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GROWTH ENVELOPES IN MUCKENHOUPT WEIGHTED FUNCTION SPACES: THE GENERAL CASE

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Abstract: We study growth envelopes of function spaces $B^s_{p,q}(\mathbb{R},w)$ and $F^s_{p,q}(\mathbb{R},w)$ where the weight belongs to some Muckenhoupt class $w \in \mathcal{A}_\infty$. This essentially extends partial forerunners in [13–16]. We also indicate some applications of these results.

Keywords: Muckenhoupt weights, function spaces, envelopes.

Introduction

The purpose of this paper is to use the recently introduced concept of growth envelopes in function spaces in order to characterize weighted spaces of type $L_p(\mathbb{R}^n,w)$, $B^s_{p,q}(\mathbb{R}^n,w)$ and $F^s_{p,q}(\mathbb{R}^n,w)$ where w belongs to some Muckenhoupt class $\mathcal{A}_p$. The idea to consider growth envelopes in (unweighted) function spaces originates from such classical results as the famous Sobolev embedding theorem [31]. Basically, the unboundedness of functions that belong to Sobolev and more general spaces is characterized as follows. Let X be a space of functions or regular distributions, $X \subset L_1^{\mathrm{loc}}$, then its *growth envelope* $\mathfrak{E}_{\mathsf{G}}(X) = (\mathcal{E}^X_{\mathsf{G}}(t), u^X_{\mathsf{G}})$ is introduced, where

$$\mathcal{E}^X_{\mathsf{G}}(t) \sim \sup\left\{f^*(t) : \|f|X\| \leqslant 1\right\}, \qquad t>0,$$

is the *growth envelope function* of X and $u^X_{\mathsf{G}} \in (0,\infty]$ is some additional index providing a finer description. Here f^* denotes the non-increasing rearrangement of f, as usual. These concepts were introduced in [38], [13], where the latter book also contains a recent survey (concerning extensions and more general approaches) as well as applications and further references.

Dealing with weighted spaces of type $B^s_{p,q}(\mathbb{R}^n,w)$ and $F^s_{p,q}(\mathbb{R}^n,w)$ first (special) results were obtained in [14, 13, 16, 15], essentially concentrating on weights from

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the Muckenhoupt $\mathcal{A}_\infty$ class. In particular, we studied the model weights

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha, & |x| < 1, \\ |x|^\beta, & |x| \geqslant 1, \end{cases} \tag{0.1}$$

and

$$w_{\log}(x) = \begin{cases} |x|^\alpha (1 - \log|x|)^\gamma, & |x| < 1, \\ |x|^\beta (1 + \log|x|)^\delta, & |x| \geqslant 1, \end{cases} \tag{0.2}$$

where $\alpha, \beta > -n$, and $\gamma, \delta \in \mathbb{R}$. Moreover, in [15] some first more general results for $L_p(\mathbb{R}^n, w)$, $w \in \mathcal{A}_\infty$, can be found.

Our main intention now is to extend this idea, that is, to study the interplay between the weight $w \in \mathcal{A}_\infty$ and the singularity behavior of spaces $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ characterized by their growth envelopes. It turns out that for the smallest weight class in this context, i.e. $\mathcal{A}_1$, we have a complete result which essentially coincides with the unweighted situation if we assume, in addition, that

$$\inf_{m \in \mathbb{Z}^n} w(Q_{0,m}) \geqslant c_w > 0, \tag{0.3}$$

where $Q_{0,m}$ are unit cubes in $\mathbb{R}^n$ centered at $m \in \mathbb{Z}^n$, and $w(\Omega) = \int_\Omega w(x)\,\mathrm{d}x$. Our main result in Theorem 3.6 below establishes that for $0 < p < \infty$, $0 < q \leqslant \infty$, $s > n \max(\frac{1}{p} - 1, 0)$, and $w \in \mathcal{A}_1$,

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(\mathbb{R}^n, w)) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right), & s < \frac{n}{p}, \\ \left(|\log t|^{\frac{1}{q'}}, q\right), & s = \frac{n}{p} \quad \text{and} \quad 1 < q \leqslant \infty, \end{cases} \tag{0.4}$$

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}(\mathbb{R}^n, w)) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right), & s < \frac{n}{p}, \\ \left(|\log t|^{\frac{1}{p'}}, p\right), & s = \frac{n}{p} \quad \text{and} \quad 1 < p < \infty, \end{cases} \tag{0.5}$$

and

$$\mathfrak{E}_{\mathsf{G}}(L_p(\mathbb{R}^n, w)) = \left(t^{-\frac{1}{p}}, p\right). \tag{0.6}$$

We also investigate the behavior of the envelope function for $t \to \infty$ in dependence on the weight. In case of $w \in \mathcal{A}_\infty$ our results are less complete so far. For instance, with the notation $r_w = \inf\{r \geqslant 1 \ : w \in \mathcal{A}_r\}$, $w \in \mathcal{A}_\infty$, we obtain in general that for any small $\varepsilon > 0$ there is some $c_\varepsilon > 0$ such that

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(\mathbb{R}^n,w)}(t) \leqslant c_\varepsilon\, t^{-\frac{r_w}{p}+\frac{s}{n}-\varepsilon}, \qquad t \to 0, \tag{0.7}$$

whereas the lower estimate reads as

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(\mathbb{R}^n,w)}(t) \geqslant c \sup_{x_0 \in \mathbb{R}^n} \left(\sum_{j=1}^{\lfloor \frac{1}{n}|\log t| \rfloor} 2^{-j(s-\frac{n}{p})q'} \left(\frac{w\left(B(x_0, 2^{-j})\right)}{|B(x_0, 2^{-j})|} \right)^{-\frac{q'}{p}} \right)^{1/q'}, \tag{0.8}$$

where $t \to 0$.

Here $B(x_0, r)$ is a ball with radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$, $|B(x_0, r)| \sim r^n$ its Lebesgue measure. It is known, that the upper estimate with $\varepsilon = 0$ cannot be true in general, and for some other reason it is more likely that the lower estimate may be sharp. But this is not yet proved in full generality.

All this will be explicated by our model weights (0.1) and (0.2) (and another one). Moreover, we briefly indicate some applications of our results.

The main tools to prove such results are unweighted counterparts, sharp embeddings and atomic decompositions of corresponding spaces. We also benefit from related observations on embeddings and local singularities $\mathbf{S}_{\mathrm{sing}}(w)$ of the weight $w \in \mathcal{A}_\infty$ contained in [18, 19, 20].

The paper is organized as follows. In Section 1 we collect all the material on Muckenhoupt weights, weighted spaces of type $B^s_{p,q}(\mathbb{R}^n, w)$, $F^s_{p,q}(\mathbb{R}^n, w)$, and embeddings that will be needed below. This is followed by a short introduction to the concept of growth envelopes in Section 2, before we deal exclusively with $w \in \mathcal{A}_1$ in Section 3 and determine the corresponding growth envelopes of $B^s_{p,q}(\mathbb{R}^n, w)$, $F^s_{p,q}(\mathbb{R}^n, w)$. Finally, in the last Section 4 we devote our attention to weights $w \in \mathcal{A}_\infty \setminus \mathcal{A}_1$.

1. Weighted function spaces

We fix some notation. By $\mathbb{N}$ we mean the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, and by $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$ having integer components. The positive part of a real function f is denoted by $f_+(x) = \max(f(x), 0)$, the integer part of $a \in \mathbb{R}$ by $\lfloor a \rfloor = \max\{k \in \mathbb{Z} : k \leqslant a\}$. If $0 < u \leqslant \infty$, the number u' is given by $\frac{1}{u'} = (1 - \frac{1}{u})_+$. For two positive real sequences $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{\beta_k\}_{k\in\mathbb{N}}$ we mean by $\alpha_k \sim \beta_k$ that there exist constants $c_1, c_2 > 0$ such that $c_1\, \alpha_k \leqslant \beta_k \leqslant c_2\, \alpha_k$ for all $k \in \mathbb{N}$; similarly for positive functions. Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. If not otherwise indicated, log is always taken with respect to base 2.

As we shall always deal with function spaces on $\mathbb{R}^n$, we may often omit the '$\mathbb{R}^n$' from their notation for convenience.

1.1. Muckenhoupt weights

We briefly recall some fundamentals on Muckenhoupt classes $\mathcal{A}_p$. By a weight w we shall always mean a locally integrable function $w \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, positive a.e. in the sequel. Let M stand for the Hardy-Littlewood maximal operator given by

$$Mf(x) = \sup_{B(x,r)\in\mathcal{B}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, \mathrm{d}y, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $\mathcal{B}$ is the collection of all open balls

$$B(x,r) = \Big\{ y \in \mathbb{R}^n : \ |y-x| < r \Big\}, \qquad r > 0.$$

Definition 1.1. *Let w be a weight on $\mathbb{R}^n$.*

(i) *Then w belongs to the Muckenhoupt class $\mathcal{A}_p$, $1 < p < \infty$, if there exists a constant $0 < A < \infty$ such that for all balls B the following inequality holds*

$$\left(\frac{1}{|B|} \int_B w(x)\,\mathrm{d}x \right)^{1/p} \cdot \left(\frac{1}{|B|} \int_B w(x)^{-p'/p}\,\mathrm{d}x \right)^{1/p'} \leqslant A. \tag{1.2}$$

(ii) *Then w belongs to the Muckenhoupt class $\mathcal{A}_1$ if there exists a constant $0 < A < \infty$ such that the inequality*

$$Mw(x) \leqslant Aw(x) \tag{1.3}$$

holds for almost all $x \in \mathbb{R}^n$.

(iii) *The Muckenhoupt class $\mathcal{A}_\infty$ is given by*

$$\mathcal{A}_\infty = \bigcup_{p>1} \mathcal{A}_p. \tag{1.4}$$

Since the pioneering work of Muckenhoupt [23, 24, 25], these classes of weight functions have been studied in great detail, we refer, in particular, to the monographs [11], [32, Ch. V], [33], and [34, Ch. IX] for a complete account on the theory of Muckenhoupt weights. As usual, we use the abbreviation

$$w(\Omega) = \int_\Omega w(x)\,\mathrm{d}x, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is some bounded, measurable set. Then a weight w on $\mathbb{R}^n$ belongs to $\mathcal{A}_p$, $1 \leqslant p < \infty$, if and only if

$$\frac{1}{|B|} \int_B f(y)\,\mathrm{d}y \leqslant \left(\frac{c}{w(B)} \int_B f^p(x) w(x)\,\mathrm{d}x \right)^{1/p}$$

holds for all nonnegative f and all balls B. In particular, with $E \subset B$ and $f = \chi_E$, this implies that

$$\frac{|E|}{|B|} \leqslant c' \left(\frac{w(E)}{w(B)} \right)^{1/r}, \qquad E \subset B, \ w \in \mathcal{A}_r, \ r \geqslant 1. \tag{1.6}$$

Another property of Muckenhoupt weights that will be used in the sequel is that $w \in \mathcal{A}_p$, $p > 1$, implies the existence of some number $r < p$ such that $w \in \mathcal{A}_r$. This is closely connected with the so-called 'reverse Hölder inequality', see [32, Ch. V, Prop. 3, Cor.]. In our case this fact will re-emerge in the number

$$r_w = \inf\{ r \geqslant 1 \ : w \in \mathcal{A}_r \}, \qquad w \in \mathcal{A}_\infty, \tag{1.7}$$

that plays an essential rôle later on. Obviously, $1 \leqslant r_w < \infty$, and $w \in \mathcal{A}_{r_w}$ implies $r_w = 1$.

Examples 1.2.

(i) One of the most prominent examples of a Muckenhoupt weight $w \in \mathcal{A}_p$, $1 \leqslant p < \infty$, is given by $w(x) = |x|^{\varrho}$, where $w \in \mathcal{A}_p$ if and only if $-n < \varrho < n(p-1)$ for $1 < p < \infty$, and $-n < \varrho \leqslant 0$ for $p = 1$. Thus $r_w = 1 + \frac{\varrho_+}{n}$ and $w \in \mathcal{A}_{r_w}$ if $\varrho \leqslant 0$, whereas $w \notin \mathcal{A}_{r_w}$ for $\varrho > 0$.

(ii) We modified this example in [18, 20] by

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^{\alpha}, & |x| < 1, \\ |x|^{\beta}, & |x| \geqslant 1, \end{cases} \tag{1.8}$$

and

$$w_{\log}(x) = \begin{cases} |x|^{\alpha}(1 - \log|x|)^{\gamma}, & |x| < 1, \\ |x|^{\beta}(1 + \log|x|)^{\delta}, & |x| \geqslant 1, \end{cases} \tag{1.9}$$

where $\alpha, \beta > -n$, and $\gamma, \delta \in \mathbb{R}$. Plainly, $w_{\alpha,\beta} = w_{\log}$ when $\gamma = \delta = 0$. Straightforward calculation shows that for $1 < r < \infty$,

$$w_{\log} \in \mathcal{A}_r \qquad \text{if and only if} \qquad -n < \alpha, \beta < n(r-1), \quad \gamma,\ \delta \in \mathbb{R},$$

such that $r_{w_{\alpha,\beta}} = r_{w_{\log}} = 1 + \frac{\max(\alpha,\beta,0)}{n}$ independent of $\gamma, \delta \in \mathbb{R}$. Moreover, $w_{\log} \in \mathcal{A}_1$ when

$$\max(\alpha, \beta) \leqslant 0 \qquad \text{and} \qquad \begin{cases} \gamma \geqslant 0 & \text{if } \alpha = 0, \\ \delta \leqslant 0 & \text{if } \beta = 0. \end{cases} \tag{1.10}$$

(iii) Finally we recall a 'fractal' example studied in [17]. Let $\Gamma \subset \mathbb{R}^n$ be a *d-set*, $0 < d < n$, in the sense of [37, Def. 3.1], [21] (which is different from [8]), i.e., there exists a Borel measure μ in $\mathbb{R}^n$ such that $\operatorname{supp} \mu = \Gamma$ compact, and there are constants $c_1, c_2 > 0$ such that for arbitrary $\gamma \in \Gamma$ and all $0 < r < 1$ holds

$$c_1 r^d \leqslant \mu(B(\gamma, r) \cap \Gamma) \leqslant c_2 r^d.$$

We proved in [17] that the weight $w_{\varkappa,\Gamma}$, given by

$$w_{\varkappa,\Gamma}(x) = \begin{cases} \operatorname{dist}(x,\Gamma)^{\varkappa}, & \text{if} \quad \operatorname{dist}(x,\Gamma) \leqslant 1, \\ 1, & \text{if} \quad \operatorname{dist}(x,\Gamma) \geqslant 1, \end{cases} \tag{1.11}$$

satisfies

$$w_{\varkappa,\Gamma} \in \mathcal{A}_p \qquad \text{if and only if} \qquad -(n-d) < \varkappa < (n-d)(p-1), \quad 1 < p < \infty,$$

and $w_{\varkappa,\Gamma} \in \mathcal{A}_1$ if $-(n-d) < \varkappa \leqslant 0$. Consequently, $r_{w_{\varkappa,\Gamma}} = 1 + \frac{\max(\varkappa,0)}{n-d}$.

For further examples we refer to [9, 18, 19].

We need some refined study of the singularity behavior of Muckenhoupt $\mathcal{A}_\infty$ weights. Let for $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$, $Q_{\nu,m}$ denote the n-dimensional cube with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$ and with side length $2^{-\nu}$. In [19] we introduced the following notion of their *set of singularities* $\mathbf{S}_{\mathrm{sing}}(w)$.

Definition 1.3. *For $w \in \mathcal{A}_\infty$ we define the* set of singularities $\mathbf{S}_{\mathrm{sing}}(w)$ *by*

$$\mathbf{S}_{\mathrm{sing}}(w) = \mathbf{S}_0(w) \;\cup\; \mathbf{S}_\infty(w),$$

where

$$\mathbf{S}_0(w) = \left\{ x_0 \in \mathbb{R}^n : \inf_{Q_{\nu,m} \ni x_0} \frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} = 0 \right\},$$

$$\mathbf{S}_\infty(w) = \left\{ x_0 \in \mathbb{R}^n : \sup_{Q_{\nu,m} \ni x_0} \frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} = \infty \right\}.$$

Remark 1.4. This is a special case of $\mathbf{S}_{\mathrm{sing}}(w_1, w_2)$ defined in [19] with $w_2 \equiv 1$, $w_1 \equiv w$.

Examples 1.5. Let $w_{\log}$ be given by (1.9) such that

$$\frac{w_{\log}(Q_{\nu,m})}{|Q_{\nu,m}|} \sim \begin{cases} 2^{-\nu\alpha}(1+\nu)^\gamma & \text{if} \quad m = 0, \\ \left|2^{-\nu}m\right|^\alpha \left(1 - \log\left|2^{-\nu}m\right|\right)^\gamma & \text{if} \quad 1 \leqslant |m| < 2^\nu, \\ \left|2^{-\nu}m\right|^\beta \left(1 + \log\left|2^{-\nu}m\right|\right)^\delta & \text{if} \quad |m| \geqslant 2^\nu. \end{cases} \tag{1.12}$$

Hence

$$\mathbf{S}_0(w_{\log}) = \begin{cases} \{0\}, & \text{if} \quad \alpha > 0 \quad \text{or} \quad \alpha = 0,\ \gamma < 0, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$\mathbf{S}_\infty(w_{\log}) = \begin{cases} \{0\}, & \text{if} \quad \alpha < 0 \quad \text{or} \quad \alpha = 0,\ \gamma > 0, \\ \emptyset, & \text{otherwise}, \end{cases}$$

such that

$$\mathbf{S}_{\mathrm{sing}}(w_{\log}) = \begin{cases} \{0\}, & \text{if} \quad \alpha \neq 0 \quad \text{or} \quad \alpha = 0,\ \gamma \neq 0, \\ \emptyset, & \text{otherwise}; \end{cases}$$

in particular,

$$\mathbf{S}_{\mathrm{sing}}(w_{\alpha,\beta}) = \begin{cases} \{0\}, & \text{if} \quad \alpha \neq 0, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{1.13}$$

In case of the weight $w_{\varkappa,\Gamma}$ introduced in (1.11) where Γ is a d-set in $\mathbb{R}^n$ with $0<d<n$ and $\varkappa>-(n-d)$, one can prove similar to our above considerations that

$$\mathbf{S}_{\mathrm{sing}}(w_{\varkappa,\Gamma})=\begin{cases}\Gamma=\mathbf{S}_0(w_{\varkappa,\Gamma}),\ \mathbf{S}_\infty(w_{\varkappa,\Gamma})=\emptyset, & \text{if } \varkappa>0,\\ \emptyset=\mathbf{S}_0(w_{\varkappa,\Gamma})=\mathbf{S}_\infty(w_{\varkappa,\Gamma}), & \text{if } \varkappa=0,\\ \Gamma=\mathbf{S}_\infty(w_{\varkappa,\Gamma}),\ \mathbf{S}_0(w_{\varkappa,\Gamma})=\emptyset, & \text{if } \varkappa<0,\end{cases}$$

based on the estimate

$$\frac{w_{\varkappa,\Gamma}(Q_{\nu,m})}{|Q_{\nu,m}|}\sim\begin{cases}1, & \text{if}\quad 2Q_{\nu,m}\cap\Gamma=\emptyset,\\ 2^{-\nu\varkappa}, & \text{otherwise},\end{cases}\tag{1.14}$$

see [17].

