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ON DILATION OPERATORS IN TRIEBEL-LIZORKIN SPACES

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Abstract: We consider dilation operators $T_k : f \to f(2^k \cdot)$ in the framework of Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$. If $s > n \max\left(\frac{1}{p}-1, 0\right)$, T_k is a bounded linear operator from $F^s_{p,q}(\mathbb{R}^n)$ into itself and there are optimal bounds for its norm. We study the situation on the line $s = n \max\left(\frac{1}{p}-1, 0\right)$, an open problem mentioned in [ET96, 2.3.1]. It turns out that the results shed new light upon the diversity of different approaches to Triebel-Lizorkin spaces on this line, associated to definitions by differences, Fourier-analytical methods and subatomic decompositions.

Keywords: Triebel-Lizorkin spaces, Besov spaces, dilation operators, moment conditions.

Introduction

In this article dilation operators acting on Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ are investigated. The idea for this paper originates from its forerunners [Vyb08] and [Sch09], where the authors studied corresponding problems for Besov spaces. Since the substantial theory of the Triebel-Lizorkin spaces is strongly linked with the theory of Besov spaces – in the sequel briefly denoted as F-spaces and B-spaces, respectively – the question came up whether those previous results could be carried over to the F-space setting. This paper aims at providing a rather final answer to this question.

We consider dilation operators of the form

$$T_k f(x) = f(2^k x), \qquad x \in \mathbb{R}^n, \quad k \in \mathbb{N}, \tag{0.1}$$

which represent bounded operators from $F^s_{p,q}(\mathbb{R}^n)$ into itself. Their behaviour is well known when $s > \sigma_p = n \max\left(\frac{1}{p} - 1, 0\right)$. Then we have for $0 < p < \infty$, $0 < q \leqslant \infty$,

$$\|T_k | \mathcal{L}(F^s_{p,q}(\mathbb{R}^n))\| \sim 2^{k(s-\frac{n}{p})}, \qquad s > \sigma_p,$$

cf. [ET96, 2.3.1, 2.3.2]. Here we investigate the situation on the line $s = \sigma_p$. For $1 < p < \infty$ and $0 < p \leqslant 1$ with $p \leqslant q$ we obtain sharp estimates for the norms of

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the operators T_k, i.e.,

$$\|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \sim 2^{k(\sigma_p-\frac{n}{p})}\cdot\begin{cases} k^{\frac{1}{q}-\frac{1}{\max(q,2)}} & \text{if } 1<p<\infty,\\ k^{1/p} & \text{if } 0<p\leqslant 1, \quad p\leqslant q,\end{cases}$$

whereas, for $0<q<p<1$, we only have

$$2^{k(\sigma_p-\frac{n}{p})}k^{1/p}\lesssim \|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\|\lesssim 2^{k(\sigma_p-\frac{n}{p})}k^{1/q}$$

or, when $0<q<p=1$,

$$2^{-kn}k^{\max(1,1/q-1/2)}\lesssim \|T_k|\mathcal{L}(F^{0}_{1,q}(\mathbb{R}^n))\|\lesssim 2^{-kn}k^{1/q}.$$

As a by-product, the results for the dilation operators lead to new insights concerning the nature of the different approaches to F-spaces with positive smoothness – namely the classical ($\mathbf{F}^s_{p,q}$), the Fourier-analytical ($F^s_{p,q}$) and the subatomic approach ($\mathfrak{F}^s_{p,q}$) – on the line $s=\sigma_p$. Recent results by Hedberg, Netrusov [HN07] on atomic decompositions and by Triebel [Tri06, Sect. 9.2] on the reproducing formula prove coincidences

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n)=\mathfrak{F}^s_{p,q}(\mathbb{R}^n),\qquad s>n\left(\frac{1}{\min(p,q)}-\frac{1}{p}\right),\ \ 0<p<\infty,\ \ 0<q\leqslant\infty,$$

and

$$F^s_{p,q}(\mathbb{R}^n)=\mathfrak{F}^s_{p,q}(\mathbb{R}^n),\qquad s>n\left(\frac{1}{\min(p,q,1)}-1\right),\ \ 0<p<\infty,\ \ 0<q\leqslant\infty,$$

resulting in

$$F^s_{p,q}(\mathbb{R}^n)=\mathbf{F}^s_{p,q}(\mathbb{R}^n)=\mathfrak{F}^s_{p,q}(\mathbb{R}^n),$$

whenever

$$0<p<\infty,\quad 0<q\leqslant\infty,\quad s>n\left(\frac{1}{\min(p,q)}-\frac{1}{\max(1,p)}\right)$$

(in terms of equivalent quasi-norms).

Furthermore, since for $s<n(\frac{1}{p}-1)$ the δ-distribution belongs to $F^s_{p,q}(\mathbb{R}^n)$ – which is a singular distribution and cannot be interpreted as a function – the spaces

$$F^s_{p,q}(\mathbb{R}^n)\quad\text{and}\quad \mathfrak{F}^s_{p,q}(\mathbb{R}^n),\qquad 0<s<\sigma_p,$$

cannot be compared. The situation on the line $s=\sigma_p$, $0<p<1$, so far remained an open problem. In this case $F^s_{p,q}(\mathbb{R}^n)$ is a subspace of $L_1^{loc}(\mathbb{R}^n)$ and the two spaces $F^{\sigma_p}_{p,q}(\mathbb{R}^n)$ and $\mathfrak{F}^{\sigma_p}_{p,q}(\mathbb{R}^n)$ can be compared. But our results yield, that they do not coincide, i.e.,

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n)\neq\mathfrak{F}^{\sigma_p}_{p,q}(\mathbb{R}^n),\qquad 0<q\leqslant\infty.$$

1. Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be euclidean n-space, $n \in \mathbb{N}$, $\mathbb{C}$ the complex plane. The set of multi-indices $\beta = (\beta_1, \dots, \beta_n)$, $\beta_i \in \mathbb{N}_0$, $i = 1, \dots, n$, is denoted by $\mathbb{N}_0^n$, with $|\beta| = \beta_1 + \cdots + \beta_n$, as usual. Moreover, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ we put $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
We use the equivalence '$\sim$' in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_k \leqslant b_k \leqslant c_2\, a_k \quad \text{or} \quad c_1\, \varphi(x) \leqslant \psi(x) \leqslant c_2\, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $\{a_k\}_k$, $\{b_k\}_k$ are non-negative sequences and φ, ψ are non-negative functions. If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$ and $[a]$ denotes the integer part of a.

All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. As we shall always deal with function spaces on $\mathbb{R}^n$, we may usually omit the '$\mathbb{R}^n$' from their notation for convenience.

Let for $0 < p, q \leqslant \infty$ the numbers σ_p and σ_{pq} be given by

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \qquad \text{and} \qquad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ . \tag{1.1}$$

Furthermore, let $Q_{\nu,m}$ with $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$, and with side length $2^{-\nu}$. For a cube Q in $\mathbb{R}^n$ and $r > 0$, we denote by rQ the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q. Moreover, $\chi^{(p)}_{\nu,m}$ stands for the p-normalized characteristic function of $Q_{\nu,m}$, i.e.,

$$\chi^{(p)}_{\nu,m}(x) = 2^{\frac{\nu n}{p}} \quad \text{if} \quad x \in Q_{\nu,m} \qquad \text{and} \qquad \chi^{(p)}_{\nu,m}(x) = 0 \quad \text{if} \quad x \notin Q_{\nu,m}.$$

Of course

$$\|\chi^{(p)}_{\nu,m}|L_p(\mathbb{R}^n)\| = 1.$$

The Fourier-analytical approach

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \qquad \text{and} \qquad \varphi(x) = 1 \quad \text{if} \quad |x| \leqslant 1\,, \tag{1.2}$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^\infty$ forms a *smooth dyadic resolution of unity*. Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$

and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then the compact support of $\varphi_j \mathcal{F} f$ implies by the Paley-Wiener-Schwartz theorem that $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$ is an entire analytic function on $\mathbb{R}^n$.

Definition 1.1. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$, and $\{\varphi_j\}_j$ a smooth dyadic resolution of unity. The space $F^s_{p,q}(\mathbb{R}^n)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\left\| f \,|F^s_{p,q}(\mathbb{R}^n)\right\| = \left\| \left\| \left\{ 2^{js} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot) \right\}_{j \in \mathbb{N}_0} |\ell_q \right\| |L_p(\mathbb{R}^n) \right\| \tag{1.3}$$

is finite.

