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ON SOME DIOPHANTINE RESULTS RELATED TO HERMITE POLYNOMIALS

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Abstract: In this paper we prove that the shifted Hermite polynomial $H_n(x) + b$ has at least three simple zeros for each complex number b, provided that $n \geqslant 7$.

Keywords: Hermite polynomials, Higher degree equations.

1. Introduction

There are several mathematicians who investigated the classical diophantine equation

$$P(x) = G(y) \qquad \text{in unknowns } x,\ y \in \mathbb{Z}, \tag{1}$$

with given polynomials $P(x)$, $G(x) \in \mathbb{Q}[x]$.

Bilu and Tichy in [2] completely characterized those polynomials $P(x)$, $G(x) \in \mathbb{Q}[x]$ for which the equation (1) has infinitely many integer solutions. Their result is ineffective so it does not give an algorithm to find all the solutions. Using the criterion of Bilu and Tichy [2] Stoll [9] as well as Stoll and Tichy [10] studied equation (1) in the special case when the polynomial $P(x)$ and $Q(x)$ are different shifted Hermite polynomials. More precisely, they proved the following theorems.

Let A, B, C denote arbitrary rational numbers with $AB \neq 0$. Stoll and Tichy [10] proved that

Theorem A (Stoll and Tichy). *Let $m > n \geqslant 4$ arbitrary rational integers. Then the number of integral (x, y) satisfying*

$$AH_m(x) + BH_n(y) = C, \tag{2}$$

is finite, where $H_n(x)$ denotes the nth Hermite polynomial which is defined below.

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For "small cases" Stoll [9] showed that

Theorem B (Stoll). *The diophantine equation*

$$AH_m(x) + BH_2(y) = C \tag{3}$$

with $m \geqslant 3$ *only admit finitely many integral solutions* (x, y) *with exception of*

$$m = 4, \qquad \frac{C}{A} - \frac{B}{2A} \in \left\{ -\frac{3}{2}, \frac{2}{3} \right\}.$$

Moreover, the solutions satisfy $\max(|x|, |y|) < C = C(A, B, C, m)$.

The Hermite polynomials $H_n(x)$ are defined by the identity

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n.$$

The Hermite polynomials play important role in numerical analysis, analytic number theory and physics (see [1], [4]). For example, the Hermite function of order n

$$\Psi_n(x) = e^{-\frac{1}{2}x^2} H_n(x)$$

occurs in the wave mechanical treatment of the harmonic oscillator [6]. However, this is a very simple mechanical system of which the analysis of its properties is of great importance because of its application to the quantum theory of radiation.

In the present paper we investigate the multiplicities of the zeros of shifted Hermite polynomials $H_n(x) + b$, where b is a complex number. We prove among other things that for given $n \geqslant 7$, the shifted Hermite polynomial $H_n(x) + b$ has at least three simple zeros for arbitrary complex number b. As an application we will give an effective finiteness theorem related to the diophantine equation

$$F(H_n(x)) = H_2(y), \tag{4}$$

where $F(x)$ is a polynomial with algebraic integer coefficients.

2. Results

A polynomial $F(x)$ with complex coefficients will be called *non-degenerate* if it has at least three zeros of odd multiplicities and *degenerate* otherwise.

For $n > 1$, L_n denotes the cardinality of the set of nonzero complex numbers b for which the nth shifted Hermite polynomial $H_n(x) + b$ is degenerate.

Theorem 1. *We have* $L_3 = L_4 = L_6 = 2$. *Further, if* $n \geqslant 7$ *then* $L_n = 0$.

The next result is a generalization of Theorem B. This is an application of our first theorem and Lemma 5 of Brindza [3].

Theorem 2. *Let $\mathbb{K}$ be an algebraic number field with ring of integers $O_{\mathbf{K}}$, and let $F(x)+2 \in O_{\mathbf{K}}[x]$ be a non-square polynomial. Then the equation*

$$F\left(H_n(x)\right) = H_2(y) \tag{5}$$

has only finitely many solutions $x,\ y \in O_{\mathbf{K}}$ which can be effectively determined, provided that $n \geqslant 7$.