Remark 1.6. Note that we always have $|\mathbf{S}_{\mathrm{sing}}(w)|=0$ for $w\in\mathcal{A}_\infty$, cf. [19].

1.2. Function spaces of type $B^s_{p,q}(\mathbb{R}^n,w)$ and $F^s_{p,q}(\mathbb{R}^n,w)$ with $w\in\mathcal{A}_\infty$

Let $w\in\mathcal{A}_\infty$ be a Muckenhoupt weight and $0<p<\infty$. Then the weighted Lebesgue space $L_p(\mathbb{R}^n,w)$ contains all measurable functions such that

$$\|f\,|L_p(\mathbb{R}^n,w)\|=\left(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}\tag{1.15}$$

is finite. For $p=\infty$ one obtains the classical (unweighted) Lebesgue space,

$$L_\infty(\mathbb{R}^n,w)=L_\infty(\mathbb{R}^n),\qquad w\in\mathcal{A}_\infty;\tag{1.16}$$

we thus mainly restrict ourselves to $p<\infty$ in what follows.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0=\varphi\in\mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\varphi\subset\{y\in\mathbb{R}^n:|y|<2\}\qquad\text{and}\qquad\varphi(x)=1\quad\text{if}\quad|x|\leqslant 1\,,$$

and for each $j\in\mathbb{N}$ let $\varphi_j(x)=\varphi(2^{-j}x)-\varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^\infty$ forms a *smooth dyadic resolution of unity.* Given any $f\in\mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively. Let $f\in\mathcal{S}'(\mathbb{R}^n)$, then the Paley-Wiener-Schwartz theorem implies that $\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)$ is an entire analytic function on $\mathbb{R}^n$.

Definition 1.7. *Let $w\in\mathcal{A}_\infty$, $0<q\leqslant\infty$, $0<p<\infty$, $s\in\mathbb{R}$ and $\{\varphi_j\}_{j\in\mathbb{N}_0}$ a smooth dyadic resolution of unity.*

(i) The weighted Besov space $B^s_{p,q}(\mathbb{R}^n,w)$ *is the set of all distributions $f\in\mathcal{S}'(\mathbb{R}^n)$ such that*

$$\big\|f|B^s_{p,q}(\mathbb{R}^n,w)\big\|=\Big\|\big\{2^{js}\big\|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)|L_p(\mathbb{R}^n,w)\big\|\big\}_{j\in\mathbb{N}_0}|\ell_q\Big\|\tag{1.17}$$

is finite.

(ii) The weighted Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n, w)$ *is the set of all distributions* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\big\| f | F^s_{p,q}(\mathbb{R}^n, w) \big\| = \Big\| \big\| \big\{ 2^{js} |\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)| \big\}_{j \in \mathbb{N}_0} | \ell_q \big\| \; \Big| L_p(\mathbb{R}^n, w) \Big\| \tag{1.18}$$

is finite.

Remark 1.8. The spaces $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, q \geqslant 1$), and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, where the first embedding is dense if $q < \infty$, similarly for the F-case; cf. [4]. Moreover, for $w_0 \equiv 1 \in \mathcal{A}_\infty$ these are the usual (unweighted) Besov and Triebel-Lizorkin spaces; we refer, in particular, to the series of monographs [35, 36, 37, 38, 39] for a comprehensive treatment of the unweighted spaces.

The above spaces with weights of type $w \in \mathcal{A}_\infty$ have been studied systematically in [4, 5], with subsequent papers [6, 7]. It turned out that many of the results from the unweighted situation have weighted counterparts: e.g., we have $F^0_{p,2}(\mathbb{R}^n, w) = h_p(\mathbb{R}^n, w)$, $0 < p < \infty$, where the latter are Hardy spaces, see [4, Thm. 1.4], and, in particular, $h_p(\mathbb{R}^n, w) = L_p(\mathbb{R}^n, w) = F^0_{p,2}(\mathbb{R}^n, w)$, $1 < p < \infty$, $w \in \mathcal{A}_p$, see [33, Ch. VI, Thm. 1]. Concerning (classical) Sobolev spaces $W^k_p(\mathbb{R}^n, w)$ (built upon $L_p(\mathbb{R}^n, w)$ in the usual way) it holds

$$W^k_p(\mathbb{R}^n, w) = F^k_{p,2}(\mathbb{R}^n, w), \qquad k \in \mathbb{N}_0, \quad 1 < p < \infty, \quad w \in \mathcal{A}_p, \tag{1.19}$$

cf. [4, Thm. 2.8]. Further details can be found in [4, 5, 11, 26, 27, 10, 2, 3]. In [28] the above class of weights was extended in order to incorporate locally regular weights, too, creating in that way the class $\mathcal{A}^{\ell oc}_p$. We partly rely on our approaches in [17, 18, 19, 20].

Remark 1.9. In [4, Thm. 2.8] it is proved that for $w \in \mathcal{A}_\infty$ the operator J_σ, given by $\mathcal{F} J_\sigma(x) = (1 + 4\pi^2 |x|^2)^{-\sigma/2}$, $\sigma \in \mathbb{R}$, is an isomorphism of $B^s_{p,q}(\mathbb{R}^n, w)$ onto $B^{s+\sigma}_{p,q}(\mathbb{R}^n, w)$ and from $F^s_{p,q}(\mathbb{R}^n, w)$ onto $F^{s+\sigma}_{p,q}(\mathbb{R}^n, w)$, parallel to the unweighted case.

We briefly recall the definition of atoms.

Definition 1.10. *Let* $K \in \mathbb{N}_0$ *and* $b > 1$.

(i) *The complex-valued function* $a \in C^K(\mathbb{R}^n)$ *is said to be an* 1_K-atom *if* $\operatorname{supp} a \subset bQ_{0,m}$ *for some* $m \in \mathbb{Z}^n$, *and* $|\mathrm{D}^\alpha a(x)| \leqslant 1$ *for* $|\alpha| \leqslant K$, $x \in \mathbb{R}^n$.

(ii) *Let* $s \in \mathbb{R}$, $0 < p \leqslant \infty$, *and* $L + 1 \in \mathbb{N}_0$. *The complex-valued function* $a \in C^K(\mathbb{R}^n)$ *is said to be an* $(s,p)_{K,L}$-atom *if for some* $\nu \in \mathbb{N}_0$,

$$\operatorname{supp} a \subset bQ_{\nu,m} \qquad \textit{for some } m \in \mathbb{Z}^n,$$

$$|\mathrm{D}^\alpha a(x)| \leqslant 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu} \qquad \textit{for } |\alpha| \leqslant K,\ x \in \mathbb{R}^n,$$

$$\int_{\mathbb{R}^n} x^\beta a(x)\, \mathrm{d}x = 0 \qquad \textit{for } |\beta| \leqslant L.$$

We shall denote an atom $a(x)$ supported in some $Q_{\nu,m}$ by $a_{\nu,m}$ in the sequel. Choosing $L = -1$ in (ii) means that no moment conditions are required. For $0 < p < \infty$, $0 < q \leqslant \infty$, $w \in \mathcal{A}_\infty$, we introduce suitable sequence spaces $b_{pq}(w)$ by

$$b_{pq}(w) = \Bigg\{ \lambda = \{\lambda_{\nu,m}\}_{\nu,m} : \ \lambda_{\nu,m} \in \mathbb{C}, \\ \|\lambda | b_{pq}(w)\| \sim \left\| \left\{ \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p 2^{\nu n} w(Q_{\nu,m}) \Big)^{\frac{1}{p}} \right\}_{\nu\in\mathbb{N}_0} |\ell_q \right\| < \infty \Bigg\}.$$

For convenience we adopt the usual notation

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ , \qquad 0 < p \leqslant \infty. \tag{1.20}$$

Then the atomic decomposition result used below reads as follows.

Proposition 1.11. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$, and $w \in \mathcal{A}_\infty$ be a weight with r_w given by* (1.7). *Let $K, L+1 \in \mathbb{N}_0$ with*

$$K \geqslant (1 + \lfloor s \rfloor)_+ \qquad \text{and} \qquad L \geqslant \max\left(-1, \lfloor \sigma_{p/r_w} - s \rfloor\right). \tag{1.21}$$

Then $f \in \mathcal{S}(\mathbb{R}^n)$ belongs to $B^s_{p,q}(\mathbb{R}^n, w)$ if and only if it can be written as a series

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu,m}(x), \qquad \text{converging in } \ \mathcal{S}'(\mathbb{R}^n), \tag{1.22}$$

where $a_{\nu,m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s,p)_{K,L}$-atoms ($\nu \in \mathbb{N}$) and $\lambda \in b_{pq}(w)$. Furthermore,

$$\inf \|\lambda | b_{pq}(w)\| \tag{1.23}$$

is an equivalent quasi-norm in $B^s_{pq}(\mathbb{R}^n, w)$, where the infimum ranges over all admissible representations (1.22).

Remark 1.12. The above result coincides with [17, Thm. 3.10], cf. also [2, Thm. 5.10]. There are parallel F-results, too.

Notational agreement. We adopt the nowadays usual custom to write $A^s_{p,q}$ instead of $B^s_{p,q}$ or $F^s_{p,q}$, respectively, when both scales of spaces are meant simultaneously in some context.

1.3. Continuous Embeddings

We collect some embedding results for weighted spaces that will be used later. First we formulate the most general criterion obtained in [18], which is afterwards specified for one-weight situations and finally explicated for our model weights. Recall that we deal with function spaces on $\mathbb{R}^n$ exclusively, and will thus omit the '$\mathbb{R}^n$' from their notation.

Proposition 1.13. *Let w_1 and w_2 be two $\mathcal{A}_\infty$ weights and let $-\infty < s_2 \leqslant s_1 < \infty$, $0 < p_1, p_2 \leqslant \infty$, $0 < q_1, q_2 \leqslant \infty$. We put*

$$\frac{1}{p^*} = \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \qquad \text{and} \qquad \frac{1}{q^*} = \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ . \tag{1.24}$$

Then

$$\mathrm{id} : B^{s_1}_{p_1,q_1}(w_1) \hookrightarrow B^{s_2}_{p_2,q_2}(w_2) \tag{1.25}$$

is continuous if and only if

$$\left\{ 2^{-\nu(s_1-s_2)} \left\| \left\{ \frac{w_2(Q_{\nu,m})^{1/p_2}}{w_1(Q_{\nu,m})^{1/p_1}} \right\}_{m\in\mathbb{Z}^n} |\ell_{p^*} \right\| \right\}_{\nu\in\mathbb{N}_0} \in \ell_{q^*} . \tag{1.26}$$

Remark 1.14. For the proof and further details, also concerning questions of compactness, we refer to [18]. In view of (1.16) it is clear that we obtain unweighted Besov spaces if $p_1 = p_2 = \infty$. Then by (1.5), $w_1(Q_{\nu,m}) = w_2(Q_{\nu,m}) = 2^{-\nu n}$ for all $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, such that (1.26) leads to $p^* = \infty$, i.e., $p_1 \leqslant p_2$, and

$$\boldsymbol{\delta}_* = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0, \tag{1.27}$$

with the extension to $\boldsymbol{\delta}_* = 0$ if $q_1 \leqslant q_2$, i.e., $q^* = \infty$.

In [18, 19] we concentrated on the interplay between smoothness parameters and properties of the weight in the following sense.

Corollary 1.15. *Let $w \in \mathcal{A}_\infty$ with r_w given by (1.7), and*

$$-\infty < s_2 \leqslant s_1 < \infty, \quad 0 < p_1 < \infty, \quad 0 < p_2 \leqslant \infty, \quad 0 < q_1, q_2 \leqslant \infty. \tag{1.28}$$

(i) *Let*

$$\boldsymbol{\delta}_* > \frac{n}{p^*} + \frac{n}{p_1}(r_w - 1). \tag{1.29}$$

Then

$$\mathrm{id}_w : B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2} \tag{1.30}$$

is continuous if and only if

$$\left\| \left\{ w(Q_{0,m})^{-1/p_1} \right\}_{m\in\mathbb{Z}^n} |\ell_{p^*} \right\| \leqslant c < \infty. \tag{1.31}$$

In particular, if $\mu > \frac{n}{p}(r_w - 1)$, then

$$B^s_{p,q}(w) \hookrightarrow B^{s-\mu}_{p,q} \qquad \text{if and only if} \qquad \inf_m w(Q_{0,m}) \geqslant c > 0. \tag{1.32}$$

(ii) *Let $\boldsymbol{\delta}_* < \frac{n}{p^*}$ or $\boldsymbol{\delta}_* = \frac{n}{p^*}$ and $q^* < \infty$, then $B^{s_1}_{p_1,q_1}(w)$ is not embedded in $B^{s_2}_{p_2,q_2}$.*

(iii) *Assume that* $w \in \mathcal{A}_1$ *such that* $r_w = 1$. *Then* id_w *in* (1.30) *is continuous if and only if*

$$\begin{cases} \inf_m w(Q_{0,m}) \geqslant c > 0, & \text{and} \\ \left\{2^{-j\boldsymbol{\delta}_*}\right\}_{j\in\mathbb{N}_0} \in \ell_{q^*}, & \text{and} \\ p_1 \leqslant p_2, \end{cases} \tag{1.33}$$

in particular,

$$B^s_{p,q}(w) \hookrightarrow B^s_{p,q} \qquad \text{if and only if} \qquad \inf_m w(Q_{0,m}) \geqslant c > 0. \tag{1.34}$$

Finally we give the complete description in case of our special weights $w_{\log}$, cf. [20], and $w_{\varkappa,\Gamma}$.

Corollary 1.16. *Let the parameters be given by* (1.28), *and* $w_{\log}$ *by* (1.9) *with* $\alpha, \beta > -n$, $\gamma, \delta \in \mathbb{R}$. *The embedding*

$$\mathrm{id}_{\log} : B^{s_1}_{p_1,q_1}(w_{\log}) \hookrightarrow B^{s_2}_{p_2,q_2} \tag{1.35}$$

is continuous if and only if

$$\begin{cases} \text{either} \ \frac{\beta}{p_1} > \frac{n}{p^*}, & \delta \in \mathbb{R}, \\ \text{or} \quad \frac{\beta}{p_1} = \frac{n}{p^*}, & \frac{\delta}{p_1} > \frac{1}{p^*} \quad \text{if} \quad p^* < \infty, \\ \qquad \beta = 0, & \delta \geqslant 0 \quad \text{if} \quad p^* = \infty, \end{cases} \tag{1.36}$$

and one of the following conditions is satisfied,

$$\begin{cases} \left\{2^{-\nu(\boldsymbol{\delta}_* - \frac{\alpha}{p_1})}(1+\nu)^{-\frac{\gamma}{p_1}}\right\}_\nu \in \ell_{q^*} & \text{if} \ \frac{\alpha}{p_1} > \frac{n}{p^*}, \ \gamma \in \mathbb{R}, \\ \left\{2^{-\nu(\boldsymbol{\delta}_* - \frac{n}{p^*})}\right\}_\nu \in \ell_{q^*} & \text{if} \ \frac{\alpha}{p_1} < \frac{n}{p^*}, \ \gamma \in \mathbb{R}, \ \text{or} \ \frac{\alpha}{p_1} = \frac{n}{p^*}, \ \frac{\gamma}{p_1} > \frac{1}{p^*}, \\ \left\{2^{-\nu(\boldsymbol{\delta}_* - \frac{n}{p^*})}(1+\nu)^{\frac{1}{p^*} - \frac{\gamma}{p_1}}\right\}_\nu \in \ell_{q^*} & \text{if} \ \frac{\alpha}{p_1} = \frac{n}{p^*}, \ \frac{\gamma}{p_1} < \frac{1}{p^*}, \\ \left\{2^{-\nu(\boldsymbol{\delta}_* - \frac{n}{p^*})}\log^{\frac{1}{p^*}}(1+\nu)\right\}_\nu \in \ell_{q^*} & \text{if} \ \frac{\alpha}{p_1} = \frac{n}{p^*}, \ \frac{\gamma}{p_1} = \frac{1}{p^*}. \end{cases}$$

In particular,

$$B^s_{p,q}(w_{\log}) \hookrightarrow B^s_{p,q} \qquad \text{if and only if} \qquad \begin{cases} \beta \geqslant 0, \ \delta \in \mathbb{R} & \text{with } \delta \geqslant 0 \text{ if } \beta = 0, \\ \text{and} \\ \alpha \leqslant 0, \ \gamma \in \mathbb{R} & \text{with } \gamma \geqslant 0 \text{ if } \alpha = 0. \end{cases} \tag{1.37}$$

Remark 1.17. Plainly, $w_{\alpha,\beta} = w_{\log}$ for $\gamma = \delta = 0$, such that (1.37) reads in this special case as

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^s_{p,q} \qquad \text{if and only if} \qquad \alpha \leqslant 0 \leqslant \beta, \tag{1.38}$$

and

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^{s - \frac{\max(\alpha,0)}{p}}_{p,q} \qquad \text{if and only if} \qquad \beta \geqslant 0. \tag{1.39}$$

Corollary 1.18. *Let $\Gamma \subset \mathbb{R}^n$ be a d-set, $0 < d < n$, and $w_{\varkappa,\Gamma}$ be given by* (1.11) *with $\varkappa > -(n-d)$. Let the parameters satisfy* (1.28). *The embedding*

$$\mathrm{id}_{\varkappa,\Gamma} : B^{s_1}_{p_1,q_1}(w_{\varkappa,\Gamma}) \hookrightarrow B^{s_2}_{p_2,q_2} \tag{1.40}$$

is continuous if and only if

$$p_1 \leqslant p_2 \qquad \text{and} \qquad \left\{2^{-\nu(\boldsymbol{\delta}_* - \frac{\max(\varkappa,0)}{p_1})}\right\}_{\nu\in\mathbb{N}_0} \in \ell_{q^*}. \tag{1.41}$$

In particular,

$$B^s_{p,q}(w_{\varkappa,\Gamma}) \hookrightarrow B^s_{p,q} \qquad \text{if and only if} \qquad \varkappa \leqslant 0, \tag{1.42}$$

and

$$B^s_{p,q}(w_{\varkappa,\Gamma}) \hookrightarrow B^{s-\frac{\max(\varkappa,0)}{p}}_{p,q}. \tag{1.43}$$

In [18, 19] we also considered situations where both source and target space are weighted with the same $w \in \mathcal{A}_\infty$. Here we shall only need the following basic observation.