Remark 1.2. The spaces $F^s_{p,q}(\mathbb{R}^n)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definition. They are quasi-Banach spaces (Banach spaces for $p, q \geqslant 1$), and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, where the first embedding is dense if $q < \infty$. An extension of Definition 1.1 to $p = \infty$ does not make sense if $0 < q < \infty$ (in particular, a corresponding space is not independent of the choice $\{\varphi_j\}_j$). The case $p = q = \infty$ yields the Besov spaces $B^s_{\infty,\infty}(\mathbb{R}^n)$.

In general, the Fourier-analytical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are defined correspondingly to the spaces $F^s_{p,q}(\mathbb{R}^n)$ by interchanging the order in which the quasi-norms are taken, i.e., first using the L_p-norm and afterwards applying the ℓ_q-norm – in view of (1.3). These B-spaces are closely linked with the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ via

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{1.4}$$

The theory of the spaces $F^s_{p,q}(\mathbb{R}^n)$ (and $B^s_{p,q}(\mathbb{R}^n)$) has been developed in detail in [Tri83] and [Tri92] (and continued and extended in the more recent monographs [Tri01], [Tri06]), but has a longer history already including many contributors; we do not further want to discuss this here.

Note that the spaces $F^s_{p,q}(\mathbb{R}^n)$ contain tempered distributions which can only be interpreted as regular distributions (functions) for sufficiently high smoothness. More precisely, we have

$$F^s_{p,q}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n) \quad \text{if, and only if,} \quad \begin{cases} s \geqslant \sigma_p, & \text{for } 0 < p < 1,\ 0 < q \leqslant \infty, \\ s > \sigma_p, & \text{for } 1 \leqslant p < \infty,\ 0 < q \leqslant \infty, \\ s = \sigma_p, & \text{for } 1 \leqslant p < \infty,\ 0 < q \leqslant 2, \end{cases} \tag{1.5}$$

cf. [ST95, Thm. 3.3.2]. In particular, for $s < \sigma_p$ one cannot interpret $f \in F^s_{p,q}(\mathbb{R}^n)$ as a regular distribution in general.

The scale $F^s_{p,q}(\mathbb{R}^n)$ contains many well-known function spaces. We list a few special cases.

Let $1 < p < \infty$, then

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \qquad s \in \mathbb{R},$$

are the (*fractional*) *Sobolev spaces* containing all $f \in S'(\mathbb{R}^n)$ with

$$\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}f \in L_p(\mathbb{R}^n).$$

In particular, for $k \in \mathbb{N}_0$, we obtain the *classical Sobolev spaces*

$$F^k_{p,2}(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \qquad \text{i.e., } F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

usually normed by

$$\|f|W^k_p(\mathbb{R}^n)\| = \left(\sum_{|\alpha|\leqslant k} \|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\|^p\right)^{1/p}.$$

Furthermore,

$$F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n), \qquad 0<p<\infty,$$

the latter being the *inhomogenoues Hardy spaces*.

Local means and atomic decompositions

There are equivalent characterizations for the F-spaces $F^s_{p,q}(\mathbb{R}^n)$ in terms of *local means* and *atomic decompositions*. We first sketch the approach via local means. For further details we refer to [BPT96], [BPT97], and [Tri06] with forerunners in [Tri92, Sect. 2.5.3].

Let $B = \{y \in \mathbb{R}^n : |y| < 1\}$ be the unit ball in $\mathbb{R}^n$ and let κ be a C^∞ function in $\mathbb{R}^n$ with supp $\kappa \subset B$. Then

$$k(t,f)(x) = \int_{\mathbb{R}^n} \kappa(y) f(x+ty)\mathrm{d}y = t^{-n}\int_{\mathbb{R}^n} \kappa\Big(\frac{y-x}{t}\Big) f(y)\mathrm{d}y \tag{1.6}$$

with $x \in \mathbb{R}^n$, and $t>0$ are *local means* (appropriately interpreted for $f \in S'(\mathbb{R}^n)$). For given $s \in \mathbb{R}$ it is assumed that the kernel κ satisfies in addition for some $\varepsilon > 0$,

$$\kappa^\vee(\xi) \neq 0 \text{ if } 0<|\xi|<\varepsilon \qquad \text{and} \qquad (\mathrm{D}^\alpha \kappa^\vee)(0) = 0 \text{ if } |\alpha| \leqslant s. \tag{1.7}$$

The second condition is empty if $s<0$. Furthermore, let κ_0 be a second C^∞ function in $\mathbb{R}^n$ with supp $\kappa_0 \subset B$ and $\kappa_0^\vee(0) \neq 0$. The meaning of $k_0(f,t)$ is defined in the same way as (1.6) with κ_0 instead of κ.

We have the following characterization in terms of local means, cf. [Tri06, Th. 1.10] and [Ryc99].

Theorem 1.3. *Let $0<p<\infty$, $0<q\leqslant\infty$ and $s \in \mathbb{R}$. Let κ_0 and κ be the above kernels of local means. Then for $f \in S'(\mathbb{R}^n)$,*

$$\|k_0(1,f)|L_p(\mathbb{R}^n)\| + \left\| \left(\sum_{j=1}^\infty 2^{jsq}|k(2^{-j},f)(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right\| \tag{1.8}$$

is an equivalent quasi-norm in $F^s_{p,q}(\mathbb{R}^n)$.

Remark 1.4. We shall only need one part of Theorem 1.3, namely that $\|f|F^s_{p,q}(\mathbb{R}^n)\|$ can be estimated from below by (1.8). In that case some of the asumptions in (1.7) may be omitted. The inspection of the proof, cf. [Ryc99, Rem. 3], shows that if κ is a C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp}\kappa \subset B \qquad \text{and} \qquad \mathrm{D}^\alpha \kappa^\vee(0) = 0, \quad |\alpha| \leqslant N,$$

where $N > s-1$, then

$$\|f|F^s_{p,q}(\mathbb{R}^n)\| \geqslant c \left\| \left(\sum_{j=1}^\infty 2^{jsq} |k(2^{-j}, f)(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\|$$

for some $c > 0$.

The following atomic characterization of function spaces of type $F^s_{p,q}(\mathbb{R}^n)$ is sometimes preferred (compared with the above Fourier-analytical approach), e.g. when establishing the lower bound for the dilation operators later on; we closely follow the presentation in [Tri97, Sect. 13].

Definition 1.5. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, and $\lambda = \{\lambda_{\nu,m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Then*

$$f_{p,q} = \left\{ \lambda : \|\lambda|f_{p,q}\| = \left\| \left(\sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}\chi^{(p)}_{\nu,m}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*).

Definition 1.6.

(i) *Let $K \in \mathbb{N}_0$ and $d > 1$. A K-times differentiable complex-valued function a on $\mathbb{R}^n$ (continuous if $K = 0$) is called a 1_K-atom if*

$$\operatorname{supp} a \subset dQ_{0,m} \qquad \text{for some } m \in \mathbb{Z}^n, \tag{1.9}$$

and

$$|\mathrm{D}^\alpha a(x)| \leqslant 1 \qquad \text{for } |\alpha| \leqslant K.$$

(ii) *Let $s \in \mathbb{R}$, $0 < p \leqslant \infty$, $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$, and $d > 1$. A K-times differentiable complex-valued function a on $\mathbb{R}^n$ (continuous if $K = 0$) is called an $(s,p)_{K,L}$-atom if for some $\nu \in \mathbb{N}_0$*

$$\operatorname{supp} a \subset dQ_{\nu,m} \qquad \text{for some } m \in \mathbb{Z}^n, \tag{1.10}$$

$$|\mathrm{D}^\alpha a(x)| \leqslant 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu} \qquad \text{for } |\alpha| \leqslant K, \tag{1.11}$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\mathrm{d}x = 0 \qquad \text{if } |\beta| \leqslant L. \tag{1.12}$$

It is convenient to write $a_{\nu,m}(x)$ instead of $a(x)$ if this atom is located at $Q_{\nu,m}$ according to (1.9) and (1.10). Assumption (1.12) is called a *moment condition*, where $L=-1$ means that there are no moment conditions. Furthermore, K denotes the smoothness of the atom, cf. (1.11). The atomic characterization of function spaces of type $F^s_{p,q}(\mathbb{R}^n)$ is given by the following result, cf. [Tri97, Thm. 13.8].