3. Auxiliary results

The following are well-known identities of Hermite polynomials (see [7], Chapter 11):

Lemma 1. *For a positive integer n, $H_n(x)$ denotes the nth Hermite polynomial. Then we have*

(i) $H_n'(x) = 2nH_{n-1}(x)$.
(ii) $H_{2n+1}(0) = 0,\ H_{2n}'(0) = 0$.
(iii) $H_n(x) = 2^n x^n + \pi_{n-2}(x)$, *where $\pi_{n-2}(x)$ is a polynomial with degree $n-2$.*
(iv) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$.
(v) $H_n(x) = 2xH_{n-1}(x) - H_{n-1}'(x)$.
(vi) $H_n(-x) = (-1)^n H_n(x)$.
(vii) $H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2$.
(viii) $H_n(x)$ *has only simple zeros.*

To prove our theorems we need some lemmas.

Lemma 2 (Stoll). *Let $n \geqslant 7$. Then $H_n(x) + \delta$ has at least three simple roots for $\delta \in \mathbb{Q}$.*

We remark that one can deduce from the proof of the Lemma 3.5 of Stoll [9] that the assertion of his result is also true if $\delta \in \mathbb{R}$. In fact, our first theorem is an extension of Lemma 2 to the case when δ is a complex number.

Lemma 3. *If n is a rational integer with $n \geqslant 5$ and a, b are complex numbers with $b \neq 0$ then the polynomial $(H_n(x)+a)^2 + b$ is non-degenerate.*

Proof. On supposing the contrary we have

$$(H_n(x)+a)^2 + b = f^2(x) \tag{6}$$

or

$$(H_n(x)+a)^2 + b = g(x)f^2(x), \tag{7}$$

for some $f(x), g(x) \in \mathbb{C}[x]$, where $g(x)$ is a quadratic polynomial with nonzero discriminant. From (6) we obtain that

$$2(H_n(x)+a)H_n'(x) = 2f(x)f'(x). \tag{8}$$

However, since $(f(x), H_n(x) + a) = 1$ we obtain that $f(x)|H_n'(x)$, but then $n = \deg f(x) \leqslant \deg H_n'(x) = n - 1$ which is impossible.

If we differentiate (7), we have

$$2(H_n(x)+a)H_n'(x) = 2g(x)f(x)f'(x)+g'(x)f^2(x) = f(x)(2g(x)f'(x)+g'(x)f(x)).$$

Since $b \neq 0$, we obtain from (7) that $(f(x)g(x), H_n(x) + a) = 1$, so

$$f(x)|H_n'(x) \qquad \text{and} \qquad H_n(x) + a|2g(x)f'(x) + g'(x)f(x).$$

It is easy to see that $\deg f(x) = n - 1$, therefore

$$H_n'(x) = c_1 f(x) \qquad \text{and} \qquad g'(x)f(x) + 2g(x)f'(x) = c_2(H_n(x) + a), \tag{9}$$

where c_1, c_2 are nonzero complex numbers. From (9) we can infer by induction that

$$c_2 H_n^{(i)}(x) = i^2 g''(x)f^{(i-1)}(x) + (2i+1)g'(x)f^{(i)}(x) + 2g(x)f^{(i+1)}(x) \tag{10}$$

and

$$f^{(i)}(x) = \frac{1}{c_1}H_n^{(i+1)}(x), \qquad i = 1, 2, \dots. \tag{11}$$

We can assume that the polynomial $g(x)$ is monic. Let a_0 denote the leading coefficient of polynomial $f(x)$. If we compare the leading coefficients in (9) we get that

$$2^n n = c_1 a_0 \qquad \text{and} \qquad c_2 2^n = 2na_0.$$

It is easy to see from the above that

$$c_1 c_2 = 2n^2. \tag{12}$$

It is not to hard to obtain from (i) by induction that

$$H_n^{(i)}(x) = 2^i n(n-1)\cdots(n-(i-1))H_{n-i}(x), \qquad i = 1, 2, \dots, n. \tag{13}$$