Proposition 1.19. *Let $0 < q \leqslant \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty$.*

(i) *Let $-\infty < s_1 \leqslant s_0 < \infty$ and $0 < q_0 \leqslant q_1 \leqslant \infty$, then*

$$A^{s_0}_{p,q}(w) \hookrightarrow A^{s_1}_{p,q}(w) \qquad \text{and} \qquad A^s_{p,q_0}(w) \hookrightarrow A^s_{p,q_1}(w).$$

(ii) *We have*

$$B^s_{p,\min(p,q)}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^s_{p,\max(p,q)}(w). \tag{1.44}$$

(iii) *Assume that there are numbers $c > 0$, $d > 0$ such that for all cubes,*

$$w\left(Q_{\nu,m}\right) \geqslant c2^{-\nu d}, \qquad \nu \in \mathbb{N}_0, \qquad m \in \mathbb{Z}^n. \tag{1.45}$$

Let $0 < p_0 < p < p_1 < \infty$, $-\infty < s_1 < s < s_0 < \infty$ satisfy

$$s_0 - \frac{d}{p_0} = s - \frac{d}{p} = s_1 - \frac{d}{p_1}. \tag{1.46}$$

Then

$$B^{s_0}_{p_0,q}(w) \hookrightarrow B^{s_1}_{p_1,q}(w), \tag{1.47}$$

and

$$B^{s_0}_{p_0,p}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,p}(w). \tag{1.48}$$

Remark 1.20. These embeddings are natural extensions from the unweighted case $w \equiv 1$, see [35, Prop. 2.3.2/2, Thm. 2.7.1] and [30, Thm. 3.2.1]. The above result essentially coincides with [4, Thm. 2.6] and can be found in [18, Prop. 1.8].

Assume that $\inf_{m\in\mathbb{Z}^n} w(Q_{0,m}) \geqslant c > 0$, then (1.6) implies $d \geqslant nr_w$ in (1.45). In particular, for our model weights $w_{\alpha,\beta}$, $w_{\log}$ and $w_{\varkappa,\Gamma}$ the embeddings (1.47) and (1.48) can be exemplified as follows, recall also (1.12) and (1.14).

Example 1.21. Let $0 < p_0 < p < p_1 < \infty$, $-\infty < s_1 < s < s_0 < \infty$, $0 < q \leqslant \infty$.

(i) Let $w_{\alpha,\beta}$ and $w_{\log}$ be given by (1.8) and (1.9), respectively, with $\alpha > -n$, $\beta \geqslant 0$, $\gamma, \delta \in \mathbb{R}$, and $\delta \geqslant 0$ if $\beta = 0$, and $\gamma \geqslant 0$ if $\alpha \geqslant 0$. Assume that

$$s_0 - \frac{\max(\alpha,0)+n}{p_0} = s - \frac{\max(\alpha,0)+n}{p} = s_1 - \frac{\max(\alpha,0)+n}{p_1}. \tag{1.49}$$

Then (1.47) and (1.48) hold for $w = w_{\alpha,\beta}$ or $w = w_{\log}$, respectively.

(ii) Let $\Gamma \subset \mathbb{R}^n$ be a d-set, $0 < d < n$, and $w_{\varkappa,\Gamma}$ be given by (1.11) with $\varkappa > -(n-d)$. Assume that

$$s_0 - \frac{\max(\varkappa,0)+n}{p_0} = s - \frac{\max(\varkappa,0)+n}{p} = s_1 - \frac{\max(\varkappa,0)+n}{p_1}. \tag{1.50}$$

Then (1.47) and (1.48) hold for $w = w_{\varkappa,\Gamma}$.

2. Envelopes

2.1. Definition and basic properties

Let for some measurable function $f : \mathbb{R}^n \to \mathbb{C}$, finite a.e., its decreasing rearrangement f^* be defined as usual,

$$f^*(t) = \inf\{s \geqslant 0 : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \leqslant t\}, \qquad t \geqslant 0 .$$

Definition 2.1. *Let X be some quasi-normed function space on $\mathbb{R}^n$.*

(i) *The* growth envelope function $\mathcal{E}_{\mathsf{G}}^X : (0,\infty) \to [0,\infty]$ *of X is defined by*

$$\mathcal{E}_{\mathsf{G}}^X(t) = \sup_{\|f|X\|\leqslant 1} f^*(t), \qquad t > 0. \tag{2.1}$$

(ii) *Assume $X \not\hookrightarrow L_\infty(\mathbb{R}^n)$. Let $\varepsilon \in (0,1)$, $H(t) = -\log \mathcal{E}_{\mathsf{G}}^X(t)$, $t \in (0,\varepsilon]$, and let μ_H be the associated Borel measure. The number u_{G}^X, $0 < u_{\mathsf{G}}^X \leqslant \infty$, is defined as the infimum of all numbers v, $0 < v \leqslant \infty$, such that*

$$\left(\int_0^\varepsilon \left(\frac{f^*(t)}{\mathcal{E}_{\mathsf{G}}^X(t)}\right)^v \mu_H(\mathrm{d}t)\right)^{1/v} \leqslant c\,\|f|X\| \tag{2.2}$$

(with the usual modification if $v = \infty$) holds for some $c > 0$ and all $f \in X$. The couple

$$\mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}_{\mathsf{G}}^X(\cdot), u_{\mathsf{G}}^X\right)$$

is called (local) growth envelope *for the function space X.*

This concept was introduced and first studied in [38, Ch. 2], [12], see also [13]. For convenience we recall some properties. In view of (i) we obtain – strictly speaking – equivalence classes of growth envelope functions when working with equivalent quasi-norms in X as we shall usually do. But we do not want to distinguish between representative and equivalence class in what follows and thus stick at the notation introduced in (i). Concerning (ii) we shall assume that we can choose a continuous representative in the equivalence class $[\mathcal{E}_{\mathsf{G}}^X]$, for convenience (but in a slight abuse of notation) denoted by $\mathcal{E}_{\mathsf{G}}^X$ again. It is obvious that (2.2) holds for $v = \infty$ and any X. Moreover, one verifies that

$$\sup_{0<t\leqslant\varepsilon} \frac{g(t)}{\mathcal{E}_{\mathsf{G}}^X(t)} \leqslant c_1 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_{\mathsf{G}}^X(t)} \right)^{v_1} \mu_H(\mathrm{d}t) \right)^{\frac{1}{v_1}} \leqslant c_2 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_{\mathsf{G}}^X(t)} \right)^{v_0} \mu_H(\mathrm{d}t) \right)^{\frac{1}{v_0}}$$

for $0 < v_0 < v_1 < \infty$ and all non-negative monotonically decreasing functions g on $(0, \varepsilon]$; cf. [38, Prop. 12.2]. So with $g = f^*$ we observe that the left-hand sides in (2.2) are monotonically ordered in v and it is natural to look for the smallest possible one.

Proposition 2.2.

(i) *Let $X_i \hookrightarrow L_\infty$, $i = 1, 2$, be some function spaces on $\mathbb{R}^n$. Then $X_1 \hookrightarrow X_2$ implies that there is some positive constant c such that for all $t > 0$,*

$$\mathcal{E}_{\mathsf{G}}^{X_1}(t) \leqslant c\,\mathcal{E}_{\mathsf{G}}^{X_2}(t). \tag{2.3}$$

(ii) *We have $X \hookrightarrow L_\infty$ if and only if $\mathcal{E}_{\mathsf{G}}^X$ is bounded.*

(iii) *Let X_i, $i = 1, 2$, be some function spaces on $\mathbb{R}^n$ with $X_1 \hookrightarrow X_2$. Assume for their growth envelope functions*

$$\mathcal{E}_{\mathsf{G}}^{X_1}(t) \sim \mathcal{E}_{\mathsf{G}}^{X_2}(t), \qquad t \in (0, \varepsilon),$$

for some $\varepsilon > 0$. Then we get for the corresponding indices $u_{\mathsf{G}}^{X_i}$, $i = 1, 2$, that

$$u_{\mathsf{G}}^{X_1} \leqslant u_{\mathsf{G}}^{X_2}. \tag{2.4}$$

This result coincides with [13, Props. 3.4, 4.5].

Remark 2.3. For rearrangement-invariant Banach function spaces X with fundamental function φ_X we proved in [13, Sect. 2.3] that

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \frac{1}{\varphi_X(t)} = \left\| \chi_{A_t} | X \right\|^{-1}, \qquad t > 0, \tag{2.5}$$

where $A_t \subset \mathbb{R}^n$ with $|A_t| = t$.

In contrast to the local characterisation in Definition 2.1(ii) it turned out, that sometimes also the *global behavior* of the envelope function,

$$\mathcal{E}_{\mathsf{G}}^X(t) \qquad \text{for } t \to \infty$$

is of interest, in particular in weighted spaces.

2.2. Growth envelopes in unweighted spaces

We briefly summarize some results for unweighted spaces, in particular, for Besov and Triebel-Lizorkin spaces and Lorentz-Zygmund spaces $L_{p,q}(\log L)_a$; for definitions and further details of the latter we refer to [1, Ch. 4, Defs. 4.1, 6.13].

Proposition 2.4.

(i) *Let* $0 < p \leqslant \infty$ *(with* $p < \infty$ *in* F*-case),* $0 < q \leqslant \infty$*,* $s \geqslant \sigma_p$*. Then*

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right), & \text{if} \quad \sigma_p < s < \frac{n}{p},\ 0 < q \leqslant \infty, \\ \left(|\log t|^{\frac{1}{q'}}, q\right), & \text{if} \quad s = \frac{n}{p},\ 1 < q \leqslant \infty, \\ \left(t^{-\frac{1}{p}+\frac{\sigma_p}{n}}, p\right), & \text{if} \quad s = \sigma_p,\ 1 \leqslant p < \infty,\ 0 < q \leqslant \min(p,2), \\ \left(t^{-\frac{1}{p}+\frac{\sigma_p}{n}}, q\right), & \text{if} \quad s = \sigma_p,\ 0 < p < 1,\ 0 < q \leqslant 1, \end{cases} \tag{2.6}$$

and

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right), & \text{if} \quad \sigma_p < s < \frac{n}{p},\ 0 < q \leqslant \infty, \\ \left(|\log t|^{\frac{1}{p'}}, p\right), & \text{if} \quad s = \frac{n}{p},\ 1 < p < \infty, \\ \left(t^{-\frac{1}{p}+\frac{\sigma_p}{n}}, p\right), & \text{if} \quad s = \sigma_p,\ 1 \leqslant p < \infty,\ 0 < q \leqslant 2, \\ \left(t^{-\frac{1}{p}+\frac{\sigma_p}{n}}, p\right), & \text{if} \quad s = \sigma_p,\ 0 < p < 1,\ 0 < q \leqslant \infty. \end{cases} \tag{2.7}$$

For the global behavior we obtain for $s > \sigma_p$ *that*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}}(t) \sim t^{-\frac{1}{p}}, \qquad t \to \infty. \tag{2.8}$$

(ii) *Let* $0 < p < \infty$*,* $0 < q \leqslant \infty$*,* $a \in \mathbb{R}$*. Then*

$$\mathfrak{E}_{\mathsf{G}}\left(L_{p,q}(\log L)_a\right) = \left(t^{-\frac{1}{p}}|\log t|^{-a},\ q\right), \tag{2.9}$$

in particular,

$$\mathfrak{E}_{\mathsf{G}}\left(L_p\right) = \left(t^{-\frac{1}{p}}, p\right), \tag{2.10}$$

with

$$\mathcal{E}_{\mathsf{G}}^{L_{p,q}(\log L)_a}(t) \sim t^{-\frac{1}{p}}|\log t|^{-a}, \qquad t \to \infty. \tag{2.11}$$

Remark 2.5. For proofs and further discussion in (i) we refer to [13, Thms. 8.1, 8.16, Props. 8.12, 8.14], [38, Sects. 13, 15], [40]; partial results for the case $s = 0$, $p = \infty$, $1 < q \leqslant 2$ are contained in [13, Prop. 8.24] and [29]. Situation (ii) is studied in [13, Thm. 4.7].

Remark 2.6. There is a number of partial results in the weighted setting: in [16] we dealt with growth envelopes of Sobolev spaces $\mathfrak{E}_{\mathsf{G}}\left(W^k_p(w_{\alpha,\beta})\right)$; there are forerunners in [14, 13, 15], which also cover the situation of $A^s_{p,q}(w_{\alpha,\beta})$ in some cases, e.g.,

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-\frac{1}{p}-\frac{\max(\alpha,0)}{np}+\frac{s}{n}}, \qquad 0 < t < 1, \tag{2.12}$$

if $s > 0$, $\beta \geqslant 0$, $-n + \frac{\max(\alpha,0)}{p} < s - \frac{n}{p} < \frac{\max(\alpha,0)}{p}$, see [15, Thm. 3.9]. In [16] first results for $\mathfrak{E}_{\mathsf{G}}\left(L_p(w_{\log})\right)$ were obtained. The so far only approach to the general situation $w \in \mathcal{A}_\infty$ can be found in [15], where we proved

$$c_1 \sup_{|E|=t} \left(\int\limits_E w(x)\,\mathrm{d}x\right)^{-1/p} \leqslant \mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \leqslant c_2 \sup_{E\subset\mathbb{R}^n, |E|=t} \frac{1}{|E|}\left(\int\limits_E w(x)^{-p'/p}\,\mathrm{d}x\right)^{1/p'}, \tag{2.13}$$

and conjectured that

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \sim \sup_{|B|=t}\left(\int_B w(x)\,\mathrm{d}x\right)^{-1/p}, \qquad w \in \mathcal{A}_p.$$

3. Growth envelope for $w \in \mathcal{A}_1$

In this section we characterize the singularity behavior of $A^s_{p,q}(\mathbb{R}^n, w) = A^s_{p,q}(w)$ where $w \in \mathcal{A}_1$. As a preparation we characterize the parameters such that $A^s_{p,q}(w) \hookrightarrow L_\infty$, i.e., where no singularity behavior in the sense of growth envelopes appears, and give a sufficient condition such that $A^s_{p,q}(w) \subset L_1^{\mathrm{loc}}$, i.e., where the concept of growth envelopes makes sense. Borderline situations $s = \sigma_p$ are out of the scope of the present approach.

Lemma 3.1. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, $w \in \mathcal{A}_1$ with*

$$\inf_m w(Q_{0,m}) \geqslant c_w > 0. \tag{3.1}$$

(i) *Let $s > \sigma_p$. Then $A^s_{p,q}(w) \subset L_1^{\mathrm{loc}}$.*
(ii) *Let $s > \frac{n}{p}$ or $s = \frac{n}{p}$ and $0 < q \leqslant 1$. Then*

$$B^s_{p,q}(w) \hookrightarrow L_\infty. \tag{3.2}$$

(iii) *Let $0 < p_0 < p < p_1 < \infty$, $s_1 < s < s_0$ satisfy*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}. \tag{3.3}$$

Then

$$B^{s_0}_{p_0,p}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,p}(w). \tag{3.4}$$

Proof. We use embedding (1.34) with (3.1), thus $B^s_{p,q} \subset L_1^{\mathrm{loc}}$ for $s > \sigma_p$ implies $B^s_{p,q}(w) \subset L_1^{\mathrm{loc}}$; similarly for (3.2) in view of the unweighted result. The extension to F-spaces in (i) is covered by (1.44). Concerning (iii) we apply (1.6) with $E = Q_{\nu,m}$ and $B = Q_{0,m'}$ for appropriate $m' \in \mathbb{Z}^n$ such that $Q_{\nu,m} \subset Q_{0,m'}$; thus (3.1) implies that

$$w(Q_{\nu,m}) \geqslant c\, \frac{w(Q_{\nu,m})}{w(Q_{0,m'})} \geqslant c'\, \frac{|Q_{\nu,m}|}{|Q_{0,m'}|} \geqslant c''\, 2^{-\nu n}, \qquad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{3.5}$$

with a constant independent of ν and m. We apply (1.48) with (1.46) and $d = n$ and obtain (3.4) with (3.3). ■

Remark 3.2. In Corollary 3.5 below we shall prove that for $w \in \mathcal{A}_1$ we have (3.2) if and only if $s > \frac{n}{p}$ or $s = \frac{n}{p}$ and $0 < q \leqslant 1$ (as in the unweighted case), and a counterpart for F-spaces. For the moment we conclude from the above result (and general facts about growth envelopes) that it makes sense to study growth envelopes in spaces $B^s_{p,q}(w)$, $w \in \mathcal{A}_1$, in case of $\sigma_p < s \leqslant \frac{n}{p}$. We do not consider the borderline situation $s = \sigma_p$ here.

3.1. Growth envelope function

We show that whenever $s > \sigma_p$ and $w \in \mathcal{A}_1$ satisfies (3.1), then

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \sim \mathcal{E}_{\mathsf{G}}^{A^s_{p,q}}(t) \qquad \text{for} \quad t \to 0 \quad \text{and} \quad t \to \infty,$$

and similarly for L_p-spaces.

Proposition 3.3. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s > \sigma_p$, $w \in \mathcal{A}_1$ *with* (3.1).

(i) *Let* $s < \frac{n}{p}$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}}, \qquad t \to 0. \tag{3.6}$$

(ii) *Let* $s = \frac{n}{p}$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{A^{n/p}_{p,q}(w)}(t) \sim \begin{cases} |\log t|^{1/q'}, & \textit{if } A^{n/p}_{p,q} = B^{n/p}_{p,q} \textit{ and } 1 < q \leqslant \infty, \\ |\log t|^{1/p'}, & \textit{if } A^{n/p}_{p,q} = F^{n/p}_{p,q} \textit{ and } 1 < p < \infty, \end{cases} \qquad t \to 0. \tag{3.7}$$

(iii) *We obtain*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \sim t^{-\frac{1}{p}}, \qquad t \to \infty. \tag{3.8}$$

(iv) *We have*

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \sim t^{-\frac{1}{p}}, \qquad t > 0. \tag{3.9}$$

Proof. *Step 1.* Note first that it is sufficient to deal with B-spaces only: Assume that we have already proved (i), (ii) and (iii) with $A^s_{p,q} = B^s_{p,q}$; then (1.44) together with (2.3) complete the argument in case of $A^s_{p,q} = F^s_{p,q}$ in (i) and (iii); concerning (ii) we apply Lemma 3.1(iii).

The estimates from above immediately follow from (1.34) together with (2.3), (2.6), (2.8), (2.10) and (2.11).

