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THE NUMBER OF CUBEFULL NUMBERS IN AN INTERVAL (SUPPLEMENT)

H.-Q. Liu

Abstract: We derive a new result which extends the range of validity of the asymptotic formula for cubefull integers in an interval.

Keywords: cubefull numbers, exponential sums

1. Introduction

In 1991 Shiu [S] first studied the asymptotic distribution of cubefull numbers in a interval (a positive integer n is called a cubefull number, if whenever $p\,|n$ we also have $p^3\,|n$ for any prime number p) by means of the techniques of estimating exponential sums, who showed that

$$Q_3(x + x^{\frac{2}{3}+\mu}) - Q_3(x) = Cx^{\mu}(1 + o(1)) \tag{1}$$

for $\frac{140}{1123} < \mu < \frac{1}{3}$, here $\frac{140}{1123} = 0.12466\ldots$ ($Q_3(x)$ is the number of cubefull numbers not exceeding x, and C is some positive constant). In 1993 in [L1] we extended Shiu's range to

$$\mu > \frac{11}{92} = 0.11956\ldots,$$

by using the current techniques of exponential sums. In 1998 Wu ([W], Theorem 2) obtained the better range $\mu > \frac{19}{159} = 0.11949\ldots$ by using a result of [SW] on multiple exponential sums. Based on the work of [L1],[L2] and [RS], in this paper we shall deduce a new result. We have

Theorem. (1) holds for any $\mu > \frac{5}{42} = 0.1190\ldots$.

(If we use Theorem 2 of [L2] then this result may also be improved slightly).

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2. Proof of our result

In view of Theorem 2 of [L1] and the treatments of p.6 of [L1], to derive the Theorem it suffices for us to show that

$$x^{-\frac{\varepsilon}{2}}S(M,N) \ll x^{\theta},$$

for any $\varepsilon > 0$, where $\theta = \frac{5}{42}$. Instead of (3) of p.9 of [L1], we now have

$$x^{-\varepsilon}S(M,N) \ll \sqrt[24]{x^2M^7N^5} + x^{\frac{1}{9}}. \tag{2}$$

To explain this, note that we can use Theorem 1 of [L2] to replace Lemma 3.2 of [L1] in (1) of [L1] for deducing Lemma 1 of [L1], which gives(note that in Theorem 1 of [L2] we have "“F” $\approx HF \gg$ “XY” $\approx HF\frac{N}{M}$" in the present situation, and thus the second term of Theorem 1 of [L2] can be removed as compared with the leading term of it)

$$\Phi(H,M,N) \ll \sqrt[6]{(HF)^2(MN)^3} + (HF)^{\frac{1}{2}} + x^{\frac{1}{12}}\log x.$$

Putting this into (0) of [L1] and choosing $K \in (0,MN)$ optimally via Lemma 3.4 of [L1], and using the fact that $F := (xM^{-b}N^{-c})^{\frac{1}{a}} \ll (xM^{-4}N^{-5})^{\frac{1}{3}}$ for any permutation (a,b,c) of $(3,4,5)$, we get our (2) of here. We can then use Theorem 1 of [RS] to replace Lemma 3.1 of [L1] for deriving Lemmas 2 and 3 of [L1], and we obtain respectively(comparing (4) and (7) of [L1])

$$x^{-\varepsilon}S(M,N) \ll \sqrt[12]{xM^{-1}N^7} + \sqrt[15]{x^2M^{-2}N^{-1}} + x^{\frac{1}{9}}, \tag{3}$$

and

$$x^{-\varepsilon}S(M,N) \ll \sqrt[7]{xM^{-1}N^{-2}} + \sqrt[6]{xM^{-1}N^{-3}} + \sqrt[15]{x^2M^{-2}N^{-1}} + x^{\frac{1}{9}}; \tag{4}$$

where we have used the assumption $M \gg N$. It is then quite easy to deduce our result from (2), (3) and (4). In fact, from (2) and (3) we get

$$x^{-\varepsilon}S(M,N) \ll R_1 + R_2 + x^{\frac{1}{9}}, \tag{5}$$

where

$$\begin{aligned} R_1 &= \min(\sqrt[24]{x^2M^7N^5},\ \sqrt[12]{xM^{-1}N^7}) \leqslant \sqrt[12]{xN^6}, \\ R_2 &= \min(\sqrt[24]{x^2M^7N^5},\ \sqrt[15]{x^2M^{-2}N^{-1}}) \\ &\ll \min(\sqrt[24]{x^2M^7N^5},\ \sqrt[60]{x^8(M^7N^5)^{-1}}) \ll x^{\theta}. \end{aligned}$$

Similarly, from (2) and (4) we get

$$x^{-\varepsilon}S(M,N) \ll R_3 + R_4 + x^{\theta}, \tag{6}$$

where

$$\begin{aligned} R_3 &= \min(\sqrt[24]{x^2M^7N^5}, \sqrt[7]{xM^{-1}N^{-2}}) \leqslant \sqrt[73]{x^9N^{-9}}, \\ R_4 &= \min(\sqrt[24]{x^2M^7N^5}, \sqrt[6]{xM^{-1}N^{-3}}) \leqslant \sqrt[66]{x^9N^{-16}}. \end{aligned}$$

Now the required estimate follows from (5) and (6) respectively according as $N < x^{\frac{2}{21}}$ or $N \geqslant x^{\frac{2}{21}}$.

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Address: Department of Mathematics, Harbin Institute of Technology, Harbin 150001, P.R.China

E-mail: mnpqr@163.com

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