Theorem 1.7. *Let $0<p<\infty$, $0<q\leqslant\infty$, and $s\in\mathbb{R}$. Let $K\in\mathbb{N}_0$ and $L+1\in\mathbb{N}_0$ with*

$$K\geqslant(1+[s])_+ \qquad \text{and} \qquad L\geqslant\max(-1,[\sigma_{pq}-s])$$

be fixed. Then $f\in S'(\mathbb{R}^n)$ belongs to $F^s_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}a_{\nu,m}(x), \qquad \text{convergence being in } S'(\mathbb{R}^n), \tag{1.13}$$

where the $a_{\nu,m}$ are 1_K-atoms ($\nu=0$) or $(s,p)_{K,L}$-atoms ($\nu\in\mathbb{N}$) with

$$\operatorname{supp}a_{\nu,m}\subset dQ_{\nu,m}, \qquad \nu\in\mathbb{N}_0,\ m\in\mathbb{Z}^n,\ d>1,$$

and $\lambda\in f_{p,q}$. Furthermore,

$$\inf\|\lambda|f_{p,q}\|,$$

where the infimum is taken over all admissible representations (1.13)*, is an equivalent quasi-norm in $F^s_{p,q}(\mathbb{R}^n)$.*

2. Dilation Operators

In this section we present our main results concerning dilation operators T_k in F-spaces when $s=\sigma_p$. We distinguish between the cases $1<p<\infty$ and $0<p\leqslant1$, when $\sigma_p=0$ and $\sigma_p=n(1/p-1)$, respectively.

Theorem 2.1. *Let $1<p<\infty$ and $0<q\leqslant\infty$. Then*

$$\|T_k|\mathcal{L}(F^0_{p,q}(\mathbb{R}^n))\|\sim2^{-k\frac{n}{p}}\cdot k^{\frac{1}{q}-\frac{1}{\max(q,2)}}, \qquad k\in\mathbb{N}.$$

Proof. *Step 1.* Recall Definition 1.1, where in particular the dyadic resolution of unity was constructed such that

$$\varphi_j(x)=\varphi(2^{-j}x)-\varphi(2^{-j+1}x), \qquad j\in\mathbb{N}.$$

Elementary calculation yields

$$(\varphi_j(\xi)\widehat{f(2^k\cdot)}(\xi))^\vee(x)=2^{-kn}(\varphi_j(\xi)\widehat{f}(2^{-k}\xi))^\vee(x)=(\varphi_j(2^k\xi)\widehat{f}(\xi))^\vee(2^kx). \tag{2.1}$$

For convenience we assume $q<\infty$ in the sequel, but the counterpart for $q=\infty$ is obvious.

From the definition of F-spaces with $f(2^k x)$ in place of $f(x)$ we obtain

$$\begin{aligned}
\|f(2^k\cdot)|F^{\sigma_p}_{p,q}(\mathbb{R}^n)\| &= \left\| \left(\sum_{j=0}^{\infty} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(2^k\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&= 2^{-k\frac{n}{p}} \left\| \left(\sum_{j=0}^{\infty} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&\sim 2^{-k\frac{n}{p}} \Bigg\{ \left\| (\varphi_0(2^k\cdot)\widehat{f}(\cdot))^\vee(\cdot)|L_p(\mathbb{R}^n) \right\| \\
&\quad + \left\| \left(\sum_{j=1}^{k} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&\quad + \left\| \left(\sum_{j=k+1}^{\infty} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \Bigg\} \qquad (2.2)
\end{aligned}$$

If $j \geqslant k+1$, then $\varphi_j(2^k x) = \varphi_{j-k}(x)$. This yields for the last term

$$\begin{aligned}
&2^{-k\frac{n}{p}} \left\| \left(\sum_{j=k+1}^{\infty} 2^{j\sigma_p q} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&\quad = 2^{-k\frac{n}{p}} \left\| \left(\sum_{j=k+1}^{\infty} 2^{(j-k)\sigma_p q} 2^{k\sigma_p q} |(\varphi_{j-k}(\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&\quad = 2^{k(\sigma_p - \frac{n}{p})} \left\| \left(\sum_{l=1}^{\infty} 2^{l\sigma_p q} |(\varphi_l(\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&\quad \leqslant 2^{-\frac{kn}{p}} \|f|F^{\sigma_p}_{p,q}(\mathbb{R}^n)\|. \qquad (2.3)
\end{aligned}$$

If $j=0$, we use the Hausdorff-Young inequality and obtain

$$\begin{aligned}
\|(\varphi_0(2^k\cdot)\widehat{f})^\vee|L_p\| &= \|(\varphi_0(2^k\cdot)\varphi_0\widehat{f})^\vee|L_p\| \\
&= \|(\varphi_0(2^k\cdot)^\vee * (\varphi_0\widehat{f})^\vee|L_p\| \\
&\leqslant \|(\varphi_0(2^k\cdot)^\vee|L_1\| \cdot \|(\varphi_0\widehat{f})^\vee|L_p\| \\
&\leqslant c\|f|F^0_{p,q}\|.
\end{aligned}$$

Step 2. In view of Step 1 it remains to consider $j=1,\ldots,k$. Using Hölder's inequality with

$$\frac{1}{u} = \frac{q}{2} \quad \text{and} \quad \frac{1}{u'} = 1 - \frac{q}{2} \qquad \text{if } q<2$$

or

$$||id : \ell_2^k \hookrightarrow \ell_q^k|| = 1 \qquad \text{when } q \geqslant 2 \quad \text{and} \quad k \in \mathbb{N}$$

together with the Littlewood-Paley theorem, we see that

$$\begin{aligned}
\left\| \left(\sum_{j=1}^{k} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p \right\| &\leqslant k^{\frac{1}{q}-\frac{1}{\max(q,2)}} \left\| \left(\sum_{j=1}^{k} |(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^2 \right)^{1/2} |L_p \right\| \\
&\leqslant k^{\frac{1}{q}-\frac{1}{\max(q,2)}} \|(\varphi_0\widehat{f})^\vee|L_p\| \\
&\leqslant k^{\frac{1}{q}-\frac{1}{\max(q,2)}} \|f|F^0_{p,q}\|,
\end{aligned}$$

giving the desired upper bound.

Step 3. In order to establish the lower bound we take $\psi \in S(\mathbb{R}^n)$ with

$$\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \leqslant 1/8\}.$$

We define the functions f_k through their Fourier transforms

$$\widehat{f_k}(\xi) = \sum_{j=1}^{k} \psi(2^k(\xi - \xi_j)), \qquad \xi \in \mathbb{R}^n, \quad k \in \mathbb{N},$$

where $\xi_j = (2^{-j}, 0, \dots, 0)$. We shall show that

$$\|f_k|F^0_{p,q}\| \lesssim k^{1/2} 2^{kn(1/p-1)}, \tag{2.4}$$

and

$$\|f_k(2^k\cdot)|F^0_{p,q}\| \gtrsim k^{1/q} 2^{-kn}. \tag{2.5}$$

We deal with (2.4) first. As the support of $\widehat{f_k}$ lies in the unit ball of $\mathbb{R}^n$, we may omit the terms with $j \geqslant 1$ in (1.3). Furthermore, since $1 < p < \infty$ we may use the Littlewood-Paley decomposition theorem to estimate

$$\begin{aligned}
\|f_k|F^0_{p,q}\| &= \|(\varphi_0\widehat{f_k})^\vee|L_p\| \\
&\lesssim \left\| \left(\sum_{j=1}^{\infty} |(\varphi_1(2^j\cdot)\cdot\varphi_0\widehat{f_k})^\vee(x)|^2 \right)^{1/2} |L_p \right\| \qquad (2.6) \\
&= \left\| \left(\sum_{j=1}^{k} |\psi(2^k(\xi-\xi_j))^\vee(x)|^2 \right)^{1/2} |L_p \right\| \\
&= \left\| \left(\sum_{j=1}^{k} |2^{-kn}\psi^\vee(2^{-k}x)e^{ix\xi_j}|^2 \right)^{1/2} |L_p \right\| \\
&= k^{1/2}2^{-kn} \left\|\psi^\vee(2^{-k}x)|L_p\right\| = k^{1/2}2^{kn(1/p-1)} \|\psi^\vee|L_p\|.
\end{aligned}$$