Step 2. As for the estimates from below we adapt the unweighted arguments appropriately and construct special functions $f_{j,x_0} \in B^s_{p,q}(w)$ with $\|f_{j,x_0}|B^s_{p,q}(w)\| \sim 1$ such that

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(2^{-jn}) \geqslant c \sup_{x_0} f^*_{j,x_0}(2^{-jn}), \qquad j \in \mathbb{N}.$$

We begin with (i). Let for $x_0 \in \mathbb{R}^n$, $j \in \mathbb{N}$,

$$f_{j,x_0}(x) = 2^{-js}\psi\left(2^j(x-x_0)\right) w\left(B(x_0,2^{-j})\right)^{-\frac{1}{p}}, \tag{3.10}$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ is given by

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{if} \quad |x|<1, \\ 0, & \text{if} \quad |x| \geqslant 1; \end{cases} \tag{3.11}$$

thus, for $j \in \mathbb{N}$, $t \sim 2^{-jn}$, we have

$$f^*_{j,x_0}(t) \sim 2^{-js}\psi^*(2^{jn}t)w\left(B(x_0,2^{-j})\right)^{-\frac{1}{p}} \sim 2^{-js}w\left(B(x_0,2^{-j})\right)^{-\frac{1}{p}}. \tag{3.12}$$

We put

$$a_j(x) = 2^{-j(s-\frac{n}{p})}\psi\left(2^j(x-x_0)\right)$$

and observe that these are special atoms according to Definition 1.10, since $\operatorname{supp} a_j \subset \operatorname{supp}\psi\left(2^j(\cdot - x_0)\right) \subset B(x_0,2^{-j})$,

$$|\mathrm{D}^\alpha a_j(x)| \leqslant c_{\alpha,\psi}2^{-j(s-\frac{n}{p})+j|\alpha|}, \qquad |\alpha| \leqslant K, \tag{3.13}$$

and our assumption on s implies that we do not need to impose moment conditions, see (1.21). Now let $\lambda_j = 2^{-j\frac{n}{p}} w\left(B(x_0,2^{-j})\right)^{-\frac{1}{p}}$, then

$$f_{j,x_0}(x) = \lambda_j a_j(x)$$

is a special atomic decomposition (1.22) and we obtain

$$\left\|f_{j,x_0}|B^s_{p,q}(w)\right\| \leqslant \|\lambda|b_{pq}(w)\| \sim \lambda_j 2^{j\frac{n}{p}} w\left(B(x_0,2^{-j})\right)^{\frac{1}{p}} = 1.$$

This leads to

$$\begin{aligned} \mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(2^{-kn}) &\geqslant c \sup_{j,x_0} f^*_{j,x_0}(2^{-kn}) \\ &\geqslant c \sup_{x_0} f^*_{k,x_0}(2^{-kn}) \\ &\geqslant c' \ \sup_{x_0} 2^{-ks} w\left(B(x_0,2^{-k})\right)^{-\frac{1}{p}} \\ &\geqslant c'' \ 2^{-k(s-\frac{n}{p})} \sup_{x_0} \left(\frac{|B(x_0,2^{-k})|}{w\left(B(x_0,2^{-k})\right)}\right)^{\frac{1}{p}}. \end{aligned} \tag{3.14}$$

In view of (3.6) it is sufficient to prove that there exists some $x_0 \in \mathbb{R}^n$ such that

$$\frac{w\left(B(x_0,2^{-k})\right)}{|B(x_0,2^{-k})|} \leqslant c \tag{3.15}$$

independent of $k \in \mathbb{N}_0$; but since $|\mathbf{S}_\infty(w)| \leqslant |\mathbf{S}_{\mathrm{sing}}(w)| = 0$, recall Remark 1.6, we can always find some $x_0 \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$ and this completes the argument for (i).

Step 3. We modify the above approach in order to prove (ii). Let again $x_0 \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$ and put

$$f_m(x) = m^{-\frac{1}{q}} \sum_{j=1}^{m} \psi\left(2^j(x-x_0)\right), \qquad m \in \mathbb{N}. \tag{3.16}$$

Similarly as above, see also [13, Thm. 8.16], we obtain

$$f_m^*(t) \sim m^{-\frac{1}{q}} \begin{cases} m, & t \leqslant 2^{-mn}, \\ |\log t|, & 2^{-mn} \leqslant t < 1. \end{cases}$$

Regarding (3.16) as an atomic decomposition of f_m (with $a_j(x) = \psi\left(2^j(x-x_0)\right)$, $\lambda_j = m^{-\frac{1}{q}}$, $j = 1, \ldots, m$), we conclude that

$$\left\| f_m | B^{n/p}_{p,q}(w) \right\| \leqslant \|\lambda | b_{pq}(w)\| \sim m^{-\frac{1}{q}} \left(\sum_{j=1}^{m} 2^{j\frac{n}{p}q} w\left(B(x_0,2^{-j})\right)^{\frac{q}{p}} \right)^{1/q} \leqslant c, \tag{3.17}$$

where we applied (3.15) with $j = k$ and $x_0 \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$. The rest is similar to Step 2,

$$\mathcal{E}_{\mathsf{G}}^{B^{n/p}_{p,q}(w)}(2^{-kn}) \geqslant c \sup_m f_m^*(2^{-kn}) \geqslant c'\ f_k^*(2^{-kn}) \geqslant c''\ k^{-\frac{1}{q}+1} \sim k^{\frac{1}{q'}}, \qquad k \in \mathbb{N}.$$

Step 4. We show that

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \geqslant\ c\, t^{-\frac{1}{p}}, \qquad t \to \infty.$$

We adapt the corresponding proof in [13, Prop. 10.21] appropriately. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n : |y| < 2\}$ with $\varphi(x) = 1$ if $|x| \leqslant 1$, and $\varrho = \varphi(2^{-1}\cdot) - \varphi$. Then $\operatorname{supp}\varrho \subset \{x \in \mathbb{R}^n :\ 1 < |x| < 4\}$, and $\varrho^*(t) \geqslant c$, $t \leqslant 1$. We consider functions

$$g_j(x) = 2^{-j\frac{n}{p}} \varrho\left(2^{-j}x\right), \qquad j \in \mathbb{N}, \tag{3.18}$$

such that $g_j^*(t) = 2^{-j\frac{n}{p}} \varrho^*(2^{-jn}t) \geqslant 2^{-j\frac{n}{p}}$ for $t \sim 2^{jn}$, $j \in \mathbb{N}$. This leads to

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \geqslant \sup_{j\in\mathbb{N}}\ g_j^*(t) \geqslant ct^{-\frac{1}{p}}, \qquad t \to \infty,$$

if we can show that $\|g_j|A^s_{p,q}(w)\| \leqslant c$, $j \in \mathbb{N}$. Let k be a compactly supported C^∞ function on $\mathbb{R}^n$ with

$$\sum_{m\in\mathbb{Z}^n} k(x-m) = 1, \qquad x \in \mathbb{R}^n.$$

Then we have for all $x \in \mathbb{R}^n$,

$$g_j(x) = 2^{-j\frac{n}{p}} \sum_{m\in\mathbb{Z}^n} k(x-m)\varrho\left(2^{-j}x\right) \sim 2^{-j\frac{n}{p}} \sum_{|m|\sim 2^j} k(x-m)\varrho\left(2^{-j}x\right), \qquad j \in \mathbb{N}. \tag{3.19}$$

On the other hand, $a_{0,m}(x) = k(x-m)\varrho\left(2^{-j}x\right)$ can be regarded as 1_K-atom located near $Q_{0,m}$, $m \in \mathbb{Z}^n$, such that (3.19) represents a special atomic representation of g_j with $\lambda_{0m} = 2^{-j\frac{n}{p}}$, $|m| \sim 2^j$ (and $\lambda_{\nu m} = 0$ otherwise). Consequently,

$$\left\|g_j|A^s_{p,q}(w)\right\|^p \leqslant c_1\, 2^{-j\frac{n}{p}p} \sum_{|m|\sim 2^j} w(Q_{0,m}) \leqslant c_2\, 2^{-jn} \sum_{|m|\sim 2^j} 1 \leqslant c_3\, 2^{-jn+jn} = c_3,$$

with a constant independent of $j \in \mathbb{N}$. Here we used that $w(Q_{0,m}) \leqslant c$, $m \in \mathbb{Z}^n$, since w is bounded a.e. in $\mathbb{R}^n$ and $Mw(x) \leqslant cw(x)$ for a.e. $x \in \mathbb{R}^n$.

Step 5. It remains to deal with (iv). The counterpart of (1.34) for L_p-spaces,

$$L_p(w) \hookrightarrow L_p,$$

follows by the definition of $w \in \mathcal{A}_1$ and (3.1), since

$$\begin{aligned}\|f|L_p\|^p &\sim \sum_{m\in\mathbb{Z}^n} \int_{Q_{0,m}} w(x)|f(x)|^p w^{-1}(x)\,\mathrm{d}x \\ &\leqslant \sum_{m\in\mathbb{Z}^n} \|w^{-1}|L_\infty(Q_{0,m})\| \int_{Q_{0,m}} w(x)|f(x)|^p\,\mathrm{d}x,\end{aligned}$$

and, by definition of $\mathcal{A}_1$ and (3.1), $w(x) \geqslant c'_w$ for a.e. $x \in Q_{0,m}$, such that we can proceed by

$$\|f|L_p\|^p \leqslant c''_w \sum_{m\in\mathbb{Z}^n} \int_{Q_{0,m}} w(x)|f(x)|^p\,\mathrm{d}x \sim \int_{\mathbb{R}^n} w(x)|f(x)|^p\,\mathrm{d}x = \|f|L_p(w)\|^p.$$

In view of (2.10) and (2.11) this yields $\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \leqslant ct^{-\frac{1}{p}}$, $t > 0$. Conversely, we may use the same extremal functions (3.10) as in Step 2 (with $s = 0$), that is

$$g_{j,x_0}(x) = \psi\left(2^j(x-x_0)\right) w\left(B(x_0,2^{-j})\right)^{-\frac{1}{p}},$$

and choose $x_0 \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$ such that

$$g^*_{j,x_0}(t) \sim w\left(B(x_0,2^{-j})\right)^{-\frac{1}{p}} \geqslant c2^{j\frac{n}{p}} \sim t^{-\frac{1}{p}}, \qquad j \in \mathbb{N}, \quad t \sim 2^{-jn}.$$

On the other hand,

$$\begin{aligned}\|g_{j,x_0}|L_p(w)\|^p &\sim w\left(B(x_0,2^{-j})\right)^{-1} \int\limits_{B(x_0,2^{-j})} \psi\left(2^j(x-x_0)\right)^p w(x)\,\mathrm{d}x \\ &\leqslant c w\left(B(x_0,2^{-j})\right)^{-1} \int\limits_{B(x_0,2^{-j})} w(x)\,\mathrm{d}x = c,\end{aligned}$$

such that, finally,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \geqslant c \sup_{j,\ t\sim 2^{-jn}} g_{j,x_0}^*(t) \sim t^{-\frac{1}{p}}, \qquad 0<t<1.$$

Otherwise, for $t \to \infty$, we adapt the approach from Step 4. Thus it is sufficient to verify that $\|g_j|L_p(w)\| \leqslant c$ uniformly in $j \in \mathbb{N}$, where g_j are given by (3.18). As above, we use that $w(Q_{0,m}) \leqslant c$, $m \in \mathbb{Z}^n$, and obtain

$$\|g_j|L_p(w)\|^p \leqslant c2^{-jn} \sum_{|m|\sim 2^j} w(Q_{0,m}) \leqslant c' 2^{-jn} \sum_{|m|\sim 2^j} 1 \leqslant c''.$$

This concludes the proof. ■

Remark 3.4. Note that we did not use the assumptions $w \in \mathcal{A}_1$ and (3.1) in Step 2 of the above proof. Hence we always obtain

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(2^{-kn}) \geqslant c2^{-k(s-\frac{n}{p})} \sup_{x_0\in\mathbb{R}^n} \left(\frac{|B(x_0,2^{-k})|}{w\left(B(x_0,2^{-k})\right)}\right)^{\frac{1}{p}}, \tag{3.20}$$

leading to

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \geqslant ct^{-\frac{1}{p}+\frac{s}{n}} \qquad \text{for} \quad t\to 0, \tag{3.21}$$

where $w \in \mathcal{A}_\infty$ and $\sigma_p < s < \frac{n}{p}$.

Corollary 3.5. *Let $0<p<\infty$, $0<q\leqslant\infty$, $s>\sigma_p$, $w\in\mathcal{A}_1$ with (3.1). Then*

$$B^s_{p,q}(w) \hookrightarrow L_\infty \qquad \textit{if and only if} \qquad \begin{cases} s>\frac{n}{p}, & \textit{or} \\ s=\frac{n}{p} & \textit{and} \quad 0<q\leqslant 1. \end{cases}$$

Similarly,

$$F^s_{p,q}(w) \hookrightarrow L_\infty \qquad \textit{if and only if} \qquad \begin{cases} s>\frac{n}{p}, & \textit{or} \\ s=\frac{n}{p} & \textit{and} \quad 0<p\leqslant 1. \end{cases}$$

Proof. Again the F-result follows from the B-assertion, embeddings (1.44) and Lemma 3.1(iii). The sufficiency is covered by Lemma 3.1(ii), so it remains to disprove $B^s_{p,q}(w) \hookrightarrow L_\infty$ when $s<\frac{n}{p}$ or $s=\frac{n}{p}$ and $1<q\leqslant\infty$. However, in these situations we have the unboundedness of $\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t)$ when $t\to 0$ in view of Proposition 3.3(i),(ii) which is by Proposition 2.2(ii) equivalent to $B^s_{p,q}(w) \not\hookrightarrow L_\infty$. ■

3.2. Growth envelopes

We complete the characterization of $A^s_{p,q}(w)$, $w \in \mathcal{A}_1$, in terms of their growth envelopes.

Theorem 3.6. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s > \sigma_p$, $w \in \mathcal{A}_1$ *with* (3.1).

(i) *Then*

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(w)) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right), & s < \frac{n}{p}, \\ \left(|\log t|^{\frac{1}{q'}}, q\right), & s = \frac{n}{p} \quad \text{and} \quad 1 < q \leqslant \infty. \end{cases} \tag{3.22}$$

(ii) *Then*

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}(w)) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right), & s < \frac{n}{p}, \\ \left(|\log t|^{\frac{1}{p'}}, p\right), & s = \frac{n}{p} \quad \text{and} \quad 1 < p < \infty, \end{cases} \tag{3.23}$$

and

$$\mathfrak{E}_{\mathsf{G}}(L_p(w)) = \left(t^{-\frac{1}{p}}, p\right). \tag{3.24}$$

Proof. In view of Proposition 3.3, (2.4), (2.6), (2.7) and (2.10) it remains to prove that $u_{\mathsf{G}}^{B^s_{p,q}(w)} \geqslant q$, $u_{\mathsf{G}}^{F^s_{p,q}(w)} \geqslant p$ and $u_{\mathsf{G}}^{L_p(w)} \geqslant p$. By Lemma 3.1(iii) and another application of (2.4) we may restrict ourselves to the B- and the L_p-case. Let first $s < \frac{n}{p}$ and $\varepsilon > 0$. We have to verify that

$$\left(\int_0^{\varepsilon} \left[t^{\frac{1}{p}-\frac{s}{n}} \, f^*(t) \right]^v \frac{\mathrm{d}t}{t} \right)^{1/v} \leqslant c \left\| f | B^s_{p,q}(w) \right\| \tag{3.25}$$

for all $f \in B^s_{p,q}(w)$ implies $v \geqslant q$. We consider a refined construction of the above extremal functions f_{j,x_0} given by (3.10). We choose $\{x_j\}_j \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$ with, say, $|x_j - x_r| \geqslant 4$, $j \neq r$, such that $\operatorname{supp} \psi\left(2^j(\cdot - x_j)\right) \cap \operatorname{supp} \psi\left(2^r(\cdot - x_r)\right) = \emptyset$ for $j \neq r$, $j, r \in \mathbb{N}_0$, and ψ is given by (3.11). Let $\{b_j\}_{j\in\mathbb{N}}$ be a sequence of non-negative numbers where we may assume, in addition, that $b_1 = \cdots = b_{J-1} = 0$, and J is suitably chosen such that $2^{-J} \sim \varepsilon$. Let

$$f_b(x) = \sum_{j=1}^{\infty} 2^{-js} b_j \psi\left(2^j(x - x_j)\right) w\left(B(x_j, 2^{-j})\right)^{-\frac{1}{p}}. \tag{3.26}$$

Seen as atomic decomposition of f_b (with $a_j = 2^{-j(s-\frac{n}{p})}\psi\left(2^j(\cdot - x_j)\right)$ and $\lambda_j =$

$2^{-j\frac{n}{p}} b_j\ w\left(B(x_j,2^{-j})\right)^{-\frac{1}{p}}$), this implies

$$\begin{aligned}\left\|f_b|B^s_{p,q}(w)\right\| &\leqslant \|\lambda|b_{pq}(w)\| \\ &\leqslant c\left(\sum_{j=J}^{\infty} 2^{-j\frac{n}{p}q} w\left(B(x_j,2^{-j})\right)^{-\frac{q}{p}} b_j^q\ 2^{j\frac{n}{p}q} w\left(B(x_j,2^{-j})\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\ &\sim \|b|\ell_q\|. \end{aligned}\tag{3.27}$$

Since

$$f_b^*(t) \geqslant c b_j\ 2^{-js} w\left(B(x_j,2^{-j})\right)^{-\frac{1}{p}} \geqslant c' b_j\ 2^{-j(s-\frac{n}{p})}, \qquad j\in\mathbb{N}, \quad t\sim 2^{-jn}, \tag{3.28}$$

inequality (3.25) can be extended on both sides to

$$\left(\sum_{j=J}^{\infty} b_j^v\right)^{\frac{1}{v}} \leqslant c_1 \left(\int_0^{\varepsilon}\left[t^{\frac{1}{p}-\frac{s}{n}} f_b^*(t)\right]^v \frac{\mathrm{d}t}{t}\right)^{\frac{1}{v}} \leqslant c_2 \left\|f_b|B^s_{p,q}(w)\right\| \leqslant c_3\|b|\ell_q\|$$

for arbitrary sequences of non-negative numbers. This obviously requires $v \geqslant q$.

As for the L_p-situation we adapt the above argument and use the functions f_b given by (3.26) with $s = 0$. All what is left to show in this case is that $\|f_b|L_p(w)\| \leqslant c\|b|\ell_p\|$, but by the disjointness of the supports in construction (3.26) and the choice of $x_j \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$ this is straightforward.

Assume now $s = \frac{n}{p}$, $1 < q \leqslant \infty$. The counterpart of (3.25) reads as

$$\left(\int_0^{\varepsilon}\left[\frac{f^*(t)}{|\log t|^{1/q'+1/v}}\right]^v \frac{\mathrm{d}t}{t}\right)^{1/v} \leqslant c\ \left\|f|B^{n/p}_{p,q}(w)\right\| \tag{3.29}$$

for all $f \in B^{n/p}_{p,q}(w)$. We want to prove $v \geqslant q$ and proceed by contradiction; that is, we assume $v < q$. We refine the approach presented in Step 3 of the proof of Proposition 3.3. Let $x_0 \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$, $m \in \mathbb{N}$, and

$$f_{m,b}(x) = \sum_{j=1}^{m} b_j \psi\left(2^j(x-x_0)\right), \qquad x\in\mathbb{R}^n, \tag{3.30}$$

where

$$b_j = j^{-\frac{1}{q}}\,(1+\log j)^{-\frac{1}{v}}, \qquad j=1,\ldots,m,$$

in (3.26). Then similar to (3.17),

$$\left\|f_{m,b}|B^{n/p}_{p,q}(w)\right\| \leqslant c\,\|b|\ell_q\| = c\left(\sum_{j=1}^{m}\frac{1}{j\,(1+\log j)^{q/v}}\right)^{1/q} \leqslant c_2$$

since $v < q$, where c_2 does not depend on $m \in \mathbb{N}$. On the other hand, by our choice of $\{b_j\}_j$,

$$f^*_{m,b}(2^{-kn}) \geqslant c \sum_{j=1}^{k} b_j \geqslant c\, k\, b_k \sim k^{\frac{1}{q'}} (1+\log k)^{-\frac{1}{v}}, \qquad k=1,\ldots,m,$$

hence for $m \geqslant J$,

$$\left(\int_0^{\varepsilon} \left[\frac{f^*_{m,b}(t)}{|\log t|^{1/q'+1/v}} \right]^v \frac{\mathrm{d}t}{t} \right)^{\frac{1}{v}} \geqslant c_1 \left(\sum_{k=1}^{m} \left[\frac{f^*_{m,b}(2^{-kn})}{k^{1/q'+1/v}} \right]^v \right)^{\frac{1}{v}}$$

$$\geqslant c_2 \left(\sum_{k=1}^{m} \frac{1}{k(1+\log k)} \right)^{\frac{1}{v}}.$$

Obviously the expression on the right-hand side diverges for $m \to \infty$, such that there are functions $f_{m,b} \in B^{n/p}_{p,q}(w)$, not satisfying (3.29). This completes the proof. ■

Remark 3.7. Let $w \in \mathcal{A}_1$ with (3.1). Then Proposition 3.3, Corollary 3.5 and Theorem 3.6 describe exactly the counterparts of the unweighted situations with $w \equiv 1$, see Proposition 2.4 (apart from borderline cases). In other words, though we only have the embedding (1.34) in this setting, the spaces are so close together that their singularity behavior (measured in growth envelopes) cannot be distinguished. This phenomenon is already known from similar studies concerning questions of compactness, cf. [19].