Substituting $i = n - 1$ and $x = 0$ into (10) and applying (vii), (13) and (11) we have that

$$0 = (2n-1)g'(0)\frac{1}{c_1}2^n n!$$

hence

$$g'(0) = 0. \tag{14}$$

After this, if we substitute $i = n - 2$ and $x = 0$ into the equation (10) and use the expressions (vii), (11), (13) and (12) we can infer that $g(0) = 1 - n$ and so

$$g(x) = x^2 + 1 - n. \tag{15}$$

Thus we get from (9) that

$$\frac{1}{c_1}2xH_n'(x) + \frac{2}{c_1}(x^2 + 1 - n)H_n''(x) = c_2(H_n(x) + a).$$

Using that $c_1c_2 = 2n^2$ and $H_n''(x) = 2xH_n'(x) - 2nH_n(x)$ we obtain after some computation that

$$\left(2nx^2 + 2n - n^2\right) H_n(x) + an^2 = \left(2x^3 + 3x - 2nx\right) H_n'(x). \tag{16}$$

If we differentiate (16) we can deduce by (i) that

$$\begin{aligned} 4nxH_n(x) + 2n\left((2n-6)x^2 + 4n - n^2 - 3\right) H_{n-1}(x) \\ = 4n(n-1)\left(2x^3 + 3x - 2nx\right) H_{n-2}(x). \end{aligned} \tag{17}$$

For odd values of n we substitute $x = 0$ into (17) and we obtain, by (ii), that

$$2n\left(4n - n^2 - 3\right) H_{n-1}(0) = 0.$$

Since now $H_{n-1}(0) \neq 0$ we can see that $4n - n^2 - 3 = 0$, that is $n = 1$ or $n = 3$. Hence our assertion is true for odd $n \geqslant 5$. In case when n is even, differentiating (17) and using (v) we have

$$\begin{aligned} &4nxH_n'(x) + 4n\left[2xH_{n-1}(x) - H_{n-1}'(x)\right] \\ &+ 4n(2n-6)xH_{n-1}(x) + 2n\left((2n-6)x^2 + 4n - n^2 - 3\right) H_{n-1}'(x) \\ &= 4n(n-1)(6x^2 + 3 - 2n)H_{n-2}(x) + 4n(n-1)\left(2x^3 + 3x - 2nx\right) H_{n-2}'(x). \end{aligned} \tag{18}$$

From Lemma 1 we know that $H_{2n}'(0) = H_{2n+1}(0) = 0$. Whence, if we substitute $x = 0$ into (18) we have

$$(-2n^3 + 8n^2 - 10n)H_{n-1}'(0) = 4n(n-1)(3-2n)H_{n-2}(0). \tag{19}$$

Using that $H_{n-1}'(x) = 2(n-1)H_{n-2}(x)$ and $H_{n-2}(0) \neq 0$ we get

$$n(n-2)(n-4) = 0. \tag{20}$$

■

In an earlier paper [8] we proved the following lemma.

Lemma 4. *Let $f(x)$ be a polynomial with complex coefficients. If $\deg f(x) \geqslant 5$ then there are at most two complex numbers b for which the polynomial $f(x) + b$ is degenerate.*

A superelliptic equation is of the form $f(x) = y^m$, where $f(x)$ is a polynomial of degree $\geqslant 3$ with integer or algebraic integer coefficients and $m \in \mathbb{N}$. LeVeque [5] gave a criterion for superelliptic equations to have only finitely many integer solutions. LeVeque's theorem is ineffective in the sense that the proof does not provide any algorithm to compute the solutions. Later, Brindza generalized this result and gave an effective upper bound for the size of solutions of the equation $f(x) = by^m$.

Let $\mathbb{K}$ be an algebraic number field with ring of integers $O_{\mathbf{K}}$.

Lemma 5 (Brindza, [3]). *Let*

$$f(x) = a_0x^N + \cdots + a_N = a_0 \prod_{i=1}^{n} (x - \alpha_i)^{r_i}$$

be a polynomial in $O_{\mathbf{K}}[x]$ *with* $a_0 \neq 0$ *and* $\alpha_i \neq \alpha_j$ *for* $i \neq j$. *Further, let* $b \in O_{\mathbf{K}}$, $m > 1$ *and* $q_i = m/(m, r_i)$, $i = 1, 2, \cdots, n$. *Suppose that* $(q_1, q_2, \cdots, q_n)$ *is not a permutation of* $(q, 1, \cdots, 1)$ *or* $(2, 2, 1, \cdots, 1)$, *where* $q \geqslant 1$. *Then the equation*

$$f(x) = by^m \qquad \text{in } x, y \in O_{\mathbf{K}}$$

has only finitely many solutions and all these can be effectively determined.