Let us mention, that (2.4) holds also for $p = 1$. In this case, the inequality on the second line of (2.6) follows (roughly speaking) by the embedding

$$F^0_{1,2}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n),$$

cf. [ST95, Th. 3.1.1]. To prove (2.5), observe that

$$\widehat{f_k(2^k\cdot)}(\xi) = 2^{-kn}\widehat{f}_k(2^{-k}\xi) = 2^{-kn}\sum_{j=1}^{k}\psi(\xi - 2^k\xi_j), \qquad \xi \in \mathbb{R}^n.$$

Using again the support properties of ψ and φ_j, we arrive at

$$\begin{aligned}\|f_k(2^k\cdot)|F^0_{p,q}\| &= \left\|\left(\sum_{j=1}^{k}|(\varphi_j\widehat{f_k(2^k\cdot)}(\xi))^\vee(x)|^q\right)^{1/q}|L_p\right\| \\ &= 2^{-kn}\left\|\left(\sum_{j=1}^{k}|(\psi(\xi - 2^k\xi_j)^\vee(x)|^q\right)^{1/q}|L_p\right\| \\ &= 2^{-kn}\left\|\left(\sum_{j=1}^{k}|(\psi^\vee(x)e^{ix2^k\xi_j}|^q\right)^{1/q}|L_p\right\| \\ &= 2^{-kn}k^{1/q}\|\psi^\vee|L_p\|.\end{aligned}$$

Observe, that also (2.5) holds even for $p = 1$.

This finally leads to

$$\|T_k|\mathcal{L}(F^0_{p,q})\| \geqslant \frac{\|f_k(2^k\cdot)|F^0_{p,q}\|}{\|f_k|F^0_{p,q}\|} \geqslant k^{1/q-1/2}2^{-\frac{kn}{p}}.$$

Step 4. Let $1 < p < \infty$ and $q \geqslant 2$. Chose an arbitrary non-vanishing $\psi \in S(\mathbb{R}^n)$. Using the trivial embedding $F^0_{p,2} \hookrightarrow F^0_{p,q}$, we obtain

$$\|T_k|\mathcal{L}(F^0_{p,q})\| \geqslant \frac{\|\psi|F^0_{p,q}\|}{\|\psi(2^{-k}\cdot)|F^0_{p,q}\|} \geqslant \frac{\|\psi|F^0_{p,q}\|}{\|\psi(2^{-k}\cdot)|F^0_{p,2}\|} \sim 2^{-k\frac{n}{p}}.$$

■

Theorem 2.2. *Let* $0 < p \leqslant 1$, $0 < p \leqslant q \leqslant \infty$. *Then*

$$\|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \sim 2^{k(\sigma_p - \frac{n}{p})}k^{1/p}, \quad k \in \mathbb{N}.$$

Proof. *Step 1.* We give an estimate for the upper bounds of the dilation operators T_k similar to Theorem 2.1. We need to find suitable substitutes when $0 < p \leqslant 1$.

For the further calculations we make use of the following Fourier multiplier theorem, cf. [Tri83, Prop. 1.5.1],

$$\|(M\widehat{h})^\vee|L_p\| \leqslant c\|M^\vee|L_p\| \cdot \|h|L_p\|, \qquad \text{if } 0<p\leqslant 1, \tag{2.7}$$

with $M^\vee \in S' \cap L_p$, and $\operatorname{supp}\widehat{h} \subset \Omega$, $\operatorname{supp} M \subset \Gamma$, where Ω and Γ are compact subsets of $\mathbb{R}^n$ (c does not depend on M and h, but may depend on Ω and Γ). Of course for $p=1$ this is just the Hausdorff-Young inequality (which was also used in Theorem 2.1). We put $h=(\varphi_0\widehat{f})^\vee$, where $\operatorname{supp}\widehat{h} \subset \operatorname{supp}\varphi_0 = \Omega$.

If $j=0$, we take $M_0=\varphi_0(2^k\cdot)$ where $\operatorname{supp} M_0 \subset \operatorname{supp}\varphi_0 = \Gamma$ and calculate

$$\begin{aligned} 2^{-k\frac{n}{p}}\|(\varphi_0(2^k\cdot)\widehat{f})^\vee|L_p\| &\leqslant c2^{-k\frac{n}{p}}\|\varphi_0(2^k\cdot)^\vee|L_p\| \cdot \|(\varphi_0\widehat{f})^\vee|L_p\|, \\ &= c2^{-k\frac{n}{p}}2^{k\sigma_p}\|{\varphi_0}^\vee|L_p\| \cdot \|(\varphi_0\widehat{f})^\vee|L_p\| \\ &= c'2^{k(\sigma_p-\frac{n}{p})}\|(\varphi_0\widehat{f})^\vee|L_p\| \\ &= c2^{-kn}\|f|F^{\sigma_p}_{p,q}\|. \end{aligned} \tag{2.8}$$

According to the observations in Step 1 of Theorem 2.1 it remains to consider $1\leqslant j\leqslant k$. This is the crucial step, leading to $k^{1/p}$. In this case $\varphi_j(x)=\bar{\varphi}(2^{-j}x)$, where $\bar{\varphi}=\varphi_0(x)-\varphi_0(2x)$. Hence

$$\begin{aligned} &\left\| \left(\sum_{j=1}^k 2^{j\sigma_p q}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n)\right\| \\ &\qquad = \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^k 2^{j\sigma_p q}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(x)|^q \right)^{p/q} \mathrm{d}x \right)^{1/p} \\ &\qquad \leqslant \left(\sum_{j=1}^k \int_{\mathbb{R}^n} 2^{j\sigma_p p}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(x)|^p \mathrm{d}x \right)^{1/p} \\ &\qquad = \left(\sum_{j=1}^k 2^{j\sigma_p p}\|(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p\|^p \right)^{1/p} \qquad (2.9) \\ &\qquad = \left(\sum_{j=1}^{k-1} 2^{j\sigma_p p}\|(\bar{\varphi}(2^{k-j}\cdot)\widehat{f})^\vee|L_p\|^p + 2^{k\sigma_p p}\|(\bar{\varphi}\widehat{f})^\vee|L_p\|^p \right)^{1/p} \end{aligned}$$

where the inequality follows from $\ell_{\frac{p}{q}} \hookrightarrow \ell_1$ since $p<q$.

The term for $j=k$ in (2.9) needs some extra care. Using (2.7) where we set $M_k=\varphi_0(2\cdot)$, $\operatorname{supp} M_k \subset \operatorname{supp}\varphi_0 = \Gamma$ we obtain

$$
\begin{aligned}
2^{k\sigma_p p}\|(\bar\varphi\widehat{f})^\vee|L_p\|^p &= 2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee-(\varphi_0(2\cdot)\widehat{f})^\vee|L_p\|^p\\
&\leqslant c2^{k\sigma_p p}\left(\|(\varphi_0\widehat{f})^\vee|L_p\|+\|(\varphi_0(2\cdot)\varphi_0\widehat{f})^\vee|L_p\|\right)^p\\
&\leqslant c'2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee|L_p\|^p\,(1+\|\varphi_0^\vee(2\cdot)|L_p\|)^p\\
&= c_1 2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee|L_p\|^p.
\end{aligned}\tag{2.10}
$$

This estimate can be incorporated into our further calculations. Now for $1\leqslant j\leqslant k-1$ we use the multiplier theorem with $M_j=\bar\varphi(2^{k-j}\cdot)$, and observe that

$$\operatorname{supp} M_j\subset\{x:|2^{k-j}x|\leqslant 2\}\subset\{x:|x|\leqslant 2\}=\Gamma.$$

Now inserting (2.10) into (2.9) yields

$$
\begin{aligned}
&\left(\sum_{j=1}^{k-1}2^{j\sigma_p p}\|(\bar\varphi(2^{k-j}\cdot)\varphi_0\widehat{f})^\vee|L_p\|^p+c_1 2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee|L_p\|^p\right)^{1/p}\\
&\leqslant c\left(\sum_{j=1}^{k-1}2^{j\sigma_p p}\|(\bar\varphi(2^{k-j}\cdot))^\vee(\cdot)|L_p\|^p\|(\varphi_0\widehat{f})^\vee|L_p\|^p+2^{k\sigma_p p}\|(\varphi_0\widehat{f})^\vee|L_p\|^p\right)^{1/p}\\
&\leqslant c\|(\varphi_0\widehat{f})^\vee|L_p\|\left(\sum_{j=1}^{k-1}2^{j\sigma_p p}\|2^{(j-k)n}\bar\varphi^\vee(2^{j-k}\cdot)|L_p\|^p+2^{k\sigma_p p}\right)^{1/p}\\
&= c\|(\varphi_0\widehat{f})^\vee|L_p\|\left(\sum_{j=1}^{k-1}2^{j\sigma_p p}2^{(j-k)np}2^{-(j-k)\frac{n}{p}p}\|\bar\varphi^\vee|L_p\|^p+2^{k\sigma_p p}\right)^{1/p}\\
&\leqslant c2^{k\sigma_p p}k^{1/p}\|F^{\sigma_p}_{p,q}(\mathbb{R}^n)\|.
\end{aligned}\tag{2.11}
$$

Now (2.2) together with (2.3), (2.8), and (2.11) give the upper estimate.