We separately formulate Theorem 3.6 for our example weights $w_{\alpha,\beta}$, $w_{\log}$ and $w_{\varkappa,\Gamma}$. Note that (3.1) requires $\beta \geqslant 0$ and $\delta \geqslant 0$ if $\beta = 0$. Hence we can apply Theorem 3.6 to $w_{\log}$ and $w_{\alpha,\beta}$ in case of $\beta = \delta = 0$ in view of (1.10), thus only the local behavior can differ from the unweighted setting (as seems reasonable when characterizing *local* singularity behavior).

Corollary 3.8. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, $s > \sigma_p$.*

(i) *Let $w_{\alpha,\beta}$ be given by (1.8) and $w_{\log}$ by (1.9) with $-n < \alpha \leqslant 0$, $\gamma \in \mathbb{R}$, with $\gamma \geqslant 0$ if $\alpha = 0$, and $\beta = \delta = 0$. Then*

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(w_{\alpha,0})) = \mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(w_{\log})) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right), & s < \frac{n}{p}, \\ \left(|\log t|^{\frac{1}{q'}}, q\right), & s = \frac{n}{p} \text{ and } 1 < q \leqslant \infty, \end{cases} \tag{3.31}$$

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}(w_{\alpha,0})) = \mathfrak{E}_{\mathsf{G}}(F^s_{p,q}(w_{\log})) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right), & s < \frac{n}{p}, \\ \left(|\log t|^{\frac{1}{p'}}, p\right), & s = \frac{n}{p} \text{ and } 1 < p < \infty, \end{cases} \tag{3.32}$$

and

$$\mathfrak{E}_{\mathsf{G}}(L_p(w_{\alpha,0})) = \mathfrak{E}_{\mathsf{G}}(L_p(w_{\log})) = \left(t^{-\frac{1}{p}}, p\right). \tag{3.33}$$

(ii) *Let* $\Gamma \subset \mathbb{R}^n$ *be a* d*-set,* $0 < d < n$*, and* $w_{\varkappa,\Gamma}$ *given by* (1.11) *with* $-(n-d) < \varkappa \leqslant 0$*. Then*

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(w_{\varkappa,\Gamma})) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right), & s < \frac{n}{p}, \\ \left(|\log t|^{\frac{1}{q'}}, q\right), & s = \frac{n}{p} \quad \text{and} \quad 1 < q \leqslant \infty, \end{cases} \tag{3.34}$$

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}(w_{\varkappa,\Gamma})) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right), & s < \frac{n}{p}, \\ \left(|\log t|^{\frac{1}{p'}}, p\right), & s = \frac{n}{p} \quad \text{and} \quad 1 < p < \infty, \end{cases} \tag{3.35}$$

and

$$\mathfrak{E}_{\mathsf{G}}(L_p(w_{\varkappa,\Gamma})) = \left(t^{-\frac{1}{p}}, p\right). \tag{3.36}$$

Remark 3.9. Plainly, the case $-n < \beta < 0$ in the above example, referring to weights $w \in \mathcal{A}_1$ which do not satisfy (3.1) is of some interest, too, but not yet covered by our above techniques, apart from lower estimates, see Remark 3.4.

3.3. Applications

We briefly present two typical applications of the preceding envelope results: Hardy type inequalities and sharp embedding criteria.

Corollary 3.10. *Let* $0 < p < \infty$*,* $0 < q \leqslant \infty$*,* $s > \sigma_p$*,* $w \in \mathcal{A}_1$ *with* (3.1)*, and* $\varepsilon > 0$ *be small.*

(i) *Let* $s < \frac{n}{p}$*,* $0 < u \leqslant \infty$ *and let* $\varkappa$ *be a positive monotonically decreasing function on* $(0, \varepsilon]$*. Then*

$$\left(\int_0^{\varepsilon} \left[\varkappa(t)\ t^{\frac{1}{p}-\frac{s}{n}} f^*(t)\right]^u \ \frac{\mathrm{d}t}{t}\right)^{1/u} \leqslant c\,\|f|A^s_{p,q}(w)\| \tag{3.37}$$

for some $c > 0$ *and all* $f \in A^s_{p,q}(w)$ *if and only if* $\varkappa$ *is bounded and*

$$\begin{cases} q \leqslant u \leqslant \infty, & \text{if} \quad A^s_{p,q} = B^s_{p,q}, \\ p \leqslant u \leqslant \infty, & \text{if} \quad A^s_{p,q} = F^s_{p,q}, \end{cases}$$

with the modification

$$\sup_{t\in(0,\varepsilon)} \varkappa(t)\ t^{\frac{1}{p}-\frac{s}{n}} f^*(t) \leqslant c\,\|f|A^s_{p,q}(w)\| \tag{3.38}$$

if $u = \infty$*. In particular, if* $\varkappa$ *is an arbitrary non-negative function on* $(0, \varepsilon]$*, then* (3.38) *holds if and only if* $\varkappa$ *is bounded.*

(ii) *Let* $s = \frac{n}{p}$*,* $1 < q \leqslant \infty$*,* $0 < u \leqslant \infty$ *and let* $\varkappa$ *be a positive monotonically decreasing function on* $(0, \varepsilon]$*. Then*

$$\left(\int_0^\varepsilon \left[\varkappa(t)\ (1 + |\log t|)^{\frac{1}{q'}} f^*(t) \right]^u \frac{\mathrm{d}t}{t\ |\log t|} \right)^{1/u} \leqslant c\ \left\| f | B^{n/p}_{p,q}(w) \right\| \tag{3.39}$$

for some $c > 0$ *and all* $f \in B^{n/p}_{p,q}(w)$ *if and only if* $\varkappa$ *is bounded and* $q \leqslant u \leqslant \infty$*, with the modification*

$$\sup_{t \in (0,\varepsilon)} \varkappa(t)\ (1 + |\log t|)^{\frac{1}{q'}}\ f^*(t) \leqslant c\ \left\| f | B^{n/p}_{p,q}(w) \right\| \tag{3.40}$$

if $u = \infty$*. In particular, if* $\varkappa$ *is an arbitrary non-negative function on* $(0, \varepsilon]$*, then* (3.40) *holds if and only if* $\varkappa$ *is bounded.*

(iii) *Let* $s = \frac{n}{p}$*,* $1 < p < \infty$*,* $0 < u \leqslant \infty$ *and let* $\varkappa$ *be a positive monotonically decreasing function on* $(0, \varepsilon]$*. Then*

$$\left(\int_0^\varepsilon \left[\varkappa(t)\ (1 + |\log t|)^{\frac{1}{p'}} f^*(t) \right]^u \frac{\mathrm{d}t}{t\ |\log t|} \right)^{1/u} \leqslant c\ \left\| f | F^{n/p}_{p,q}(w) \right\| \tag{3.41}$$

for some $c > 0$ *and all* $f \in F^{n/p}_{p,q}(w)$ *if and only if* $\varkappa$ *is bounded and* $p \leqslant u \leqslant \infty$*, with the modification*

$$\sup_{t \in (0,\varepsilon)} \varkappa(t)\ (1 + |\log t|)^{\frac{1}{p'}}\ f^*(t) \leqslant c\ \left\| f | F^{n/p}_{p,q}(w) \right\| \tag{3.42}$$

if $u = \infty$*. In particular, if* $\varkappa$ *is an arbitrary non-negative function on* $(0, \varepsilon]$*, then* (3.42) *holds if and only if* $\varkappa$ *is bounded.*

(iv) *Let* $0 < u \leqslant \infty$ *and let* $\varkappa$ *be a positive monotonically decreasing function on* $(0, \varepsilon]$*. Then*

$$\left(\int_0^\varepsilon \left[\varkappa(t)\ t^{\frac{1}{p}}\ f^*(t) \right]^u \frac{\mathrm{d}t}{t} \right)^{1/u} \leqslant c\ \| f | L_p(w) \| \tag{3.43}$$

for some $c > 0$ *and all* $f \in L_p(w)$ *if and only if* $\varkappa$ *is bounded and* $p \leqslant u \leqslant \infty$*, with the modification*

$$\sup_{t \in (0,\varepsilon)} \varkappa(t)\ t^{\frac{1}{p}}\ f^*(t) \leqslant c\ \| f | L_p(w) \| \tag{3.44}$$

if $u = \infty$*. In particular, if* $\varkappa$ *is an arbitrary non-negative function on* $(0, \varepsilon]$*, then* (3.44) *holds if and only if* $\varkappa$ *is bounded.*

This follows immediately from Definition 2.1 and Theorem 3.6. Of course, the above Hardy-type inequalities can be explicated for the particular weights $w_{\log}$, $w_{\alpha,\beta}$ and $w_{\varkappa,\Gamma}$ considered in Corollary 3.8.

Another type of application concerns sharp (or limiting) embeddings which naturally can be understood as sharp inequalities, too. In addition to Corollary 3.5 we now restrict ourselves to a few model cases only to demonstrate the method.

Corollary 3.11. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $w \in \mathcal{A}_1$ *with* (3.1).

(i) *Let* $0 < p_0 < p < p_1 < \infty$, $s_1 < s < s_0$ *satisfy*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}. \tag{3.45}$$

Then for $0 < u, v \leqslant \infty$,

$$B^{s_0}_{p_0,u}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,v}(w) \qquad \text{if and only if} \qquad u \leqslant p \leqslant v. \tag{3.46}$$

(ii) *Let* $1 < r < \infty$, $0 < u \leqslant \infty$, $\sigma_p < s < \frac{n}{p}$, *with*

$$s - \frac{n}{p} = -\frac{n}{r}.$$

Then

$$A^s_{p,q}(w) \hookrightarrow L_{r,u} \qquad \text{if and only if} \qquad \begin{cases} q \leqslant u \leqslant \infty, & \text{if} \quad A^s_{p,q} = B^s_{p,q}, \\ p \leqslant u \leqslant \infty, & \text{if} \quad A^s_{p,q} = F^s_{p,q}. \end{cases} \tag{3.47}$$

Proof. The sufficiency parts of (i) and (ii) are covered by Lemma 3.1(iii) together with (1.34) and the well-known unweighted counterpart of (ii), cf. [13, p. 120]. It remains to show that the embeddings imply the corresponding parameter estimates, but (ii) is an immediate consequence of Corollary 3.10(i) with $\varkappa \equiv 1$.

We complete the proof of (i). Let $B^{s_0}_{p_0,u}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,v}(w)$ with (3.45). Assume first that $s_1 > \sigma_{p_1}$ (which implies $s_0 > \sigma_{p_0}$ and $s > \sigma_p$) and $s_i - \frac{n}{p_i} = s - \frac{n}{p} < 0$, $i = 0, 1$, such that we can apply Theorem 3.6 to all spaces involved and obtain by (3.45)

$$\mathcal{E}_{\mathsf{G}}^{B^{s_0}_{p_0,u}(w)}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w)}(t) \sim \mathcal{E}_{\mathsf{G}}^{B^{s_1}_{p_1,v}(w)}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}}, \qquad t \to 0.$$

Now Proposition 2.2(iii) implies that $u_{\mathsf{G}}^{B^{s_0}_{p_0,u}(w)} \leqslant u_{\mathsf{G}}^{F^s_{p,q}(w)} \leqslant u_{\mathsf{G}}^{B^{s_1}_{p_1,v}(w)}$ which is by Theorem 3.6 equivalent to the desired result $u \leqslant p \leqslant v$. It remains to consider situations $s_1 \leqslant \sigma_{p_1}$ or $s_i - \frac{n}{p_i} = s - \frac{n}{p} \geqslant 0$, $i = 0, 1$. This is done by the lifting argument mentioned in Remark 1.9: we can always choose some number $\mu \in \mathbb{R}$ in such a way that $\tilde{s}_i = s_i - \mu$, $i = 0, 1$, $\tilde{s} = s - \mu$ satisfy $\tilde{s}_1 > \sigma_{p_1}$ and $\tilde{s}_i - \frac{n}{p_i} = \tilde{s} - \frac{n}{p} < 0$, $i = 0, 1$. Since $B^{s_0}_{p_0,u}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,v}(w)$ implies $B^{\tilde{s}_0}_{p_0,u}(w) \hookrightarrow F^{\tilde{s}}_{p,q}(w) \hookrightarrow B^{\tilde{s}_1}_{p_1,v}(w)$, the preceding observation concludes the argument. ■

Remark 3.12. It is obvious that also parts (ii) and (iii) of Corollary 3.10(i) can be reformulated in the sense of Corollary 3.11(ii), dealing with spaces of exponential type accordingly. We shall return to Corollary 3.10(iv) in Remark 4.8 below.

4. Growth envelope functions for $w \in \mathcal{A}_\infty$

We turn to the general situation now where the results are less complete so far.

Lemma 4.1. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $w \in \mathcal{A}_\infty$ *with* (3.1).

(i) *Let* $s > n\left(\frac{r_w}{p} - \frac{1}{\max(p,1)}\right)$. *Then* $A^s_{p,q}(w) \subset L_1^{\mathrm{loc}}$.

(ii) *Assume that*

$$\left\{2^{-\nu(s-\frac{n}{p})} \sup_{m\in\mathbb{Z}^n} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|}\right)^{-\frac{1}{p}}\right\}_{\nu\in\mathbb{N}_0} \in \ell_{q'}. \tag{4.1}$$

Then

$$B^s_{p,q}(w) \hookrightarrow L_\infty, \tag{4.2}$$

in particular, if $s > \frac{n}{p} r_w$, *then*

$$A^s_{p,q}(w) \hookrightarrow L_\infty. \tag{4.3}$$

(iii) *Let* $d > n r_w$, *and assume that* $0 < p_0 < p < p_1 < \infty$, $s_1 < s < s_0$ *satisfy*

$$s_0 - \frac{d}{p_0} = s - \frac{d}{p} = s_1 - \frac{d}{p_1}. \tag{4.4}$$

Then

$$B^{s_0}_{p_0,p}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,p}(w). \tag{4.5}$$

Proof. Note that the extension to F-spaces in (i) is a direct consequence of (1.44), so it is sufficient to consider B-spaces. In view of our assumptions we can choose $\varepsilon > 0$ sufficiently small such that

$$s - \frac{n}{p}(r_w - 1) - \varepsilon n > \sigma_p. \tag{4.6}$$

Now we use embedding (1.32) with $\mu = \frac{n}{p}(r_w - 1) + \varepsilon n$, hence (3.1) implies

$$B^s_{p,q}(w) \hookrightarrow B^{s-\frac{n}{p}(r_w-1)-\varepsilon n}_{p,q} \tag{4.7}$$

and the unweighted result gives (i). We come to (ii). Since (4.1) implies $B^s_{p,q}(w) \hookrightarrow B^0_{\infty,1}$ in view of (1.26), the classical result $B^0_{\infty,1} \hookrightarrow L_\infty$ leads to (4.2). In particular, if $s > \frac{n}{p} r_w$ one may choose $\varepsilon > 0$ sufficiently small such that $s - \mu > \frac{n}{p}$ and (4.7) can be extended by the unweighted embedding $B^{s-\frac{n}{p}(r_w-1)-\varepsilon n}_{p,q} \hookrightarrow L_\infty$. Alternatively, using (1.6) for some number r such that $s > n\frac{r}{p} > n\frac{r_w}{p}$ one obtains convergence in (4.1) for any q, since

$$2^{-\nu(s-\frac{n}{p})} \sup_{m\in\mathbb{Z}^n} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|}\right)^{-\frac{1}{p}} \leqslant c'_w\, 2^{-\nu(s-n\frac{r}{p})};$$

hence (4.2) with Proposition 1.19(i), (ii) imply (4.3).

Finally, concerning (iii) we apply (1.6) with $\frac{d}{n} > r_w$, such that $w \in \mathcal{A}_{d/n}$. Together with (3.1) this gives for $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and appropriate $m' \in \mathbb{Z}^n$ with $Q_{\nu,m} \subset Q_{0,m'}$, that

$$w(Q_{\nu,m}) \geqslant c_w \, \frac{w(Q_{\nu,m})}{w(Q_{0,m'})} \geqslant c_w' \, \frac{|Q_{\nu,m}|^{d/n}}{|Q_{0,m'}|^{d/n}} \geqslant c 2^{-\nu d}, \qquad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{4.8}$$

with a constant independent of $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Application of (1.48) with (1.46) gives (iii). ∎

Remark 4.2. In contrast to Lemma 3.1(ii) and Remark 3.2 we only obtain a sufficient condition for $B^s_{p,q}(w) \hookrightarrow L_\infty$ here when $w \in \mathcal{A}_\infty$; borderline situations corresponding to (i) are out of the frame of the present approach again. Moreover, inspired by Lemma 3.1(iii) one might be tempted to take the limiting case $d = nr_w$ in (iii), but sharp embedding criteria for $w_{\alpha,\beta}$ and $w_{\log}$ studied in [18, 20] disprove this assumption. On the other hand, condition (4.1) may be sharp in certain cases, also for $w \notin \mathcal{A}_1$: take $w_{\alpha,\beta}$ with (3.1) (thus $\beta \geqslant 0$), then we have by Corollary 1.16 for any $p < r < \infty$ the continuous embedding

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^\sigma_{r,q}, \qquad s - \frac{n}{p} = \sigma - \frac{n}{r} + \frac{\max(\alpha, 0)}{p}, \tag{4.9}$$

in particular, with $\sigma = \frac{n}{r}$,

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^{n/r}_{r,q}, \qquad s - \frac{n}{p} = \frac{\max(\alpha, 0)}{p}, \tag{4.10}$$

such that $B^{n/r}_{r,q} \hookrightarrow L_\infty$ for $0 < q \leqslant 1$ implies

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow L_\infty, \qquad s = \frac{n}{p} + \frac{\max(\alpha, 0)}{p}, \quad 0 < q \leqslant 1, \tag{4.11}$$

in accordance with (4.1), recall (1.12). As it turns out in Corollary 4.20 below, this is indeed sharp.