An easy consequence of this result is that the hyperelliptic equation

$$f(x) = y^2 \qquad \text{in } x, y \in O_{\mathbf{K}}$$

has only finitely many solutions x, y provided that the polynomial $f(x)$ is non-degenerated.

4. Proofs

Proof of Theorem 1. First of all, we remark that the discriminant of the polynomial $H_n(x) + b$ is a polynomial in b of degree $n - 1$. Hence $H_n(x) + b$ has only simple zeros apart from at most $n - 1$ distinct values of b.

For values of $n < 7$ we can use the above observation. First assume that $n \geqslant 7$ is an odd integer and there is a nonzero value b for which the polynomial $H_n(x)+b$ is degenerate. Then the polynomial $H_n(-x) + b = -H_n(x) + b$ is also degenerate by (vi). It is easy to see that the polynomial

$$(H_n(x) + b)(H_n(x) - b) = (H_n(x))^2 - b^2 \tag{21}$$

is also degenerate. However, from Lemma 3 we obtain that it is possible only if $b = 0$. But this is a contradiction.

Now suppose that n is even. From Lemma 4 we know that there are at most two complex numbers for which the shifted Hermite polynomials are degenerate. Suppose that b_1 and b_2 are two distinct complex numbers for which $H_n(x) + b_i$ are degenerate $i = 1, 2$. In this case there are the following four possibilities:

(a) $H_n(x) + b_1 = f_1(x)^2$ and $H_n(x) + b_2 = f_2(x)^2$,
(b) $H_n(x) + b_1 = f_1(x)^2$ and $H_n(x) + b_2 = g_2(x)f_2(x)^2$,
(c) $H_n(x) + b_1 = g_1(x)f_1(x)^2$ and $H_n(x) + b_2 = f_2(x)^2$,
(d) $H_n(x) + b_1 = g_1(x)f_1(x)^2$ and $H_n(x) + b_2 = g_2(x)f_2(x)^2$,

where $f_1(x)$, $f_2(x) \in \mathbb{C}[x]$ and the polynomials $g_1(x)$, $g_2(x) \in \mathbb{C}[x]$ are quadratic polynomials with nonzero discriminant. Further, $g_1(x)$, $g_2(x)$, $f_1(x)$, $f_2(x)$ are pairwise coprime polynomials in $\mathbb{C}[x]$. Study the cases (a), (b) and (c) we can infer that the polynomial

$$(H_n(x)+b_1)(H_n(x)+b_2)=\left(H_n(x)+\frac{b_1+b_2}{2}\right)^2-\left(\frac{b_1+b_2}{2}\right)^2+b_1b_2$$

is degenerate. However, it is possible only if that $((b_1+b_2)/2)^2-b_1b_2=0$ and so $b_1=b_2$. Investigate now the last case (d). We can suppose that

$$g_1(x)=(x-\alpha_1)(x-\alpha_2) \qquad \text{and} \qquad f_1(x)=2^{n/2}\prod_{i=1}^{n/2-1}(x-\beta_i) \tag{22}$$

and

$$g_2(x)=(x-\gamma_1)(x-\gamma_2) \qquad \text{and} \qquad f_2(x)=2^{n/2}\prod_{i=1}^{n/2-1}(x-\delta_i), \tag{23}$$

where $\alpha_j,\gamma_j,\beta_i,\delta_i\in\mathbb{C}$, $j=1,2$, $i=1,\dots,n/2-1$ and $\alpha_1\neq\alpha_2$, $\gamma_1\neq\gamma_2$.