Step 2. We construct a function that gives the lower bound. Let $\psi\in S(\mathbb{R})$ be a non-negative function with $\operatorname{supp}\psi\subset\{x\in\mathbb{R}^n:|x|\leqslant 1/8\}$ and $\int_{\mathbb{R}^n}\psi(x)\mathrm{d}x=1$. We show that

$$\|\psi(2^k\cdot)|F^{\sigma_p}_{p,q}(\mathbb{R}^n)\|\geqslant c2^{-kn}k^{1/p},\qquad k\in\mathbb{N},\quad 0<q\leqslant\infty.$$

Let us take a function $\kappa\in S(\mathbb{R}^n)$ with

$$(\mathrm{D}^\alpha\kappa^\vee)(0)=0,\qquad |\alpha|\leqslant r,\tag{2.12}$$

where $r>\sigma_p-1$, according to [Ryc99, Th. BPT]. In particular, by [Ryc99, Rem. 3] these conditions on κ are sufficient for our purposes. Furthermore, we require

$$\kappa(x)=1\qquad\text{if}\quad x\in M=\{z\in\mathbb{R}^n:|z-(1/2,0\dots,0)|<1/4\}.\tag{2.13}$$

Such a function κ was constructed in [Sch09, Th. 2.1].

Simple calculation shows that if $j = 1, 2, \ldots, k$ and $|x-(-\frac{1}{2}\cdot\frac{1}{2^j}, 0\ldots, 0)| < \frac{1}{2^j}\frac{1}{8}$, which is equivalent to writing

$$x \in B_{2^{-(j+3)}}(x_j), \qquad x_j = (-2^{-(j+1)}, 0, \ldots, 0),$$

then

$$\operatorname{supp}_y \psi(2^k x + 2^{k-j} y) \subset M.$$

For these x we get

$$\mathcal{K}(2^{-j}, \psi(2^k\cdot))(x) = \int_{\mathbb{R}^n} \kappa(y)\psi(2^k x + 2^{k-j}y)\mathrm{d}y = \int_{\mathbb{R}^n} \psi(2^k x + 2^{k-j}y)\mathrm{d}y = 2^{(j-k)n}.$$

Note that the for different values of j, the balls $B_{2^{-(j+3)}}(x_j)$ are pairwise disjoint. Hence we calculate

$$\begin{aligned}
\|\psi(2^k\cdot)|F^{\sigma_p}_{p,q}\| &\geqslant \left\| \left(\sum_{j=1}^k 2^{j\sigma_p q}|\mathcal{K}(2^{-j}, \psi(2^k\cdot))(\cdot)|^q \right)^{1/q} |L_p \right\| \\
&= \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^k 2^{j\sigma_p q}|\mathcal{K}(2^{-j}, \psi(2^k\cdot))(x)|^q \right)^{p/q} \mathrm{d}x \right)^{1/p} \\
&\geqslant \left(\sum_{l=1}^k \int_{B_{2^{-(l+3)}}(x_l)} \left(\sum_{j=1}^k \delta_{lj} 2^{j\sigma_p q}|\mathcal{K}(2^{-j}, \psi(2^k\cdot))(x)|^q \right)^{p/q} \mathrm{d}x \right)^{1/p} \\
&\geqslant \left(\sum_{j=1}^k 2^{j\sigma_p p} 2^{(j-k)np} 2^{-jn} \right)^{1/p} = 2^{-kn} \left(\sum_{j=1}^k 2^{jn(\frac{1}{p}-1)p} 2^{jnp} 2^{-jn} \right)^{1/p} \\
&= 2^{-kn} k^{1/p},
\end{aligned}$$

which gives the desired result. Our estimate holds as well in the case $p = 1$. ■

Refining the methods used in Theorem 2.2 we obtain the following generalization. However, our estimates are not sharp and might still be improved.

Theorem 2.3. *Let* $0 < q < p < 1$. *Then*

$$2^{k(\sigma_p - \frac{n}{p})} k^{1/p} \lesssim \|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \lesssim 2^{k(\sigma_p - \frac{n}{p})} k^{1/q}.$$

Furthermore, if $0 < q < p = 1$ *we have*

$$2^{-kn} k^{\max(1, 1/q - 1/2)} \lesssim \|T_k|\mathcal{L}(F^0_{1,q}(\mathbb{R}^n))\| \lesssim 2^{-kn} k^{1/q}.$$

Proof. *Step 1.* Refining the estimates for the upper bound used in Step 1 of Theorem 2.2 we see that we only need to consider the 'critical terms' when $j = 1, \dots, k$. In this case we now calculate

$$\begin{aligned}
&\left\| \left(\sum_{j=1}^{k} 2^{j\sigma_p q} |(\varphi_j(2^k \cdot)\widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \\
&\quad = \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{k} 2^{j\sigma_p q} |(\varphi_j(2^k \cdot)\widehat{f})^\vee(x)|^q \right)^{p/q} \mathrm{d}x \right)^{1/p} \\
&\quad = \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^{k} 2^{j\sigma_p q} |(\varphi_j(2^k \cdot)\widehat{f})^\vee(x)|^q \right)^{p/q} \mathrm{d}x \right)^{\frac{q}{p}\cdot\frac{1}{q}} \\
&\quad \leqslant \left(\sum_{j=1}^{k} \left(\int_{\mathbb{R}^n} 2^{j\sigma_p p} |(\varphi_j(2^k \cdot)\widehat{f})^\vee(x)|^p \mathrm{d}x \right)^{q/p} \right)^{1/q} \\
&\quad = \left(\sum_{j=1}^{k} 2^{j\sigma_p q} \|(\varphi_j(2^k \cdot)\widehat{f})^\vee |L_p\|^q \right)^{1/q} \\
&\quad \leqslant c \left(\sum_{j=1}^{k} 2^{j\sigma_p q} \|\bar{\varphi}(2^{k-j} \cdot)^\vee |L_p\|^q \cdot \|(\varphi_0\widehat{f})^\vee |L_p\|^q \right)^{1/q} \\
&\quad \leqslant c \|(\varphi_0\widehat{f})^\vee |L_p\| \left(\sum_{j=1}^{k} 2^{j\sigma_p q} \|\bar{\varphi}(2^{k-j} \cdot)^\vee |L_p\|^q \right)^{1/q} \\
&\quad \leqslant c \|(\varphi_0\widehat{f})^\vee |L_p\| \left(\sum_{j=1}^{k} 2^{j\sigma_p q} 2^{(j-k)nq} 2^{-(j-k)\frac{n}{p}q} \|\bar{\varphi}(\cdot)^\vee |L_p\|^q \cdot \right)^{1/q} \\
&\quad \leqslant c' \|(\varphi_0\widehat{f})^\vee |L_p\| 2^{k\sigma_p n} k^{1/q} \\
&\quad \leqslant c'' 2^{k\sigma_p n} k^{1/q} \|f|F^{\sigma_p}_{p,q}\|,
\end{aligned}$$

where in the third step we used the generalized triangle inequality, cf. [HLP52, p. 148], since $\frac{p}{q} > 1$, before applying the Fourier Multiplier theorem (2.7).

Step 2. The proof of the lower bound

$$\|T_k|\mathcal{L}(F^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \gtrsim k^{1/p} 2^{k(\sigma_p - \frac{n}{p})}, \qquad k \in \mathbb{N}$$

is the same as in Step 2 of Theorem 2.2.