In view of Lemma 4.1(i), (ii) and general facts about growth envelopes we mainly restrict ourselves to $w \in \mathcal{A}_\infty$ and parameters

$$0 < p < \infty, \qquad 0 < q \leqslant \infty, \qquad n\left(\frac{r_w}{p} - \frac{1}{\max(p,1)}\right) < s \leqslant \frac{n}{p} r_w \tag{4.12}$$

in the sequel. Plainly, (4.12) implies $s > \sigma_{p/r_w}$ which is needed to avoid moment conditions for the atoms in the corresponding Besov spaces, see (1.21). If $r_w = 1$, e.g. for $w \in \mathcal{A}_1$, then (4.12) reduces to $\sigma_p < s \leqslant \frac{n}{p}$, corresponding to the unweighted setting and Remark 3.2.

4.1. Estimates from above

Here we mainly apply (sharp) embeddings together with results from the unweighted setting.

Proposition 4.3. *Let $w \in \mathcal{A}_\infty$ with* (3.1) *and let* (4.12) *be satisfied.*

(i) *Assume that for μ with $0 \leqslant \mu < s - \sigma_p$ there exists some $c > 0$ such that*

$$\inf_{\nu\in\mathbb{N}_0} 2^{\nu\mu} \inf_{m\in\mathbb{Z}^n} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|}\right)^{\frac{1}{p}} \geqslant c. \tag{4.13}$$

Then

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \leqslant c' \begin{cases} t^{-\frac{1}{p}+\frac{s}{n}-\frac{\mu}{n}}, & \text{if} \quad \mu > s - \frac{n}{p}, \\ |\log t|^{\frac{1}{q'}}, & \text{if} \quad \mu = s - \frac{n}{p},\ 1 < q \leqslant \infty \ \text{and} \ A^s_{p,q} = B^s_{p,q}, \\ |\log t|^{\frac{1}{p'}}, & \text{if} \quad \mu = s - \frac{n}{p},\ 1 < p < \infty \ \text{and} \ A^s_{p,q} = F^s_{p,q}, \end{cases} \tag{4.14}$$

for $t \to 0$. In particular, for any small $\varepsilon > 0$ there is some $c_\varepsilon > 0$ such that

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \leqslant c_\varepsilon\, t^{-\frac{r_w}{p}+\frac{s}{n}-\varepsilon}, \qquad t \to 0. \tag{4.15}$$

(ii) *Assume that for some u with $0 < u \leqslant \infty$ and $s > n\left(\frac{1}{p}+\frac{1}{u}-1\right)_+$ there is some $c > 0$ such that*

$$\sup_{\nu\in\mathbb{N}_0} 2^{-\nu n\left(\frac{1}{p}+\frac{1}{u}\right)} \left\| w(Q_{\nu,m})^{-\frac{1}{p}} |\ell_u \right\| \leqslant c < \infty. \tag{4.16}$$

Then

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \leqslant c t^{-\frac{1}{p}-\frac{1}{u}}, \qquad t \to \infty. \tag{4.17}$$

In any case we have

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \leqslant c t^{-\frac{1}{p}}, \qquad t \to \infty. \tag{4.18}$$

Proof. Let for $\varepsilon > 0$ the number $r_\varepsilon = r_w + \varepsilon > r_w$, hence $w \in \mathcal{A}_{r_\varepsilon}$. Then by (3.1) and (1.6),

$$\inf_{m\in\mathbb{Z}^n} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|}\right)^{\frac{1}{p}} \geqslant c 2^{-\nu\frac{n}{p}(r_\varepsilon-1)},$$

such that (4.13) is always satisfied with $\mu_\varepsilon = \frac{n}{p}(r_\varepsilon - 1)$. Moreover, assumption (4.12) guarantees that $\mu_\varepsilon < s - \sigma_p$ for sufficiently small $\varepsilon > 0$. Thus

$$B^s_{p,q}(w) \hookrightarrow B^{s-\mu_\varepsilon}_{p,q}$$

is always true for small ε which immediately leads to (4.15) and (4.18) in view of the unweighted results (2.6), (2.8) together with (2.3). Moreover, in case of (4.13) we have

$$B^s_{p,q}(w) \hookrightarrow B^{s-\mu}_{p,q}$$

and in case of (4.16),

$$B^s_{p,q}(w) \hookrightarrow B^s_{r,q} \qquad \text{with} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{u}.$$

Thus (4.14) and (4.17) are again consequences of (2.3), (2.6), and (2.8), at least in case of B-spaces. The F-case in the first line of (4.14) and (4.17) can be easily obtained in view of (1.44), only the last line in (4.14) needs some more care. Assume that $\mu = s - \frac{n}{p}$, let $d = sp$, then (4.13) reads as

$$\inf_{\nu\in\mathbb{N}_0} 2^{\nu d} \inf_{m\in\mathbb{Z}^n} w(Q_{\nu,m}) \geqslant c'. \tag{4.19}$$

Since $s \geqslant \frac{n}{p}$ we may choose $s_1 < s$ and $p_1 > p$ such that $s_1 p_1 = d = sp$ and $\mu_1 = s_1 - \frac{n}{p_1} \geqslant 0$. On the other hand, (4.19) together with the second line of (4.14) yield

$$\mathcal{E}_{\mathsf{G}}^{B^{s_1}_{p_1,p}(w)}(t) \leqslant c|\log t|^{\frac{1}{p'}}, \qquad t \to 0,$$

such that Proposition 1.19(iii), that is $F^s_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,p}(w)$, and (2.3) conclude the argument. ■

Remark 4.4. Clearly, one is interested in the smallest numbers μ in (i) and u in (ii) to obtain sharp upper estimates for $t \to 0$ and $t \to \infty$, respectively. Note that $\varepsilon > 0$ in (4.15) cannot be omitted in general, though in special cases $\varepsilon = 0$ is possible: Let $w_{\alpha,\beta}$ be given by (1.8) with $\beta = 0$ such that (3.1) is satisfied. Then by (2.12) for $0 < p < \infty$, $0 < q \leqslant \infty$, $s - \frac{n}{p} < \frac{\max(\alpha,0)}{p}$,

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}-\frac{\max(\alpha,0)}{np}}, \qquad \text{for} \quad t \to 0. \tag{4.20}$$

Since $r_{w_{\alpha,\beta}} = 1 + \frac{\max(\alpha,0)}{n}$, this corresponds to (4.15) with $\varepsilon = 0$. We discuss this point in further detail below.

We collect consequences of Proposition 4.3 for our example weights $w_{\alpha,\beta}$, $w_{\log}$ and $w_{\varkappa,\Gamma}$.

Corollary 4.5. *Let $0 < p < \infty$ and $0 < q \leqslant \infty$.*

(i) *Let $w_{\alpha,\beta}$ be given by (1.8) with $\beta \geqslant 0$ and assume $\sigma_p < s - \frac{\max(\alpha,0)}{p} \leqslant \frac{n}{p}$. Then*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\alpha,\beta})}(t) \leqslant c \begin{cases} t^{-\frac{1}{p}+\frac{s}{n}-\frac{\max(\alpha,0)}{np}}, & \text{if} \quad s < \frac{n}{p} + \frac{\max(\alpha,0)}{p}, \\ |\log t|^{\frac{1}{q'}}, & \text{if} \quad s = \frac{n}{p} + \frac{\max(\alpha,0)}{p},\ 1 < q \leqslant \infty \\ & \text{and} \quad A^s_{p,q} = B^s_{p,q}, \\ |\log t|^{\frac{1}{p'}}, & \text{if} \quad s = \frac{n}{p} + \frac{\max(\alpha,0)}{p},\ 1 < p < \infty \\ & \text{and} \quad A^s_{p,q} = F^s_{p,q}, \end{cases} \tag{4.21}$$

for $t \to 0$. Moreover, let $\alpha < \beta$, then for any small $\varepsilon > 0$ there is some $c_\varepsilon > 0$ such that

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\alpha,\beta})}(t) \leqslant c_\varepsilon \, t^{-\min(\frac{1}{p}+\frac{\beta}{np},\frac{s}{n}+1)+\varepsilon}, \qquad t \to \infty. \tag{4.22}$$

If $1 < p < \infty$, $0 < \beta < n(p-1)$, $\alpha \leqslant \beta$, then (4.22) can be improved by

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\alpha,\beta})}(t) \leqslant c t^{-\frac{1}{p}-\frac{\beta}{np}}, \qquad t \to \infty. \tag{4.23}$$

(ii) *Let $w_{\log}$ be given by (1.9) with $\beta \geqslant 0$ and $\delta \geqslant 0$ if $\beta = 0$. Assume $\sigma_p < s - \frac{\max(\alpha,0)}{p} \leqslant \frac{n}{p}$, and $\gamma \geqslant 0$ if $\alpha \geqslant 0$. Then $\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\log})}(t)$ can be estimated by the right-hand side of (4.21) for $t \to 0$, and by the right-hand side of (4.22) for $t \to \infty$. In case of $s > n(\frac{1}{p} + \frac{\beta}{np} - 1)_+$ and $\delta > \frac{\beta}{n}$ we have the sharper estimate (4.23).*

(iii) *Let $w_{\varkappa,\Gamma}$ be given by (1.11), and $\sigma_p < s - \frac{\max(\varkappa,0)}{p} \leqslant \frac{n}{p}$. Then*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\varkappa,\Gamma})}(t) \leqslant c \begin{cases} t^{-\frac{1}{p}+\frac{s}{n}-\frac{\max(\varkappa,0)}{np}}, & \text{if} \quad s < \frac{n}{p} + \frac{\max(\varkappa,0)}{p}, \\ |\log t|^{\frac{1}{q'}}, & \text{if} \quad s = \frac{n}{p} + \frac{\max(\varkappa,0)}{p}, \ 1 < q \leqslant \infty \\ & \text{and} \quad A^s_{p,q} = B^s_{p,q}, \\ |\log t|^{\frac{1}{p'}}, & \text{if} \quad s = \frac{n}{p} + \frac{\max(\varkappa,0)}{p}, \ 1 < p < \infty \\ & \text{and} \quad A^s_{p,q} = F^s_{p,q}, \end{cases} \tag{4.24}$$

for $t \to 0$. Moreover, we have for $s - \frac{\max(\varkappa,0)}{p} > \sigma_p$ that

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\varkappa,\Gamma})}(t) \leqslant c \, t^{-\frac{1}{p}}, \qquad t \to \infty. \tag{4.25}$$

Proof. Part (i) follows from Proposition 4.3(i) with $\mu = \max(\frac{\alpha}{p}, 0)$ in the local case, recall $\beta \geqslant 0$ and (1.12), whereas u in Proposition 4.3(ii) can be chosen such that $\frac{\alpha}{np} < \frac{1}{u} < \frac{\beta}{np}$. This leads to (4.22) if $s > (\frac{\beta}{p} + \frac{n}{p} - n)_+$. In case of $w_{\log}$ and $\delta > \frac{\beta}{n}$ we may even take $\frac{1}{u} = \frac{\beta}{np}$ and arrive at (4.23).

Concerning (4.22) in case of $s \leqslant (\frac{\beta}{p} + \frac{n}{p} - n)_+$ note first that our general assumption $s > \sigma_p + \max(\frac{\alpha}{p}, 0)$ then implies $\alpha < \beta$ and $\frac{\beta}{p} + \frac{n}{p} > n$. The idea is to find the smallest number r such that for suitable $\mu > 0$,

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^{s-\mu}_{r,q} \qquad \text{with} \quad s - \mu \ > n\left(\frac{1}{r} - 1\right)_+. \tag{4.26}$$

By Corollary 1.16 this requires $\frac{1}{r} < \frac{\beta}{np} + \frac{1}{p}$ (or $\frac{1}{r} = \frac{\beta}{np} + \frac{1}{p}$ with $\delta > \frac{\beta}{n}$ in case of $w_{\log}$), and $\mu > (\frac{\alpha}{p} - \frac{n}{r} + \frac{n}{p})_+$. But if $s \leqslant (\frac{\beta}{p} + \frac{n}{p} - n)_+ = \frac{\beta}{p} + \frac{n}{p} - n$, we may always choose r such that

$$\max\left(1, \frac{\alpha}{np} + \frac{1}{p}\right) < \frac{1}{r} < \min\left(\frac{\beta}{np} + \frac{1}{p}, \frac{s}{n} + 1\right) = \frac{s}{n} + 1.$$

Thus for any (sufficiently small) $\varepsilon > 0$ we take $\mu_\varepsilon < \frac{\varepsilon}{n}$, and $\frac{1}{r_\varepsilon} = \frac{s}{n} + 1 - \varepsilon$, such that (4.26) becomes

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^{s-\mu_\varepsilon}_{r_\varepsilon,q} .$$

Finally, (2.3) and (2.8) lead to (4.22). The argument for $w_{\varkappa,\Gamma}$ is parallel, where (4.24) corresponds to (4.14) with $\mu = \max(\frac{\varkappa}{p}, 0)$, and (4.25) follows from (4.18). ■

Remark 4.6. We already obtained the non-limiting case in (4.21) and (4.23) in [15]. It seems natural to assume that (4.23) describes the correct upper bound in all cases of s, but this is not yet verified.

In case of $L_p(w)$ we generalize the interpolation idea used in [16, Lemma 3.3].

Lemma 4.7. *Let* $0 < p < \infty$ *and* $w \in \mathcal{A}_\infty$.

(i) *If* $w^{-\frac{1}{p}} \in L_{q,\infty}(\log L)_\varkappa$ *for some* $0 < q < \infty$ *and* $\varkappa \in \mathbb{R}$, *then*

$$L_p(w) \hookrightarrow L_{r,p}(\log L)_\varkappa \qquad \text{for} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \tag{4.27}$$

and, consequently,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \leqslant c t^{-\frac{1}{p}-\frac{1}{q}} (1 + |\log t|)^{-\varkappa}, \qquad t > 0. \tag{4.28}$$

(ii) *If* $w^{-1} \in L_\infty$, *then*

$$L_p(w) \hookrightarrow L_p, \tag{4.29}$$

and, consequently,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \leqslant c t^{-\frac{1}{p}}, \qquad t > 0. \tag{4.30}$$

Proof. Similar to [13, Lemma 3.33] and [16, Lemma 3.3] we use interpolation arguments and the result of Merucci [22, Ex. 3] to extend Hölder's inequality to

$$L_{p,u}(\log L)_a \cdot L_{q,v}(\log L)_b \hookrightarrow L_{r,u}(\log L)_{a+b},$$
$$0 < u, v \leqslant \infty,\ a, b \in \mathbb{R},\ \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \tag{4.31}$$

in the sense that $f \in L_{p,u}(\log L)_a$ and $g \in L_{q,v}(\log L)_b$ implies $fg \in L_{r,u}(\log L)_{a+b}$ with

$$\|fg|L_{r,u}(\log L)_{a+b}\| \leqslant c \|f|L_{p,u}(\log L)_a\| \, \|g|L_{q,v}(\log L)_b\| . \tag{4.32}$$

We apply (4.32) with $u = p$, $v = \infty$, $a = 0$, $b = \varkappa$, $g = w^{-\frac{1}{p}}$ and $f = h w^{\frac{1}{p}}$ which yields

$$\|h|L_{r,p}(\log L)_\varkappa\| \leqslant c \|h|L_p(w)\| \left\| w^{-\frac{1}{p}} | L_{q,\infty}(\log L)_\varkappa \right\|,$$

that is, (4.27). Embedding (4.29) is obvious. The estimates (4.28) and (4.30) are consequences of these embeddings together with (2.3) and Proposition 2.4(ii). ■

Remark 4.8. If $w \in \mathcal{A}_1$ with (3.1), then Lemma 4.7(ii) corresponds to Proposition 3.3(iv). Moreover, Corollary 3.10(iv) with $\varkappa \equiv 1$ confirms that the target space L_p is optimal within the scale of Lorentz spaces.

We explicate Lemma 4.7 for $w = w_{\log}$.

Corollary 4.9. *Let $0 < p < \infty$, $\varkappa \in \mathbb{R}$, and $w_{\log}$ given by (1.9) with $-n < \alpha \leqslant \beta$, $\beta > 0$, $\gamma, \delta \in \mathbb{R}$. Then*

$$L_p\left(w^{\log}\right) \hookrightarrow L_{r,p}(\log L)_{\varkappa} \tag{4.33}$$

if (at least) one of the following conditions is satisfied:

(a) $\frac{1}{p} + \frac{\max(\alpha,0)}{np} < \frac{1}{r} < \frac{1}{p} + \frac{\beta}{np}$, *or*

(b) $\frac{1}{r} = \frac{1}{p} + \frac{\beta}{np}$ *and* $\varkappa \leqslant \frac{\delta}{p}$, *or*

(c) $\frac{1}{r} = \frac{1}{p} + \frac{\max(\alpha,0)}{np}$ *and* $\varkappa \leqslant \begin{cases} \frac{\gamma}{p}, & \text{if } \alpha > 0 \text{ or } \alpha = 0 > \gamma, \\ 0, & \text{if } \alpha < 0 \text{ or } \alpha = 0 \leqslant \gamma. \end{cases}$

Consequently we obtain for any $t > 0$ and $\frac{1}{p} + \frac{\max(\alpha,0)}{np} < \frac{1}{r} < \frac{1}{p} + \frac{\beta}{np}$, $\varkappa \in \mathbb{R}$,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\log})}(t) \leqslant c t^{-\frac{1}{r}} (1 + |\log t|)^{-\varkappa}, \tag{4.34}$$

and, in the limiting cases, for $\alpha < \beta$,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\log})}(t) \leqslant c t^{-\frac{1}{p} - \frac{\beta}{np}} (1 + |\log t|)^{-\frac{\delta}{p}}, \tag{4.35}$$

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\log})}(t) \leqslant c \begin{cases} t^{-\frac{1}{p} - \frac{\max(\alpha,0)}{np}} (1 + |\log t|)^{-\frac{\gamma}{p}}, & \alpha > 0 \text{ or } \alpha = 0 > \gamma, \\ t^{-\frac{1}{p}}, & \alpha < 0 \text{ or } \alpha = 0 \leqslant \gamma, \end{cases} \tag{4.36}$$

whereas $\alpha = \beta$ leads to

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\log})}(t) \leqslant c \begin{cases} t^{-\frac{1}{p} - \frac{\alpha}{np}} (1 + |\log t|)^{-\frac{\min(\gamma,\delta)}{p}}, & \alpha = \beta > 0, \\ t^{-\frac{1}{p}}, & \alpha = \beta = 0 \leqslant \min(\gamma, \delta). \end{cases} \tag{4.37}$$

Proof. The embedding result can be found in [20] (with a forerunner for $\alpha > 0$ in [16, Lemma 3.3]), it is based on Lemma 4.7(i). Plainly, (4.34)-(4.37) are consequences of (a)-(c) together with (2.3) and Proposition 2.4(ii). ∎

Remark 4.10. Note that for $\alpha < 0$ or $\alpha = 0 \leqslant \gamma$ embedding (4.33) remains valid for $\beta = 0$ and $\delta \geqslant 0$, that is, $L_p(w_{\log}) \hookrightarrow L_p \hookrightarrow L_p(\log L)_{\varkappa}$, $\varkappa \leqslant 0$, leading again to

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\log})}(t) \leqslant c t^{-\frac{1}{p}}, \qquad t > 0. \tag{4.38}$$

Of course, $w_{\log} = w_{\alpha,\beta}$ if $\gamma = \delta = 0$, such that Corollary 4.9 contains sharp embeddings and upper estimates for the growth envelope function of $L_p(w_{\alpha,\beta})$ as special case.

