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ARITHMETIC PROGRESSIONS OF SQUARES, CUBES AND n-TH POWERS

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To Professors K. Győry and A. Sárközy
on their 70th birthdays

Abstract: In this paper we continue the investigations about unlike powers in arithmetic progression. We provide sharp upper bounds for the length of primitive non-constant arithmetic progressions consisting of squares/cubes and n-th powers.

Keywords: perfect powers, arithmetic progressions

1. Introduction

It was claimed by Fermat and proved by Euler (see [10] pp. 440 and 635) that four distinct squares cannot form an arithmetic progression. It was shown by Darmon and Merel [9] that, apart from trivial cases, there do not exist three-term arithmetic progressions consisting of n-th powers, provided $n \geqslant 3$. An arithmetic progression $a_1, a_2, \ldots, a_t$ of integers is called primitive if $\gcd(a_1, a_2) = 1$. A recent result of Hajdu [11] implies that if

$$x_1^{l_1}, \ldots, x_t^{l_t} \tag{1}$$

is a primitive arithmetic progression in $\mathbb{Z}$ with $2 \leqslant l_i \leqslant L$ $(i = 1, \ldots, t)$, then t is bounded by some constant $c(L)$ depending only on L. Note that $c(L)$ is effective, but it is not explicitly given in [11], and it is a very rapidly growing function of L.

On the other hand, it is known (see e.g. [12], [8], [14] and the references given there) that there exist exponents $l_1, l_2, l_3 \geqslant 2$ for which there are infinitely many primitive arithmetic progressions of the form (1). In this case the exponents in question satisfy the condition

$$\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} \geqslant 1.$$

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In [7] Bruin, Győry, Hajdu and Tengely among other things proved that for any $t \geqslant 4$ and $L \geqslant 3$ there are only finitely many primitive arithmetic progressions of the form (1) with $2 \leqslant l_i \leqslant L$ $(i = 1, \dots, t)$. Furthermore, they showed that in case of $L = 3$ we have $x_i = \pm 1$ for all $i = 1, \dots, t$.

The purpose of the present paper is to give a good, explicit upper bound for the length t of the progression (1) under certain restrictions. More precisely, we consider the cases when the set of exponents is given by $\{2, n\}$, $\{2, 5\}$ and $\{3, n\}$, and (excluding the trivial cases) we show that the length of the progression is at most six, four and four, respectively.

2. Results

Theorem 2.1. *Let n be a prime and $x_1^{l_1}, \dots, x_t^{l_t}$ be a primitive non-constant arithmetic progression in $\mathbb{Z}$ with $l_i \in \{2, n\}$ $(i = 1, \dots, t)$. Then we have $t \leqslant 6$. Further, if $t = 6$ then*

$$(l_1, l_2, l_3, l_4, l_5, l_6) = (2, n, n, 2, 2, 2),\ (2, 2, 2, n, n, 2).$$

In the special case $n = 5$ we are able to prove a sharper result.

Theorem 2.2. *Let $x_1^{l_1}, \dots, x_t^{l_t}$ be a primitive non-constant arithmetic progression in $\mathbb{Z}$ with $l_i \in \{2, 5\}$ $(i = 1, \dots, t)$. Then we have $t \leqslant 4$. Further, if $t = 4$ then*

$$(l_1, l_2, l_3, l_4) = (2, 2, 2, 5),\ (5, 2, 2, 2).$$

Theorem 2.3. *Let n be a prime and $x_1^{l_1}, \dots, x_t^{l_t}$ be a primitive non-constant arithmetic progression in $\mathbb{Z}$ with $l_i \in \{3, n\}$ $(i = 1, \dots, t)$. Then we have $t \leqslant 4$. Further, if $t = 4$ then*

$$(l_1, l_2, l_3, l_4) = (3, 3, n, n), (n, n, 3, 3),\ (3, n, n, 3),\ (n, 3, 3, n).$$

Note that Theorems 2.2 and 2.3 are almost best possible. This is demonstrated by the primitive non-constant progression $-1, 0, 1$. (In fact one can easily give infinitely many examples of arithmetic progressions of length three, consisting of squares and fifth powers.)

We also remark that by a previously mentioned result from [7], the number of progressions of length at least four is finite in each case occurring in the above theorems.

3. Proofs of Theorems 2.1 and 2.3

In the proof of these theorems we need several results about ternary equations of signatures $(n, n, 2)$ and $(n, n, 3)$, respectively. We start this section with summarizing these statements. The first three lemmas are known from the literature, while the fourth one is new.

Lemma 3.1. *Let n be a prime. Then the Diophantine equations*

$$X^n + Y^n = 2Z^2 \quad (n \geqslant 5),$$
$$X^n + Y^n = 3Z^2 \quad (n \geqslant 5),$$
$$X^n + 4Y^n = 3Z^2 \quad (n \geqslant 7)$$

have no solutions in nonzero pairwise coprime integers (X, Y, Z) with $XY \neq \pm 1$.