Step 3. Finally, the estimate

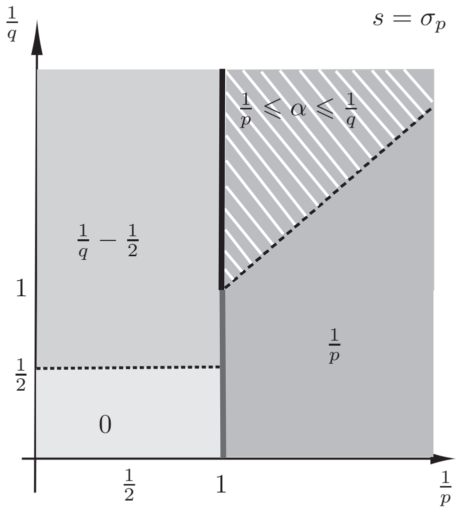
$$\|T_k|\mathcal{L}(F^0_{1,q}(\mathbb{R}^n))\| \gtrsim k^{1/q-1/2}2^{-kn}, \qquad k \in \mathbb{N}$$

for $0 < q < p = 1$ follows from the Step 3 of Theorem 2.1. ■

Remark 2.4. The picture aside summarizes our results and illustrates the dependency of the additional factors k^α on p and q that were obtained for upper bounds of the dilation operators when $s = \sigma_p$, i.e.

$$T_k \sim 2^{k(\sigma_p - n/p)} \cdot k^\alpha.$$

There is a jump at $p = 1$ in the exponent of k caused by the absence of the Littlewood-Paley assertion in this case. Furthermore, our estimates when $0 < q < p < 1$ and $0 < q < p = 1$ are not sharp and might be improved.



3. Applications

3.1. F-spaces with positive smoothness on $\mathbb{R}^n$

In this section we want to discuss the connection and diversity of three different approaches to F-spaces with positive smoothness, using the previous results on dilation operators.

In addition to the Fourier-analytical approach, cf. Definition 1.1, we now present two further characterizations – associated to definitions by differences and subatomic decompositions – before we come to some comparisions.

The classical approach: Triebel-Lizorkin spaces $\mathbf{F}^{s}_{p,q}(\mathbb{R}^n)$

If f is an arbitrary function on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $r \in \mathbb{N}$, then

$$(\Delta^1_h f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta^{r+1}_h f)(x) = \Delta^1_h(\Delta^r_h f)(x), \qquad x \in \mathbb{R}^n.$$

For convenience we may write Δ_h instead of Δ^1_h. Furthermore, for a function $f \in L_p(\mathbb{R}^n)$, $0 < p < \infty$, $r \in \mathbb{N}$, the *ball means* are denoted by

$$d^r_{t,p}f(x) = \left(t^{-n}\int_{|h|\leqslant t} |(\Delta^r_h f)(x)|^p \mathrm{d}h\right)^{1/p}, \qquad x \in \mathbb{R}^n, \ t > 0. \tag{3.1}$$

Definition 3.1. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, $s > 0$, and $r \in \mathbb{N}$ such that $r > s$. Then $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that*

$$\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left\| \left(\int_0^1 t^{-sq} d^r_{t,p} f(\cdot)^q \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{3.2}$$

(*with the usual modification if $q = \infty$*) *is finite.*

Remark 3.2. The approach by differences for the spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ has been described in detail in [Tri83, 2.5.10] for those spaces which can also be considered as subspaces of $S'(\mathbb{R}^n)$. Otherwise one finds in [Tri06, 9.2.2] the necessary explanations and references to the relevant literature. In particular, the spaces in Definition 3.1 are independent of r, meaning that different values of $r > s$ result in quasi-norms which are equivalent. Furthermore, the spaces are quasi-Banach spaces (Banach spaces, if $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$). Recall that we deal with subspaces of $L_p(\mathbb{R}^n)$, in particular, we have the embedding

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \qquad s > 0,\ 0 < q \leqslant \infty,\ 0 < p < \infty.$$

Further information on the classical approach to F-spaces – treated in a more general context – may be found in [HN07].

We add the following homogeneity estimate, which will serve us later on. Let $s > 0$, $0 < p < \infty$, $0 < q \leqslant \infty$, and $k \in \mathbb{N}_0$. Then for all $f \in \mathbf{F}^s_{p,q}(\mathbb{R}^n)$

$$\|f(2^k\cdot)|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\| \leqslant 2^{k(s-\frac{n}{p})}\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|. \tag{3.3}$$

Let $f \in \mathbf{F}^s_{p,q}(\mathbb{R}^n)$. For the proof we observe that

$$\begin{aligned} \|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\| &= \|f|L_p(\mathbb{R}^n)\| \\ &+ \left(\int_{\mathbb{R}^n} \left(\int_0^1 t^{-(s+\frac{n}{p})q} \left(\int_{|h|\leqslant t} |\Delta^r_h f(x)|^p \mathrm{d}h \right)^{q/p} \frac{\mathrm{d}t}{t} \right)^{p/q} \mathrm{d}x \right)^{1/p}, \end{aligned}$$

where $\displaystyle\int_0^1 \dots \frac{\mathrm{d}t}{t}$ can be replaced by $\displaystyle\int_0^\lambda \dots \frac{\mathrm{d}t}{t}$ with arbitrary $0 < \lambda \leqslant \infty$ in the sense of equivalent quasi-norms.

Now straightforward calculation yields

$$\begin{aligned}
&\|f(2^k\cdot)|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\| = \|f(2^k\cdot)|L_p(\mathbb{R}^n)\| \\
&\quad + \left(\int_{\mathbb{R}^n}\left(\int_0^1 t^{-(s+\frac{n}{p})q}\left(\int_{|h|\leqslant t}|\Delta^r_h f(2^k x)|^p \mathrm{d}h\right)^{q/p}\frac{\mathrm{d}t}{t}\right)^{p/q}\mathrm{d}x\right)^{1/p} \\
&\leqslant 2^{-k\frac{n}{p}}\|f|L_p(\mathbb{R}^n)\| \\
&\quad + 2^{k(s-\frac{n}{p})}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty t'^{-(s+\frac{n}{p})q}\left(\int_{|h'|\leqslant t'}|\Delta^r_{h'} f(x')|^p \mathrm{d}h'\right)^{q/p}\frac{\mathrm{d}t'}{t'}\right)^{p/q}\mathrm{d}x\right)^{1/p} \\
&\leqslant \max\left(2^{-k\frac{n}{p}}, 2^{k(s-\frac{n}{p})}\right)\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\| \\
&= 2^{k(s-\frac{n}{p})}\|f|\mathbf{F}^s_{p,q}(\mathbb{R}^n)\|,
\end{aligned}$$

where we used in the second step that

$$\Delta^r_h f(2^k x) = \sum_{l=0}^{r}\binom{r}{l}(-1)^{r-l} f(2^k x + l2^k h) =: \Delta^r_{h'} f(x'),$$

by substituting $x' = 2^k x$, $h' = 2^k h$, and $t' = 2^k t$.

The subatomic approach: Triebel-Lizorkin spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$

We complement our notation. Let

$$\mathbb{R}^n_{++} := \{y \in \mathbb{R}^n : y = (y_1, \dots, y_n),\ y_j > 0\}.$$

Moreover, $\chi_{\nu,m}$ denotes the characteristic function of the cube $Q_{\nu,m}$. The subatomic approach provides a constructive definition for Triebel-Lizorkin spaces, expanding functions f via building blocks and suitable coefficients, where the latter belong to certain sequence spaces $f^{s,\varrho}_{p,q}$.

Definition 3.3. *Let k be a non-negative C^∞-function in $\mathbb{R}^n$ with*

$$\operatorname{supp} k \subset \left\{y \in \mathbb{R}^n : |y| < 2^{J-\varepsilon}\right\} \cap \mathbb{R}^n_{++} \tag{3.4}$$

for some fixed $\varepsilon > 0$ and some fixed $J \in \mathbb{N}$, satisfying

$$\sum_{m\in\mathbb{Z}^n} k(x-m) = 1, \qquad x \in \mathbb{R}^n. \tag{3.5}$$

Let $\beta \in \mathbb{N}^n_0$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and set $k^\beta(x) = (2^{-J}x)^\beta k(x)$. Then

$$k^\beta_{\nu,m}(x) = k^\beta(2^\nu x - m) \tag{3.6}$$

denote the building blocks related to $Q_{\nu,m}$.