We use Corollary 4.9 to improve our results in Corollary 4.5 partially.

Corollary 4.11. *Let* $w_{\log}$ *be given by* (1.9) *with* $\alpha > -n$, $\beta \geqslant 0$, $\gamma, \delta \in \mathbb{R}$, *and* $\delta \geqslant 0$ *if* $\beta = 0$. *Assume that* $0 < p < \infty$, $0 < q \leqslant \infty$, *and* $s > \sigma_p + \frac{\max(\alpha,0)}{p}$.

(i) *If* $0 < \alpha \leqslant \beta$, $\gamma \geqslant 0$, $\delta \geqslant \gamma$ *if* $\alpha = \beta$, *and*

$$n\left(\frac{1}{p} - \frac{n+\alpha}{n+\beta}\right)_+ < s - \frac{\alpha}{p} < \frac{n}{p}, \tag{4.39}$$

then (4.21) *can be refined by*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\log})}(t) \leqslant c t^{\frac{s}{n} - \frac{1}{p} - \frac{\alpha}{np}} (1 + |\log t|)^{-\frac{\gamma}{p} + \frac{\gamma s}{n+\alpha}}, \qquad t \to 0, \tag{4.40}$$

where in case of $A = B$ *we have, in addition, to assume that* $\frac{1}{q} \geqslant \frac{1}{p} - \frac{s}{n+\alpha}$.

(ii) *For the global behavior we obtain that if* $1 < p < \infty$, $0 < \beta < n(p-1)$, $\alpha \leqslant \beta$, *and* $\delta \leqslant \gamma$ *if* $\alpha = \beta$, *then* (4.23) *can be improved by*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\log})}(t) \leqslant c t^{-\frac{1}{p} - \frac{\beta}{np}} (1 + |\log t|)^{-\frac{\delta}{p}}, \qquad t \to \infty. \tag{4.41}$$

Proof. Plainly, the results in case of F-spaces result from their B-counterparts in view of the embeddings (1.44) and Example 1.21 together with (2.3).

Concerning (4.40) we apply (4.36), (4.37) together with Example 1.21 and $L_r(w) = F^0_{r,2}(w)$, $1 < r < \infty$, $w \in \mathcal{A}_r$, cf. (1.19). Note that (4.39) implies $s - \frac{\alpha}{p} > \sigma_p$ such that $A^s_{p,q}(w_{\log}) \subset L^{\mathrm{loc}}_1$, as desired. Beginning with the F-case we have

$$F^s_{p,q}(w_{\log}) \hookrightarrow F^0_{r,2}(w_{\log}), \qquad s - \frac{n+\alpha}{p} = -\frac{n+\alpha}{r}, \tag{4.42}$$

since $\gamma \geqslant 0$, recall Example 1.21. Moreover, our assumptions (4.39) ensure that $1 < r < \infty$ and $w_{\log} \in \mathcal{A}_r$ such that (4.42) can be continued by

$$F^s_{p,q}(w_{\log}) \hookrightarrow L_r(w_{\log}). \tag{4.43}$$

Thus (4.36), (4.37) and (2.3) lead to (4.40). In case of $A = B$ and $q \leqslant p$ the result is a consequence of the F-case and (1.44), whereas for $\frac{1}{p} > \frac{1}{q} \geqslant \frac{1}{r} = \frac{1}{p} - \frac{s}{n+\alpha}$ (4.42) has to be replaced by

$$B^s_{p,q}(w_{\log}) \hookrightarrow B^s_{p,r}(w_{\log}) \hookrightarrow F^0_{r,2}(w_{\log}), \qquad s - \frac{n+\alpha}{p} = -\frac{n+\alpha}{r}. \tag{4.44}$$

We come to (ii). If $1 < p < \infty$ and $w_{\log} \in \mathcal{A}_p$, then similar to (4.42) we use

$$A^s_{p,q}(w_{\log}) \hookrightarrow F^0_{p,2}(w_{\log}) = L_p(w_{\log}),$$

since $s > 0$. Thus (4.35) and (4.37) yield (4.41) in view of (2.3). ∎

4.2. Estimates from below

We refine previous arguments and constructions of (atomic) extremal functions to obtain lower bounds for $\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t)$ and $\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t)$. First we study the general situation before we list consequences for our model weights afterwards.

Proposition 4.12. *Let $w \in \mathcal{A}_\infty$.*

(i) *Let $0 < p < \infty$, $0 < q \leqslant \infty$, and $s > n\left(\frac{r_w}{p} - \frac{1}{\max(p,1)}\right)$. Then, if $t \to 0$,*

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \geqslant c \sup_{x_0 \in \mathbb{R}^n} \left(\sum_{j=1}^{\lfloor \frac{1}{n} |\log t| \rfloor} 2^{-j(s-\frac{n}{p})q'} \left(\frac{w\left(B(x_0, 2^{-j})\right)}{|B(x_0, 2^{-j})|} \right)^{-\frac{q'}{p}} \right)^{1/q'}, \tag{4.45}$$

(*usual modification if $q' = \infty$*), *and*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \geqslant c\Big(\sum_{|m| \sim t^{1/n}} w(Q_{0,m}) \Big)^{-\frac{1}{p}}, \qquad t \to \infty. \tag{4.46}$$

(ii) *Let $0 < p < \infty$. Then we obtain*

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \geqslant c t^{-\frac{1}{p}} \sup_{\substack{Q_{\nu,m} \\ t \sim 2^{-\nu n}}} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} \right)^{-\frac{1}{p}}, \qquad t \to 0, \tag{4.47}$$

and

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \geqslant c\Big(\sum_{|m| \sim t^{1/n}} w(Q_{0,m}) \Big)^{-\frac{1}{p}}, \qquad t \to \infty. \tag{4.48}$$

Proof. We refine the approach from Step 3 in the proof of Proposition 3.3. Let for $x_0 \in \mathbb{R}^n$,

$$f_m(x) = c_m \sum_{j=1}^{m} b_j \ \psi\left(2^j (x - x_0)\right), \qquad m \in \mathbb{N}, \tag{4.49}$$

where $b_j \geqslant 0$ and c_m will be suitably chosen later. Similarly as above, see also [13, Thm. 8.16], we obtain

$$f_m^*(t) \geqslant \ c c_m \sum_{j=1}^{m} b_j, \qquad t \sim 2^{-mn}.$$

Regarding (4.49) as an atomic decomposition of f_m with $\lambda_j = c_m\ b_j\ 2^{j(s-\frac{n}{p})}$, $a_j(x) = 2^{-j(s-\frac{n}{p})}\psi\left(2^j(x-x_0)\right)$, $j = 1,\ldots,m$, we conclude that

$$\left\|f_m|B^s_{p,q}(w)\right\| \leqslant \|\lambda|b_{pq}(w)\| \sim c_m \left(\sum_{j=1}^m b_j^q\ 2^{j(s-\frac{n}{p})q}\left(\frac{w\left(B(x_0,2^{-j})\right)}{|B(x_0,2^{-j})|}\right)^{\frac{q}{p}}\right)^{1/q} \tag{4.50}$$

(modification if $q=\infty$). Thus, if we choose

$$c_m \sim \left(\sum_{j=1}^m b_j^q\ 2^{j(s-\frac{n}{p})q}\left(\frac{w\left(B(x_0,2^{-j})\right)}{|B(x_0,2^{-j})|}\right)^{\frac{q}{p}}\right)^{-1/q}, \tag{4.51}$$

we obtain

$$\left\|f_m|B^s_{p,q}(w)\right\| \leqslant c,$$

and

$$f_m^*(t) \geqslant\ c\left(\sum_{j=1}^m b_j^q\ 2^{j(s-\frac{n}{p})q}\left(\frac{w\left(B(x_0,2^{-j})\right)}{|B(x_0,2^{-j})|}\right)^{\frac{q}{p}}\right)^{-1/q}\left(\sum_{j=1}^m b_j\right), \qquad t\sim 2^{-mn}. \tag{4.52}$$

We choose now

$$b_j \sim 2^{-j(s-\frac{n}{p})q'}\left(\frac{w\left(B(x_0,2^{-j})\right)}{|B(x_0,2^{-j})|}\right)^{-\frac{q'}{p}}, \qquad j=1,\ldots,m,$$

thus

$$f_m^*(t) \geqslant\ c\left(\sum_{j=1}^m 2^{-j(s-\frac{n}{p})q'}\left(\frac{w\left(B(x_0,2^{-j})\right)}{|B(x_0,2^{-j})|}\right)^{-\frac{q'}{p}}\right)^{1/q'}, \qquad t\sim 2^{-mn}. \tag{4.53}$$

The modifications in case of $q'=\infty$ (i.e., $0<q\leqslant 1$) are obvious. Thus (4.50) and (4.53) imply (4.45). As for (4.47) it is sufficient to consider $f_m(x) = \psi\left(2^m(x-x_0)\right) w\left(B(x_0,2^{-m})\right)^{-1/p}$ which yields

$$\|f_m|L_p(w)\| \leqslant c \qquad \text{and} \qquad f_m^*(t) \geqslant\ c2^{m\frac{n}{p}}\left(\frac{w\left(B(x_0,2^{-m})\right)}{|B(x_0,2^{-m})|}\right)^{-\frac{1}{p}}, \quad t\sim 2^{-mn}.$$

Concerning (4.46) we adapt the argument used in Step 4 of the proof of Proposition 3.3. Note that it is sufficient to deal with B-spaces now, since the extension to F-spaces comes from (1.44) combined with (2.3). Using the same function ϱ as there we now consider functions

$$g_j(x) = \Big(\sum_{|m|\sim 2^j} w(Q_{0,m})\Big)^{-\frac{1}{p}}\varrho\left(2^{-j}x\right), \qquad j\in\mathbb{N},$$

such that

$$g_j^*(t) = \Big(\sum_{|m|\sim 2^j} w(Q_{0,m}) \Big)^{-\frac{1}{p}} \varrho^*(2^{-jn}t) \geqslant \Big(\sum_{|m|\sim 2^j} w(Q_{0,m}) \Big)^{-\frac{1}{p}}, \quad t \sim 2^{jn}, \ j \in \mathbb{N}.$$

This implies (4.46) if $\|g_j|B^s_{p,q}(w)\| \leqslant c$, $j \in \mathbb{N}$. Let $k \in C_0^\infty(\mathbb{R}^n)$ be as above, then for $x \in \mathbb{R}^n$,

$$g_j(x) \sim \Big(\sum_{|l|\sim 2^j} w(Q_{0,l}) \Big)^{-\frac{1}{p}} \sum_{|m|\sim 2^j} k(x-m)\varrho\left(2^{-j}x\right), \qquad j \in \mathbb{N}. \tag{4.54}$$

Regarding this as atomic decomposition again, we arrive at

$$\big\|g_j|B^s_{p,q}(w)\big\| \leqslant c_1 \Big(\sum_{|l|\sim 2^j} w(Q_{0,l}) \Big)^{-\frac{1}{p}} \Big(\sum_{|m|\sim 2^j} w(Q_{0,m}) \Big)^{\frac{1}{p}} \leqslant c_2,$$

with a constant independent of $j \in \mathbb{N}$. The modifications of the above argument in case of (4.47) are parallel to the proofs of Proposition 3.3 and Theorem 3.6. ■

Remark 4.13. Plainly, if $\mathbf{S}_0(w) \neq \emptyset$, then (4.45) can be replaced by

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \geqslant c \sup_{x_0 \in \mathbf{S}_0(w)} \left(\sum_{j=1}^{\lfloor \frac{1}{n} |\log t| \rfloor} 2^{-j(s-\frac{n}{p})q'} \left(\frac{w\left(B(x_0, 2^{-j})\right)}{|B(x_0,2^{-j})|} \right)^{-\frac{q'}{p}} \right)^{1/q'}, \quad t \to 0, \tag{4.55}$$

and, if $\mathbf{S}_0(w) \subset \mathbb{Z}^n$ (as is the case for our examples $w_{\alpha,\beta}$ and $w_{\log}$), then it looks even more natural to replace (4.45) by

$$\begin{aligned} \mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) &\geqslant c \sup_{m \in \mathbb{Z}^n} \left(\sum_{\nu=1}^{\lfloor \frac{1}{n} |\log t| \rfloor} 2^{-\nu(s-\frac{n}{p})q'} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} \right)^{-\frac{q'}{p}} \right)^{1/q'} \\ &\sim \sup_{m \in \mathbb{Z}^n} \left(\sum_{\nu=1}^{\lfloor \frac{1}{n} |\log t| \rfloor} 2^{-\nu s q'} \left(w(Q_{\nu,m}) \right)^{-\frac{q'}{p}} \right)^{1/q'}, \qquad t \to 0. \end{aligned}$$

We briefly discuss the compatibility of (4.15) and (4.45) if $s \leqslant \frac{n}{p} r_w$: we apply (3.1) and (1.6) with $r > r_w$,

$$w(B(x_0, 2^{-j})) \geqslant c_w \, 2^{-jnr},$$

where c is independent of $j \in \mathbb{N}_0$ and $x_0 \in \mathbb{R}^n$. Thus

$$\sup_{x_0\in\mathbb{R}^n} \left(\sum_{j=1}^{\lfloor \frac{1}{n}|\log t|\rfloor} 2^{-j(s-\frac{n}{p})q'} \left(\frac{w\left(B(x_0,2^{-j})\right)}{|B(x_0,2^{-j})|} \right)^{-\frac{q'}{p}} \right)^{1/q'} \leqslant c \left(\sum_{j=1}^{\lfloor \frac{1}{n}|\log t|\rfloor} 2^{-j(s-\frac{n}{p}r)q'} \right)^{1/q'} \sim t^{-\frac{r}{p}+\frac{s}{n}},$$

since $s < \frac{n}{p}r$. Writing $r = r_w + \varepsilon p$, it is obvious that (4.15) and (4.45) do not contradict each other. The compatibility of (4.18) and (4.46) is immediate in view of (3.1). Since the sum on the right-hand side of (4.45) converges for $s > \frac{n}{p}r_w$ the local estimate is only reasonable for $s \leqslant \frac{n}{p}r_w$, recall also (4.3).

Remark 4.14. If one uses the extremal functions constructed in (3.10), then in the same way as in Step 2 of Proposition 3.3 one obtains

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(2^{-kn}) \geqslant c2^{-k(s-\frac{n}{p})} \sup_{x_0\in\mathbb{R}^n} \left(\frac{|B(x_0,2^{-k})|}{w\left(B(x_0,2^{-k})\right)} \right)^{\frac{1}{p}},$$

see Remark 3.4 and (3.21). For $w \in \mathcal{A}_\infty$ and $\mathbf{S}_0(w) \neq \emptyset$ this can be strengthened by taking the sup over all $x_0 \in \mathbf{S}_0(w)$, and leads to

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \geqslant ct^{-\frac{1}{p}+\frac{s}{n}} \sup_{x_0\in\mathbf{S}_0(w)} \sup_{\substack{Q_{\nu,m}\ni x_0 \\ t\sim 2^{-\nu n}}} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} \right)^{-\frac{1}{p}}, \qquad t\to 0, \tag{4.56}$$

(the extension to F-spaces results from (1.44) and (2.3) again), whereas for $\mathbf{S}_0(w) = \emptyset$,

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \geqslant ct^{-\frac{1}{p}+\frac{s}{n}}, \qquad t\to 0.$$

In general, this estimate is for $q' < \infty$ weaker than (4.45).

Next we return to our special examples (1.8) and (1.9) and conclude from Proposition 4.12 that

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\log})}(t) \geqslant ct^{-\frac{1}{p}} \max\left(1, t^{-\frac{\alpha}{np}}(1+|\log t|)^{-\frac{\gamma}{p}}\right), \qquad t\to 0, \tag{4.57}$$

and

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\log})}(t) \geqslant c'\, t^{-\frac{1}{p}-\frac{\beta}{np}}(1+|\log t|)^{-\frac{\delta}{p}}, \qquad t\to\infty, \tag{4.58}$$

in view of Example 1.5. Combined with Corollary 4.9 this leads to the following.

Corollary 4.15. *Let $-n < \alpha \leqslant \beta$, $\beta \geqslant 0$, $\gamma, \delta \in \mathbb{R}$ with $\gamma \geqslant 0$ if $\alpha = 0$ and $\delta \geqslant 0$ if $\beta = 0$, and $0 < p < \infty$.*

(i) *Assume that* $\gamma \leqslant \delta$ *if* $\alpha = \beta$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\log})}(t) \sim t^{-\frac{1}{p}} \max\left(1, t^{-\frac{\alpha}{np}}(1+|\log t|)^{-\frac{\gamma}{p}}\right), \qquad t \to 0, \tag{4.59}$$

in particular,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \sim t^{-\frac{1}{p}-\frac{\max(\alpha,0)}{np}}, \qquad t \to 0. \tag{4.60}$$

(ii) *Assume that* $\gamma \geqslant \delta$ *if* $\alpha = \beta$, *and* $\delta = 0$ *if* $\beta = 0$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\log})}(t) \sim t^{-\frac{1}{p}-\frac{\beta}{np}}(1+|\log t|)^{-\frac{\delta}{p}}, \qquad t \to \infty, \tag{4.61}$$

in particular,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \sim t^{-\frac{1}{p}-\frac{\beta}{np}}, \qquad t \to \infty. \tag{4.62}$$

The above estimates extend previous results in [15, 16].

We collect the consequences of Proposition 4.12 and Corollaries 4.5 and 4.11 for $A^s_{p,q}(w_{\log})$; recall also Corollary 3.8 for $w_{\log} \in \mathcal{A}_1$.

Corollary 4.16. *Let* $w_{\log}$ *be given by* (1.9) *with* $\alpha > -n$, $\beta \geqslant 0$, $\gamma, \delta \in \mathbb{R}$. *Assume that* $0 < p < \infty$, $0 < q \leqslant \infty$, *and* $s > \sigma_p + \frac{\max(\alpha,0)}{p}$.