Applying (vi) we get that

$$(x-\alpha_1)(x-\alpha_2)\prod_{i=1}^{n/2-1}(x-\beta_i)^2=(x+\alpha_1)(x+\alpha_2)\prod_{i=1}^{n/2-1}(x+\beta_i)^2. \tag{24}$$

It follows from (24) that

$$\alpha_1,\alpha_2\in\left\{-\alpha_1,-\alpha_2,-\beta_1,\dots,-\beta_{n/2-1}\right\}.$$

If $\alpha_1=-\beta_j$ for some $j\in\{1,2,\dots,n/2-1\}$, then $x-\alpha_1=x+\beta_j$ and

$$(x-\alpha_2)\prod_{i=1}^{n/2-1}(x-\beta_i)^2=(x+\alpha_1)(x+\alpha_2)(x+\beta_j)\prod_{i=1,i\neq j}^{n/2-1}(x+\beta_i)^2. \tag{25}$$

Hence $-\beta_j\in\left\{\alpha_2,\beta_1,\dots,\beta_{n/2-1}\right\}$. When $-\beta_j=\alpha_2$ then $\alpha_1=\alpha_2$ which contradicts our assumption that $g_1(x)$ has non-zero discriminant. Thus $\alpha_1=-\beta_j=\beta_k$ for some $k\in\{1,\dots,n/2-1\}$. But, then $(g_1(x),f_1(x))\neq 1$ and so $H_n'(x)=2nH_{n-1}(x)$ has a multiple root, however we know that $H_{n-1}(x)$ has only simple zeros. We obtain from the above that

$$\alpha_1,\alpha_2\in\{-\alpha_1,-\alpha_2\}.$$

If $\alpha_1=-\alpha_1$ then $\alpha_2=-\alpha_2$ and $\alpha_1=\alpha_2=0$ which is impossible. So we get

$$\alpha_1+\alpha_2=0 \qquad \text{and} \qquad \beta_j\in\left\{-\beta_1,\dots,-\beta_{n/2-1}\right\}, \quad j=1,\dots,n/2-1. \tag{26}$$

Using the same argument as above we can infer that

$$\gamma_1+\gamma_2=0 \qquad \text{and} \qquad \delta_j\in\left\{-\delta_1,\dots,-\delta_{n/2-1}\right\}, \quad j=1,\dots,n/2-1. \tag{27}$$

If $n/2-1$ is odd then this implies that $0 \in \{\beta_1, \beta_2, \dots \beta_{n/2-1}\} \bigcap \{\delta_1, \dots, \delta_{n/2-1}\}$. But it is impossible because then $f_1(0) = f_2(0) = 0$ and so $b_1 = b_2$.

Suppose that $n/2-1$ is even. Then $f_1(0)f_2(0) \neq 0$. From (d) we obtain that

$$(H_n(x)+a)^2 + b = g(x)f(x)^2, \tag{28}$$

where

$$a = \frac{b_1+b_2}{2}, \qquad b = b_1b_2 - \left(\frac{b_1+b_2}{2}\right)^2, \qquad f(x) = f_1(x)f_2(x)$$

and

$$g(x) = (x^2+c_1)(x^2+c_2) \tag{29}$$

where $c_1 = \alpha_1\alpha_2$, $c_2 = \gamma_1\gamma_2$. It is not too hard to see from (28) that

$$2\left(H_n(x)+a\right)2nH_{n-1}(x) = f(x)\left(g'(x)f(x) + 2g(x)f'(x)\right). \tag{30}$$

Since $(H_n(x)+a, f(x)) = 1$, we have $f(x)|2H_{n-1}(x)$. This fact and property (iii) yield that

$$2H_{n-1}(x) = xf(x). \tag{31}$$

Substituting this expression into (30) we have

$$2n\left(H_n(x)+a\right)x = g'(x)f(x) + 2g(x)f'(x). \tag{32}$$

After the $i-1$-th differentiation ($i \geqslant 4$) from (32) we obtain that

$$\begin{aligned}(i-1)2^{i-1}n^2(n-1)\cdots(n-(i-3))H_{n-(i-2)}(x)&\\ +2^i n^2(n-1)\cdots(n-(i-2))xH_{n-(i-1)}(x) &= \sum_{j=0}^{4} a_{ij}g^{(j)}(x)f^{(i-j)}(x),\end{aligned} \tag{33}$$

where

$$a_{i0} = 2, \quad a_{i1} = 2i-1, \quad a_{i2} = (i-1)^2, \quad a_{i3} = \frac{(i-1)(i-2)(2i-3)}{6}$$

and

$$a_{i4} = \frac{(i-1)(i-2)^2(i-3)}{12}.$$

One can obtain from (i) and (31) by induction that

$$\begin{aligned}2^{i+1}(n-1)(n-2)\cdots(n-i)H_{n-(i+1)}(x)&\\ = if^{(i-1)}(x) + xf^{(i)}(x),& \qquad i = 1, 2, \dots, n-2.\end{aligned} \tag{34}$$