Remark 3.4. The above definition implies that the building blocks are bounded by

$$0 \leqslant k^{\beta}_{\nu,m}(x) \leqslant 2^{-\varepsilon|\beta|}, \qquad x \in \mathbb{R}^n, \tag{3.7}$$

uniformly in $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and for their supports we observe that

$$\operatorname{supp} k^{\beta}_{\nu,m} \subset 2^{J-\varepsilon} Q_{\nu,m} \tag{3.8}$$

uniformly in $\beta \in \mathbb{N}^n_0$.

Definition 3.5. *Let $\varrho \geqslant 0$, $s \in \mathbb{R}$, $0 < p, q \leqslant \infty$ and*

$$\lambda = \left\{ \lambda^{\beta}_{\nu,m} \in \mathbb{C} : \beta \in \mathbb{N}^n_0,\, m \in \mathbb{Z}^n,\, \nu \in \mathbb{N}_0 \right\}.$$

Then the sequence space $f^{s,\varrho}_{p,q}$ is defined as

$$f^{s,\varrho}_{p,q} := \left\{ \lambda : \|\lambda | f^{s,\varrho}_{p,q}\| < \infty \right\}, \tag{3.9}$$

where

$$\|\lambda | f^{s,\varrho}_{p,q}\| = \sup_{\beta \in \mathbb{N}^n_0} 2^{\varrho|\beta|} \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{\nu s q} |\lambda^{\beta}_{\nu,m}|^q \chi_{\nu,m}(\cdot) \right)^{1/q} | L_p(\mathbb{R}^n) \right\| \tag{3.10}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*).

We now define the related function spaces.

Definition 3.6. *Let $s > 0$, $0 < p < \infty$, $0 < q \leqslant \infty$, and $\varrho \geqslant 0$. Then $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ contains all $f \in L_p(\mathbb{R}^n)$ which can be represented as*

$$f(x) = \sum_{\beta \in \mathbb{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^{\beta}_{\nu,m} k^{\beta}_{\nu,m}(x), \qquad x \in \mathbb{R}^n, \tag{3.11}$$

with coefficients $\lambda = \left\{ \lambda^{\beta}_{\nu,m} \right\}_{\beta \in \mathbb{N}^n_0, \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in f^{s,\varrho}_{p,q}$. Then

$$\left\| f | \mathfrak{F}^s_{p,q}(\mathbb{R}^n) \right\| = \inf \left\| \lambda | f^{s,\varrho}_{p,q} \right\|, \tag{3.12}$$

where the infimum is taken over all possible representations (3.11).

Remark 3.7. The definitions given above follow closely [Tri06, Sect. 9.2]. The spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces for $p, q \geqslant 1$) and independent of k and ϱ (in terms of equivalent quasi-norms). Furthermore, for all admitted parameters p, q, s, we have

$$\mathfrak{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \qquad .$$

see [Tri06, Th. 9.8]. Concerning the convergence of (3.11) one obtains as a consequence of $\lambda \in f^{s,\varrho}_{p,q}$, that the series on the right-hand sides converge absolutely in $L_p(\mathbb{R}^n)$ if $p < \infty$. Since this implies unconditional convergence we may simplify (3.11) and write in the sequel

$$f = \sum_{\beta,\nu,m} \lambda^{\beta}_{\nu,m} k^{\beta}_{\nu,m}.$$

Remark 3.8. Considering the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ we obtain the following upper bounds for the dilation operators T_k. Let $s > 0$, $0 < p < \infty$, $0 < q \leqslant \infty$, and $k \in \mathbb{N}_0$. Then for all $f \in \mathfrak{F}^s_{p,q}(\mathbb{R}^n)$

$$\|f(2^k\cdot)|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| \leqslant 2^{k(s-\frac{n}{p})}\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\|. \tag{3.13}$$

The proof is fairly simple. We take $f \in \mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ with optimal representation

$$f(x) = \sum_{\beta,\nu,m} \lambda^\beta_{\nu,m} k^\beta_{\nu,m}(x),$$

i.e.,

$$\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| \sim \|\lambda|f^{s,\varrho}_{p,q}\| = \sup_\beta 2^{\varrho|\beta|}\left\|\left(\sum_\nu\sum_m 2^{\nu sq}|\lambda^\beta_{\nu,m}|^q\chi_{\nu,m}(\cdot)\right)^{1/q}|L_p\right\|,$$

where $\chi_{\nu,m}(\cdot)$ is the characteristic function of $Q_{\nu,m}$. Put

$$g(x) := f(2^k\cdot) = \sum_{\beta,\nu,m} \lambda^\beta_{\nu,m} k^\beta_{\nu,m}(2^k x) = \sum_{\beta,m}\sum_{l=k}^\infty \lambda^\beta_{l-k,m} k^\beta_{l,m}(x),$$

where $l := \nu + k$, since $k^\beta_{\nu,m}(2^k x) = (2^{\nu+k}x - m)^\beta k(2^{\nu+k}x - m) = k^\beta_{l,m}(x)$. This yields

$$\begin{aligned}
\|f(2^k\cdot)|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\| &\leqslant \sup_\beta 2^{\varrho|\beta|}\left\|\left(\sum_{l=k}^\infty\sum_m 2^{lsq}|\lambda^\beta_{l-k,m}|^q\chi_{l,m}(\cdot)\right)^{1/q}|L_p\right\| \\
&= \sup_\beta 2^{\varrho|\beta|}\left\|\left(\sum_{l=k}^\infty\sum_m 2^{ksq}2^{(l-k)sq}|\lambda^\beta_{l-k,m}|^q\chi_{l-k,m}(2^k\cdot)\right)^{1/q}|L_p\right\| \\
&= 2^{k(s-\frac{n}{p})}\sup_\beta 2^{\varrho|\beta|}\left\|\left(\sum_\nu\sum_m 2^{\nu sq}|\lambda^\beta_{\nu,m}|^q\chi_{\nu,m}(\cdot)\right)^{1/q}|L_p\right\| \\
&= 2^{k(s-\frac{n}{p})}\|f|\mathfrak{F}^s_{p,q}(\mathbb{R}^n)\|.
\end{aligned}$$

Connections and diversity

We now discuss the coincidence and diversity of the above presented concepts of F-spaces and may restrict ourselves to positive smoothness $s > 0$. In view of our Remarks 1.2, 3.2 and 3.7 concerning the different nature of these spaces, it is obvious that there cannot be established a complete coincidence of all approaches when $s < \sigma_p$.

In particular, no equivalent quasi-norms of type (3.2) can be expected for the spaces $F^s_{p,q}(\mathbb{R}^n)$ if $s < \sigma_p$. It seems to be clear that such a characterization is also impossible if $\sigma_p < s < \sigma_{pq}$ (in particular, when $0 < q < p$), i.e.

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) \neq F^s_{p,q}(\mathbb{R}^n), \qquad 0 < p < \infty,\ 0 < q \leqslant \infty,\ 0 < s < \sigma_{pq},$$

cf. [Tri06, Rem. 9.15], based on [CS06] – so the situation is even more complicated. Nevertheless, under certain restrictions on the smoothness parameter s, the above approaches result in the same F-space.

Theorem 3.9. *Let $s > 0$, $0 < p < \infty$, $0 < q \leqslant \infty$.*

(i) *Then*

$$\mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{p}\right), \tag{3.14}$$

and

$$F^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > \sigma_{pq} \tag{3.15}$$

(*in the sense of equivalent quasi-norms*).

(ii) *Furthermore,*

$$F^s_{p,q}(\mathbb{R}^n) = \mathbf{F}^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n), \qquad s > n\left(\frac{1}{\min(p,q)} - \frac{1}{\max(1,p)}\right) \tag{3.16}$$

(*in the sense of equivalent quasi-norms*).

Remark 3.10. The first equality in (3.16) is longer known, see [Tri83, Section 2.5.11], [Tri92, Thm. 3.5.3], whereas the second equality in (3.16) is a consequence of the recently proved coincidence (3.14), see [Tri06, Prop. 9.14] (with forerunners in [Tri97, Sect. 13.8], [Tri01, Thm. 2.9]). In the figures aside and below we have indicated the situation in the usual $(\frac{1}{p}, s)$-diagram for different values of q.