(i) *Then for* $s < \frac{n}{p} + \frac{\max(\alpha,0)}{p}$,

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\log})}(t) \geqslant c t^{\frac{s}{n}-\frac{1}{p}} \max\left(1, t^{-\frac{\alpha}{np}}|\log t|^{-\frac{\gamma}{p}}\right), \qquad t \to 0, \tag{4.63}$$

in particular,

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{\frac{s}{n}-\frac{1}{p}-\frac{\max(\alpha,0)}{np}}, \qquad t \to 0. \tag{4.64}$$

(ii) *If* $s = \frac{n}{p} + \frac{\max(\alpha,0)}{p}$, *and* $\frac{\gamma}{p} \neq \frac{1}{q'}$ *for* $\alpha > 0$, *then*

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\log})}(t) \geqslant c t^{\frac{s}{n}-\frac{1}{p}} \max\left(|\log t|^{\frac{1}{q'}}, t^{-\frac{\alpha}{np}}|\log t|^{(\frac{1}{q'}-\frac{\gamma}{p})_+}\right), \qquad t \to 0, \tag{4.65}$$

whereas in case of $\alpha > 0$ *and* $\frac{\gamma}{p} = \frac{1}{q'}$,

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\log})}(t) \geqslant c(\log|\log t|)^{\frac{1}{q'}}, \qquad t \to 0. \tag{4.66}$$

In particular, we obtain for $s = \frac{n}{p} + \frac{\max(\alpha,0)}{p}$ *that*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\alpha,\beta})}(t) \sim \begin{cases} |\log t|^{\frac{1}{q'}}, & \text{if } A^s_{p,q} = B^s_{p,q} \text{ and } 1 < q \leqslant \infty, \\ |\log t|^{\frac{1}{p'}}, & \text{if } A^s_{p,q} = F^s_{p,q} \text{ and } 1 < p < \infty, \end{cases} \qquad t \to 0. \tag{4.67}$$

(iii) *For the global estimates we get*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\log})}(t) \geqslant c t^{-\frac{\beta}{np}-\frac{1}{p}}(1+|\log t|)^{-\frac{\delta}{p}}, \qquad t\to\infty, \tag{4.68}$$

that is, for $1<p<\infty$, $0<\beta<n(p-1)$, $\alpha\leqslant\beta$ *and* $\delta\leqslant\gamma$ *if* $\alpha=\beta$,

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\log})}(t) \sim t^{-\frac{\beta}{np}-\frac{1}{p}}(1+|\log t|)^{-\frac{\delta}{p}}, \qquad t\to\infty, \tag{4.69}$$

and for $1<p<\infty$, $0<\beta<n(p-1)$, $\alpha\leqslant\beta$,

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\log})}(t) \sim t^{-\frac{\beta}{np}-\frac{1}{p}}, \qquad t\to\infty. \tag{4.70}$$

Remark 4.17. Plainly, estimate (4.63) is first proved for B-spaces in view of (4.45) and afterwards extended to F-spaces by (1.44). If we apply a similar argument to (4.65) and (4.66), we obtain parallel F-results where $\frac{1}{q'}$ is replaced by $(1-\frac{1}{\min(p,q)})_+$. Alternatively, for $\gamma\geqslant 0$ one can use Example 1.21, but the outcome is not sharp as well. We return to Remark 4.4. It is obvious that the above corollary disproves the assumption that (4.15) with $\varepsilon=0$ characterizes the local behavior in general; similarly for the global behavior and (the counterpart of) (4.23).

We give the counterpart for $w_{\varkappa,\Gamma}$.

Corollary 4.18. *Let* $w_{\varkappa,\Gamma}$ *be given by* (1.11) *with* $0<d<n$, $\varkappa>-(n-d)$, $0<p<\infty$, $0<q\leqslant\infty$, *and* $s>\sigma_p+\frac{\max(\varkappa,0)}{p}$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\varkappa,\Gamma})}(t) \sim \begin{cases} t^{-\frac{1}{p}+\frac{s}{n}-\frac{\max(\varkappa,0)}{np}}, & \text{if} \quad s<\frac{n}{p}+\frac{\max(\varkappa,0)}{p},\\ |\log t|^{\frac{1}{q'}}, & \text{if} \quad s=\frac{n}{p}+\frac{\max(\varkappa,0)}{p},\ 1<q\leqslant\infty \\ & \quad\text{and} \quad A^s_{p,q}=B^s_{p,q},\\ |\log t|^{\frac{1}{p'}}, & \text{if} \quad s=\frac{n}{p}+\frac{\max(\varkappa,0)}{p},\ 1<p<\infty \\ & \quad\text{and} \quad A^s_{p,q}=F^s_{p,q}, \end{cases} \tag{4.71}$$

for $t\to 0$, *and*

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w_{\varkappa,\Gamma})}(t) \sim t^{-\frac{1}{p}}, \qquad \text{for} \quad t\to\infty. \tag{4.72}$$

Some consequence of Proposition 4.12 is a necessary condition for the embedding $B^s_{p,q}(w)\hookrightarrow L_\infty$ which is by Proposition 2.2(ii) equivalent to the boundedness of $\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t)$ when $t\to 0$.

Corollary 4.19. *Let* $w\in\mathcal{A}_\infty$ *and let* (4.12) *be satisfied. Assume that for some* $x_0\in\mathbb{R}^n$,

$$\left\{ 2^{-j(s-\frac{n}{p})}\left(\frac{w\left(B(x_0,2^{-j})\right)}{|B(x_0,2^{-j})|}\right)^{-\frac{1}{p}} \right\}_{j\in\mathbb{N}} \notin \ell_{q'} \tag{4.73}$$

(*usual modification if* $q' = \infty$). *Then*

$$B^s_{p,q}(w) \not\hookrightarrow L_\infty. \tag{4.74}$$

In case of our model weights Corollary 4.19 can be refined to criteria for $A^s_{p,q}(w) \hookrightarrow L_\infty$ in some cases.

Corollary 4.20. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$.

(i) *Assume that* $\alpha > -n$, $\beta \geqslant 0$. *Then*

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow L_\infty \qquad \textit{if and only if} \qquad \begin{cases} \textit{either} & s > \frac{n}{p} + \frac{\max(\alpha,0)}{p}, \\ \textit{or} & s = \frac{n}{p} + \frac{\max(\alpha,0)}{p} \\ & \textit{and} \quad 0 < q \leqslant 1, \end{cases} \tag{4.75}$$

and

$$F^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow L_\infty \qquad \textit{if and only if} \qquad \begin{cases} \textit{either} & s > \frac{n}{p} + \frac{\max(\alpha,0)}{p}, \\ \textit{or} & s = \frac{n}{p} + \frac{\max(\alpha,0)}{p} \\ & \textit{and} \quad 0 < p \leqslant 1. \end{cases} \tag{4.76}$$

(ii) *Let* $\alpha > -n$, $\beta \geqslant 0$, $\gamma, \delta \in \mathbb{R}$ *with* $\delta \geqslant 0$ *if* $\beta = 0$. *Then*

$$B^s_{p,q}(w_{\log}) \hookrightarrow L_\infty \qquad \textit{if and only if}$$

$$\begin{cases} s > \frac{n}{p} + \frac{\max(\alpha,0)}{p}, & \textit{or} \\ s = \frac{n}{p} + \frac{\max(\alpha,0)}{p} & \textit{and} \quad 0 < q \leqslant 1 \quad \textit{if} \quad \alpha < 0, \\ & \textit{or} \quad 0 < q \leqslant 1, \gamma \geqslant 0 \quad \textit{if} \quad \alpha \geqslant 0, \\ & \textit{or} \quad \frac{\gamma}{p} > \frac{1}{q'} \quad \textit{if} \quad \alpha > 0. \end{cases}$$

(iii) *Let* $\Gamma \subset \mathbb{R}^n$ *be a* d*-set,* $0 < d < n$, *and* $w_{\varkappa,\Gamma}$ *be given by* (1.11) *with* $\varkappa > -(n-d)$. *Then*

$$B^s_{p,q}(w_{\varkappa,\Gamma}) \hookrightarrow L_\infty \qquad \textit{if and only if} \qquad \begin{cases} \textit{either} & s > \frac{n}{p} + \frac{\max(\varkappa,0)}{p}, \\ \textit{or} & s = \frac{n}{p} + \frac{\max(\varkappa,0)}{p} \\ & \textit{and} \quad 0 < q \leqslant 1, \end{cases} \tag{4.77}$$

and

$$F^s_{p,q}(w_{\varkappa,\Gamma}) \hookrightarrow L_\infty \qquad \textit{if and only if} \qquad \begin{cases} \textit{either} & s > \frac{n}{p} + \frac{\max(\varkappa,0)}{p}, \\ \textit{or} & s = \frac{n}{p} + \frac{\max(\varkappa,0)}{p} \\ & \textit{and} \quad 0 < p \leqslant 1. \end{cases} \tag{4.78}$$

Proof. The sufficiency in (4.75) is covered by (4.11) and monotonicity, whereas the necessity comes from Corollary 4.22(i): for assume that $B^{\sigma}_{p,u}(w_{\alpha,\beta}) \hookrightarrow L_\infty$ for some $\sigma < \frac{n}{p} + \frac{\max(\alpha,0)}{p}$ and $0 < u \leqslant \infty$, then monotonicity implies this for $B^s_{p,q}(w_{\alpha,\beta})$ with $s = \frac{n}{p} + \frac{\max(\alpha,0)}{p}$ and $1 < q \leqslant \infty$ which contradicts (4.80) in view of the unboundedness of $\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})}(t)$ for $t \to 0$ (which is by Proposition 2.2(ii) equivalent to $B^s_{p,q}(w_{\alpha,\beta}) \not\hookrightarrow L_\infty$). The F-case (4.76) as well as (ii) and (iii) can be treated in a parallel way. ■

Remark 4.21. We do not yet have a precise characterization of $A^s_{p,q}(w) \hookrightarrow L_\infty$ apart from the cases $w \in \mathcal{A}_1$ dealt with in Corollary 3.5 and those considered above. However, the gap between (4.1) and (4.73) seems not too large.

4.3. Growth envelopes

Since we have no complete characterizations for $\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t)$, $t \to 0$, $w \in \mathcal{A}_\infty \setminus \mathcal{A}_1$, in general, it makes no sense to study the corresponding indices $u_{\mathsf{G}}^{A^s_{p,q}(w)}$. However, dealing with our examples $w_{\alpha,\beta}$ and $w_{\varkappa,\Gamma}$, we may combine Corollaries 4.5 and 4.16 and extend it to the counterpart of Corollary 3.8.

Corollary 4.22. *Let $\alpha > -n$, $\beta \geqslant 0$, and $w_{\alpha,\beta}$ given by* (1.8). *Let $0 < p < \infty$, $0 < q \leqslant \infty$, and $s - \frac{\max(\alpha,0)}{p} > \sigma_p$. Then*

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(w_{\alpha,\beta})) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}-\frac{\max(\alpha,0)}{np}}, q\right), & s < \frac{n}{p} + \frac{\max(\alpha,0)}{p}, \\ \left(|\log t|^{\frac{1}{q'}}, q\right), & s = \frac{n}{p} + \frac{\max(\alpha,0)}{p} \quad \textit{and} \quad 1 < q \leqslant \infty, \end{cases} \tag{4.79}$$

and

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}(w_{\alpha,\beta})) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}-\frac{\max(\alpha,0)}{np}}, p\right), & s < \frac{n}{p} + \frac{\max(\alpha,0)}{p}, \\ \left(|\log t|^{\frac{1}{p'}}, p\right), & s = \frac{n}{p} + \frac{\max(\alpha,0)}{p} \quad \textit{and} \quad 1 < p < \infty, \end{cases} \tag{4.80}$$

and

$$\mathfrak{E}_{\mathsf{G}}(L_p(w_{\alpha,\beta})) = \left(t^{-\frac{1}{p}-\frac{\max(\alpha,0)}{np}}, p\right). \tag{4.81}$$

Proof. All assertions concerning the growth envelope functions are covered by our above considerations, that is, it remains to deal with the indices $u_{\mathsf{G}}^{A^s_{p,q}(w_{\alpha,\beta})}$. Moreover, the F-case (ii) follows from the B-case (i) together with Example 1.21 and Proposition 2.2. The upper estimates for $u_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})}$ follow from embeddings

(4.9), (4.10) together with Proposition 2.2(iii) and the unweighted result (2.6). We have to show that $u_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})} \geqslant q$ if $\sigma_p + \frac{\max(\alpha,0)}{p} < s \leqslant \frac{n}{p} + \frac{\max(\alpha,0)}{p}$. We proceed similar to the approach in the proof of Theorem 3.6 and consider the function

$$f_b(x) = \sum_{j=1}^{\infty} 2^{-j(s-\frac{n}{p}-\frac{\max(\alpha,0)}{p})}\ b_j\ \psi(2^j(x-x_0)), \tag{4.82}$$

where ψ is given by (3.11), $\{b_j\}_{j\in\mathbb{N}}$ a sequence of non-negative numbers, and $x_0 \in \mathbb{R}^n$ will be chosen later. Since $2^{-j(s-\frac{n}{p})}\ \psi(2^j(x-x_0))$ are atoms according to Definition 1.10 (no moment conditions needed), we obtain by Proposition 1.11 that

$$\left\| f_b | B^s_{p,q}(w_{\alpha,\beta}) \right\| \leqslant c \left(\sum_{j=1}^{\infty} 2^{j\frac{n+\max(\alpha,0)}{p}q}\ b_j^q\ w(B(x_0,2^{-j}))^{\frac{q}{p}} \right)^{\frac{1}{q}}, \tag{4.83}$$

(with obvious modification if $q=\infty$). If $\alpha \geqslant 0$, we choose $x_0 = 0$ such that $w(B(x_0,2^{-j})) \sim 2^{-j(\alpha+n)}$, for $\alpha < 0$ we take $|x_0| = 2$ such that $w(B(x_0,2^{-j})) \sim 2^{-jn}$. Thus (4.83) implies that

$$\left\| f_b | B^s_{p,q}(w_{\alpha,\beta}) \right\| \leqslant c \left\| b|\ell_q \right\|. \tag{4.84}$$

We follow the same line of arguments as in the proof of Theorem 3.6. The counterpart of (3.25) reads for $s < \frac{n}{p} + \frac{\max(\alpha,0)}{p}$ as

$$\left(\int_0^{\varepsilon} \left[t^{\frac{1}{p}-\frac{s}{n}+\frac{\max(\alpha,0)}{np}}\ f^*(t) \right]^v \frac{\mathrm{d}t}{t} \right)^{1/v} \leqslant c\ \left\| f | B^s_{p,q}(w_{\alpha,\beta}) \right\| \tag{4.85}$$

and can thus be extended on both sides to

$$\begin{aligned} \left(\sum_{j=J}^{\infty} b_j^v \right)^{\frac{1}{v}} &\leqslant c_1 \left(\int_0^{\varepsilon} \left[t^{\frac{1}{p}-\frac{s}{n}+\frac{\max(\alpha,0)}{np}}\ f_b^*(t) \right]^v \frac{\mathrm{d}t}{t} \right)^{\frac{1}{v}} \\ &\leqslant c_2 \left\| f_b | B^s_{p,q}(w_{\alpha,\beta}) \right\| \leqslant c_3 \| b|\ell_q \| \end{aligned}$$

for arbitrary sequences of non-negative numbers with, say, $b_1 = \cdots = b_J = 0$ for some $J \in \mathbb{N}$ with $2^{-J} \sim \varepsilon$. This follows by (4.84), (4.85) and the counterpart of (3.28),

$$f_b^*(t) \geqslant c b_j\ 2^{-j(s-\frac{n}{p}-\frac{\max(\alpha,0)}{p})}, \qquad t \sim 2^{-jn},$$

and leads to $v \geqslant q$.

In case of $s = \frac{n}{p} + \frac{\max(\alpha,0)}{p}$ we take $b_j = 0$, $j > m$, such that f_b obtains the special form

$$f_b(x) = \sum_{j=1}^{m} b_j \psi\left(2^j(x-x_0)\right), \qquad m \in \mathbb{N}.$$

The rest is now completely parallel to the end of the proof of Theorem 3.6. As for the L_p-part, the upper estimate for $u_{\mathsf{G}}^{L_p(w_{\alpha,\beta})} = p$ is covered by (4.33) together with (2.9). The lower estimate can be seen by the following simplified version of (4.82). We have to disprove that

$$\left(\int_0^{\varepsilon} \left[t^{\frac{1}{p}+\frac{\max(\alpha,0)}{np}} f^*(t) \right]^v \frac{\mathrm{d}t}{t} \right)^{1/v} \leqslant c \, \|f|L_p(w_{\alpha,\beta})\| \tag{4.86}$$

holds for $v < p$ and all $f \in L_p(w_{\alpha,\beta})$. Consider

$$h_\eta(x) = |x-x_0|^{-\frac{n}{p}-\frac{\max(\alpha,0)}{p}} \left(1+|\log|x-x_0||\right)^{-\mu} \chi_{B(x_0,\eta^{1/n})}(x), \qquad 0<\eta<\varepsilon<1, \quad \frac{1}{p}<\mu<\frac{1}{v}.$$

Again we choose $x_0 = 0$ if $\alpha \geqslant 0$, and $|x_0| = 2$ if $\alpha < 0$, such that $\|h_\eta|L_p(w_{\alpha,\beta})\| \leqslant c$ since $\mu > \frac{1}{p}$, but the left-hand side of (4.86) diverges since $\mu < \frac{1}{v}$. ■

Remark 4.23. The above corollary extends previous results in [15, 16]. Corollaries 4.11 and 4.16 also provide two-sided estimates in case of $w_{\log}$, but we restricted ourselves to those cases with precise asymptotic results.

We come to $w_{\varkappa,\Gamma}$; note that the case $\varkappa \leqslant 0$ is already covered by Corollary 3.8(ii).

Corollary 4.24. *Let $\Gamma \subset \mathbb{R}^n$ be a d-set, $0<d<n$, and $w_{\varkappa,\Gamma}$ be given by* (1.11) *with $\varkappa > -(n-d)$. Let $0<p<\infty$, $0<q\leqslant\infty$, and $s - \frac{\max(\varkappa,0)}{p} > \sigma_p$. Then*

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}(w_{\varkappa,\Gamma})) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}-\frac{\max(\varkappa,0)}{np}}, q\right), & s < \frac{n}{p} + \frac{\max(\varkappa,0)}{p}, \\ \left(|\log t|^{\frac{1}{q'}}, q\right), & s = \frac{n}{p} + \frac{\max(\varkappa,0)}{p} \quad \text{and} \quad 1<q\leqslant\infty, \end{cases} \tag{4.87}$$

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}(w_{\varkappa,\Gamma})) = \begin{cases} \left(t^{-\frac{1}{p}+\frac{s}{n}-\frac{\max(\varkappa,0)}{np}}, p\right), & s < \frac{n}{p} + \frac{\max(\varkappa,0)}{p}, \\ \left(|\log t|^{\frac{1}{p'}}, p\right), & s = \frac{n}{p} + \frac{\max(\varkappa,0)}{p} \quad \text{and} \quad 1<p<\infty, \end{cases} \tag{4.88}$$

and

$$\mathfrak{E}_{\mathsf{G}}(L_p(w_{\varkappa,\Gamma})) = \left(t^{-\frac{1}{p}-\frac{\max(\varkappa,0)}{np}}, p\right). \tag{4.89}$$

Proof. This works completely parallel to the proof of Corollary 4.22. Since we now have $\mathbf{S}_{\text{sing}}(w_{\varkappa,\Gamma}) = \Gamma$, $\varkappa \neq 0$, instead of $\mathbf{S}_{\text{sing}}(w_{\alpha,\beta}) = \{0\}$, $\alpha \neq 0$, there, one has to choose either $x_0 \in \Gamma$ or $x_0 \in \mathbb{R}^n$ with dist$\,(x_0,\Gamma) \geqslant c$ accordingly. ■

Remark 4.25. Concerning applications, it is obvious that Corollaries 4.22 and 4.24 admit Hardy inequalities in the sense of Corollary 3.10. The same applies to Corollary 3.11, but we shall not present it explicitly here; partial results can be found in [16] already.

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