If we substitute $x = 0$ into (34) we can deduce that

$$f^{(i)}(0) = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ \frac{2^{i+2}}{i+1}(n-1)(n-2)\cdots(n-(i+1))H_{n-(i+2)}(0), & \text{if } i \text{ is even.} \end{cases} \tag{35}$$

Substitute $i = n$, $n-2$, $n-4$ and $x = 0$ into (33). Using (35), (29) and $\deg f(x) = n-2$ we obtain that

$$\begin{aligned}(n-1)2^{n-1}n^2(n-1)(n-2)\cdots 3H_2(0)\\ = (n-1)^2 2(c_1+c_2)\frac{2^n}{n-1}(n-1)(n-2)\cdots 2\\ + 2(n-1)(n-2)^2(n-3)\frac{2^{n-2}}{n-3}(n-1)(n-2)\cdots 3H_2(0),\end{aligned} \tag{36}$$

$$\begin{aligned}(n-3)2^{n-3}n^2(n-1)(n-2)\cdots 5H_4(0)\\ = 2c_1c_2\frac{2^n}{n-1}(n-1)(n-2)\cdots 2\\ + (n-3)^2 2(c_1+c_2)\frac{2^{n-2}}{n-3}(n-1)(n-2)\cdots 3H_2(0)\\ + 2(n-3)(n-4)^2(n-5)\frac{2^{n-4}}{n-5}(n-1)(n-2)\cdots 5H_4(0),\end{aligned} \tag{37}$$

$$\begin{aligned}(n-5)2^{n-5}n^2(n-1)(n-2)\cdots 7H_6(0)\\ = 2c_1c_2\frac{2^{n-2}}{n-3}(n-1)(n-2)\cdots 3H_2(0)\\ + (n-5)^2 2(c_1+c_2)\frac{2^{n-4}}{n-5}(n-1)(n-2)\cdots 5H_4(0)\\ + 2(n-5)(n-6)^2(n-7)\frac{2^{n-6}}{n-7}(n-1)(n-2)\cdots 7H_6(0).\end{aligned} \tag{38}$$

It is well known that

$$H_2(0) = -2, \qquad H_4(0) = 12, \qquad H_6(0) = -120.$$

We compute from (36) and (37) that

$$c_1 + c_2 = 1 - n,$$

$$c_1c_2 = -\frac{(n-1)(n-3)}{4}.$$

Using these expressions, from (38) we can infer the following contradiction:

$$96 = 0.$$

Consequently, there is at most one complex number for which the polynomial $H_n(x) + b$ is degenerate when n is even. However, if b is a non-real complex number for which $H_n(x)+b$ is degenerate then $H_n(x)+\overline{b}$ is also degenerate, where $\overline{b}$ is the complex conjugate of b. Hence b must be a real number. But from Lemma 2 we know that in this case the polynomial $H_n(x) + b$ has at least three simple roots. ∎

Proof of Theorem 2. Let $\mathbb{K}$ be an algebraic number field with ring of integers $O_{\mathbf{K}}$ and let $F(x)+2 \in O_{\mathbf{K}}[x]$ be a non-square polynomial. Thus we can write the following

$$F(x)+2=\prod_{i=1}^{t}(x-\alpha_i)^{k_i}, \tag{39}$$

where some exponent k_i is odd. Since $H_n(x)+b$ is non-degenerate for every complex number b, provided that $n \geqslant 7$, by Theorem 1 the polynomial

$$F(H_n(x))+2=\prod_{i=1}^{t}(H_n(x)-\alpha_i)^{k_i}$$

is also non-degenerate. From equation

$$F(H_n(x))=H_2(y)=4y^2-2 \tag{40}$$

we can deduce that

$$F(H_n(x))+2=(2y)^2.$$

Now the assertion follows from Lemma 5. ∎

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