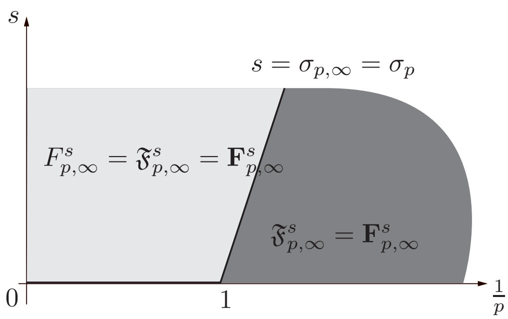


Figure 1: Parameter $q = \infty$

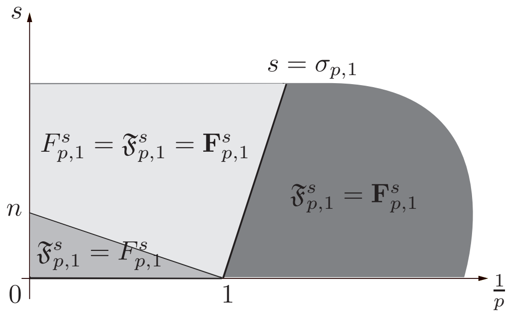


Figure 2: Parameter $q = 1$

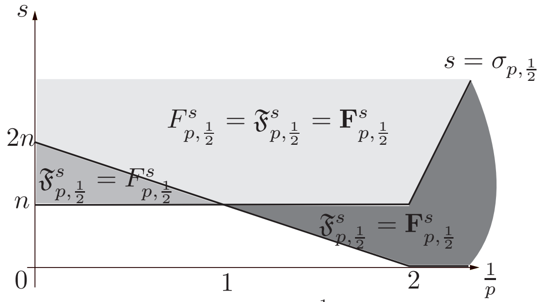


Figure 3: Parameter $q = \frac{1}{2}$

Our new results concerning the norms of the dilation operators T_k established in Section 2 now lead to new insights when dealing with different approaches for F-spaces in the limiting case $s = \sigma_p$. We obtain the following assertions which are especially interesting when $p < q$.

Corollary 3.11. *Let* $0 < p < 1$ *and* $0 < q \leqslant \infty$. *Then*

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathbf{F}^{\sigma_p}_{p,q}(\mathbb{R}^n)$$

and

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathfrak{F}^{\sigma_p}_{p,q}(\mathbb{R}^n)$$

(*in terms of equivalent quasi-norms*) *as sets of measurable functions.*

Proof. We use the homogeneity estimate (3.3),

$$\left\| f(2^k\cdot)|\mathbf{F}^s_{p,q}\right\| \leqslant 2^{k(s-\frac{n}{p})} \left\| f|\mathbf{F}^s_{p,q}\right\|,$$

where $s > 0$, $0 < p < \infty$, and $0 < q \leqslant \infty$. We proceed indirectly, assuming that $F^{\sigma_p}_{p,q}(\mathbb{R}^n) = \mathbf{F}^{\sigma_p}_{p,q}(\mathbb{R}^n)$ for $0 < q \leqslant \infty$. But then using Theorem 2.2 when $p \leqslant q$ or Theorem 2.3 for $q < p$, together with (3.3) we could find a function $\psi \in F^{\sigma_p}_{p,q}$ such that

$$\begin{aligned} 2^{k(\sigma_p-\frac{n}{p})}k^{1/p}\|\psi|F^{\sigma_p}_{p,q}\| &\leqslant c\|\psi(2^k\cdot)|F^{\sigma_p}_{p,q}\| \sim \|\psi(2^k\cdot)|\mathbf{F}^{\sigma_p}_{p,q}\| \\ &\leqslant 2^{k(\sigma_p-\frac{n}{p})}\|\psi|\mathbf{F}^{\sigma_p}_{p,q}\| \sim 2^{k(\sigma_p-\frac{n}{p})}\|\psi|F^{\sigma_p}_{p,q}\|, \end{aligned}$$

which leads to

$$k^{1/p} \leqslant c, \qquad k \in \mathbb{N}.$$

This gives the desired contradiction.

The proof for the spaces $\mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ is the same; we only need to use the estimate (3.13) instead of (3.3). We give an alternative proof of this result in the next subsection. ■

Remark 3.12. We know that $F^s_{p,q}(\mathbb{R}^n) = \mathfrak{F}^s_{p,q}(\mathbb{R}^n)$ if $s > \sigma_{pq}$. Corollary 3.11 yields $F^{\sigma_{pq}}_{p,q}(\mathbb{R}^n) \neq \mathfrak{F}^{\sigma_{pq}}_{p,q}(\mathbb{R}^n)$ if $p \leqslant q$ since in this case $\sigma_{pq} = \sigma_p$. If $p > q$, then $\sigma_{pq} > \sigma_p$ and the sharp estimates for the norms of the dilation operators T_k in $F^{\sigma_{pq}}_{p,q}(\mathbb{R}^n)$, cf. [ET96, 2.3.1], coincide with the bounds for spaces $\mathfrak{F}^{\sigma_{pq}}_{p,q}(\mathbb{R}^n)$ as given in (3.13). So in this case studying dilation operators will not help solving the problem. It does not seem unlikely that the approaches coincide in this case.

3.2. A comment on atomic expansion

It might not be obvious immediately, but the building blocks $k^\beta_{\nu,m}$ in our subatomic approach differ from the atoms $a_{\nu,m}$ – used to characterize the spaces $F^s_{p,q}(\mathbb{R}^n)$ in Theorem 1.7 – mainly by the imposed moment conditions on the latter and some unimportant technicalities. In particular, the normalizing factors $2^{\nu(s-\frac{n}{p})}$ are incorporated in the sequence spaces $f^{s,\varrho}_{p,q}$ in the subatomic approach; recall Definition 1.5. We refer as well to [Tri01, Th. 3.6]. Now following [Sch09, Sect. 3.2] one can show that first moment conditions on the line $s = \sigma_{pq}$ are necessary. This immediately leads to

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathfrak{F}^{\sigma_p}_{p,q}(\mathbb{R}^n),$$

yielding an alternative proof of Corollary 3.11. We present the main ideas. Every $f \in F^{\sigma_p}_{p,q}(\mathbb{R}^n)$ may be represented by optimal atomic decompositions

$$f(x) = \sum_{\nu,m} \lambda_{\nu,m} a_{\nu,m}(x), \qquad x \in \mathbb{R}^n,$$

with

$$\|\lambda|f_{p,q}\| \leqslant c\|f|F^{\sigma_p}_{p,q}\|, \qquad f \in F^{\sigma_p}_{p,q}(\mathbb{R}^n),$$

see [Tri06, Ch. 1.5] for details. If no moment conditions were required here, then

$$g_k(x) = f(2^k x) = \sum_{\nu,m} \lambda_{\nu,m} a_{\nu,m}(2^k x), \qquad x \in \mathbb{R}^n$$

would represent an atomic decomposition of $f(2^k x)$. This can be seen by setting

$$g_k(x) = \sum_{\nu,m} \lambda_{\nu,m} 2^{k(\sigma_p - \frac{n}{p})} 2^{-k(\sigma_p - \frac{n}{p})} a_{\nu,m}(2^k x) = \sum_{\nu,m} \lambda^k_{\nu,m} a^k_{\nu,m}(x),$$

where $a^k_{\nu,m}(x) = 2^{-k(\sigma_p - \frac{n}{p})} a_{\nu,m}(2^k x) \sim \tilde{a}_{\nu+k,m}(x)$, since

$$\operatorname{supp} a^k_{\nu,m} \subset Q_{\nu+k,m},$$

$$|\mathrm{D}^\alpha a^k_{\nu,m}(x)| = 2^{-k(\sigma_p-\frac{n}{p})+k|\alpha|}|\mathrm{D}^\alpha a_{\nu,m}(x)| \leqslant 2^{-(\nu+k)(\sigma_p-\frac{n}{p})+(\nu+k)|\alpha|}.$$

Therefore we obtain

$$\|g_k|F^{\sigma_p}_{p,q}\| \leqslant \|\lambda^k|f_{p,q}\| = 2^{k(\sigma_p-\frac{n}{p})}\|\lambda|f_{p,q}\| = 2^{-nk}\|\lambda|f_{p,q}\|,$$

resulting in

$$\|f(2^k\cdot)|F^{\sigma_p}_{p,q}\| \leqslant c2^{-nk}\|f|F^{\sigma_p}_{p,q}\|.$$

But we know by Theorem 2.2 and Theorem 2.3 that this is *not* true in general when $0 < p < \infty$.

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