\in E$;*
(FG2) $|P_kx|_{k-1}\leqslant\alpha_k|x|_k$ *for all $x\in E$ and $k\geqslant 2$;*

(FG3) *for all* $k \in \mathbb{N}$ *and* $j > k$ *there exists* $\lambda_{jk} \geqslant 1$ *such that* $|P_k x|_j \leqslant \lambda_{jk} |x|_k$ *for all* $x \in E$;

(FG4) $P_i \circ P_j = \delta_{ij} P_j$ *for all* $i, j \in \mathbb{N}$.

Actually, $(FG1)$ *and* $(FG4)$ *mean that* $\{P_n\}_{n\in\mathbb{N}}$ *is a Schauder decomposition of* E.

Examples of decomposable (FG)-spaces are the Köthe sequences spaces of order $1 \leqslant p < \infty$ or $p = 0$ (also Banach-valued); the Köthe space $\lambda_\infty(A)$ with the density condition; the Fréchet-Schwarz spaces with a finite-dimensional decomposition and a continuous norm; the space $\mathcal{H}_b(X)$ of entire holomorphic functions of bounded type endowed with the topology of uniform convergence on the bounded subsets of a Banach space X; the weigthed Fréchet spaces of continuous functions $CA_0(X)$.

We need also the following notion which was introduced in [18].

Definition 4.4. *A decomposition* $\{P_n\}_{n\in\mathbb{N}}$ *of a Fréchet space* E *is said to have the property* (M) *if*

$$\lim_{n\to\infty} \sum_{k=1}^{n} P_k = I \qquad \text{in} \quad \mathcal{L}_b(E).$$

Let E and F be Fréchet spaces and $\{P_n\}_{n\in\mathbb{N}}$ be a decomposition of E. Then, we can define a canonical decomposition $\{P_n \widehat{\otimes}_\pi I\}_{n\in\mathbb{N}}$ of $E\widehat{\otimes}_\pi F$ in the canonical way. Moreover, by [19, Theorem 5] we have

Theorem 4.5. *Let* E *be a decomposable (FG)-space having a decomposition* $\{P_n\}_{n\in\mathbb{N}}$ *with the property (M). Then, for any Fréchet space* F*, the canonical decomposition* $\{P_n \widehat{\otimes}_\pi I\}_{n\in\mathbb{N}}$ *of* $E\widehat{\otimes}_\pi F$ *has the property (M).*

Finally, we recall the following result given in [1, Theorem 3.7].

Theorem 4.6. *Let* E *be any quasicomplete, barrelled lcHs and* $\{P_n\}_{n\in\mathbb{N}}$ *a Schauder decomposition of* E. *Then the following holds.*

(i) *If* E *is GDP, then* $\{P_n\}_{n\in\mathbb{N}}$ *has the property (M).*

(ii) *If* $\{P_n\}_{n\in\mathbb{N}}$ *has the property (M) and each complemented subspace* $P_n(E)$ *of* E *is a Grothendieck space (has the DP-property, has the Schur property, is Montel, resp.), then* E *is a Grothendieck space (has the DP-property, has the Schur property, is Montel, resp.).*

By combining the previous results we obtain the following stability theorem for the projective tensor product of GDP spaces.

Theorem 4.7. *Let* E *be a Montel decomposable (FG)-space and let* F *be a Fréchet space which has the DP-property (has the Schur property, is a Grothendieck space, is a GDP space, resp.). Then* $E\widehat{\otimes}_\pi F$ *has the same property as* F.

Proof. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers in $]0, 1]$. Let $\{P_n\}_{n\in\mathbb{N}}$ be a $\{\alpha_n\}_{n\in\mathbb{N}}$ decomposition of E. Since E is Montel and hence, it is a GDP space,

$\{P_n\}_{n\in\mathbb{N}}$ has the property (M) and $\dim P_n(E) = h < \infty$ by $(FG3)$. So, the canonical decomposition $\{P_n\widehat{\otimes}_\pi I\}_{n\in\mathbb{N}}$ has also the property (M) by Theorem 4.5 and, for each $n \in \mathbb{N}$ the space $(P_n\widehat{\otimes}_\pi I)(E\widehat{\otimes}_\pi F)$ is isomorphic to F^h so that it has the same properties as F. The assertion then follows by applying Theorem 4.6(ii). ■

Corollary 4.8. *Let F be a Fréchet GDP space. If $E = \lambda^p(A)$, with $1 \leqslant p \leqslant +\infty$ and $\lambda^p(A)$ is Montel, or E is a Fréchet-Schwarz spaces with a finite-dimensional decomposition and a continuous norm, then $E\widehat{\otimes}_\pi F$ is a GDP space.*

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