Proof. The statement follows from results of Bennett and Skinner [1], and Bruin [6]. ∎

Lemma 3.2. *Let $n \geqslant 5$ be a prime. Then the Diophantine equation*

$$X^n + Y^n = 2Z^3$$

has no solutions in coprime nonzero integers X, Y, Z with $XYZ \neq \pm 1$.

Proof. The result is due to Bennett, Vatsal and Yazdani [2]. ∎

Lemma 3.3. *Let $n \geqslant 3$ be a prime. Then the Diophantine equation*

$$X^n + Y^n = 2Z^n$$

has no solutions in coprime nonzero integers X, Y, Z with $XYZ \neq \pm 1$.

Proof. The result is due to Darmon and Merel [9]. ∎

Lemma 3.4. *Let $n \geqslant 3$ be a prime. Then the Diophantine equation*

$$X^3 + Y^3 = 2Z^n$$

has no solutions in coprime nonzero integers X, Y, Z with $XYZ \neq \pm 1$ and $3 \nmid Z$.

Proof. First note that in case of $n = 3$ the statement follows from Lemma 3.3. Let $n \geqslant 5$, and assume to the contrary that (X, Y, Z) is a solution to the equation with $\gcd(X, Y, Z) = 1$, $XYZ \neq \pm 1$ and $3 \nmid Z$. Note that the coprimality of X, Y, Z shows that XY is odd. We have

$$(X + Y)(X^2 - XY + Y^2) = 2Z^n.$$

Our assumptions imply that $\gcd(X + Y, X^2 - XY + Y^2) \mid 3$, whence $2 \nmid XY$ and $3 \nmid Z$ yield that

$$X + Y = 2U^n \text{ and } X^2 - XY + Y^2 = V^n$$

hold, where $U, V \in \mathbb{Z}$ with $\gcd(U, V) = 1$. Combining these equations we get

$$f(X) := 3X^2 - 6U^n X + 4U^{2n} - V^n = 0.$$

Clearly, the discriminant of f has to be a square in $\mathbb{Z}$, which leads to an equality of the form

$$V^n - U^{2n} = 3W^2$$

with some $W \in \mathbb{Z}$. However, this is impossible by Lemma 3.1. ∎

Now we are ready to prove our Theorems 2.1 and 2.3.

Proof of Theorem 2.1. Suppose that we have an arithmetic progression (1) of the desired form, with $t = 6$. In view of a result from [7] about the case $n = 3$ and Theorem 2.2, without loss of generality we may assume that $n \geqslant 7$.

First note that the already mentioned classical result of Fermat and Euler implies that we cannot have four consecutive squares in our progression. Further, observe that Lemmas 3.1 and 3.3 imply that we cannot have three consecutive terms with exponents $(n, 2, n)$ and (n, n, n), respectively, and further that $(l_1, l_3, l_5) = (n, 2, n)$, (n, n, n) are also impossible.

If $(l_1, l_2, l_3, l_4, l_5) = (n, 2, 2, n, 2)$ or $(2, n, 2, 2, n)$, then we have

$$4x_4^n - x_1^n = 3x_5^2 \qquad \text{or} \qquad 4x_2^n - x_5^n = 3x_1^2,$$

respectively, both equations yielding a contradiction by Lemma 3.1.

To handle the remaining cases, let d denote the common difference of the progression. Let $(l_1, l_2, l_3, l_4, l_5) = (2, 2, n, 2, 2)$. Then (as clearly $x_1 \neq 0$) we have

$$(1 + X)(1 + 3X)(1 + 4X) = Y^2$$

where $X = d/x_1$ and $Y = x_2x_4x_5/x_1$. However, a simple calculation with Magma [3] shows that the rank of this elliptic curve is zero, and it has exactly eight torsion points. However, none of these torsion points gives rise to any appropriate arithmetic progression.

When $(l_1, l_2, l_3, l_4, l_5, l_6) = (2, 2, n, n, 2, 2)$, then in a similar manner we get

$$(1 + X)(1 + 4X)(1 + 5X) = Y^2$$

with $X = d/x_1$ and $Y = x_2x_5x_6/x_1$, and just as above, we get a contradiction.

In view of the above considerations, a simple case-by-case analysis yields that the only remaining possibilities are the ones listed in the theorem. Hence to complete the proof we need only to show that the possible six-term progressions cannot be extended to seven-term ones. Using symmetry it is sufficient to deal with the case given by

$$(l_1, l_2, l_3, l_4, l_5, l_6) = (2, n, n, 2, 2, 2).$$

However, one can easily verify that all the possible extensions lead to a case treated before, and the theorem follows. ■

Proof of Theorem 2.3. In view of Lemma 3.3 and the previously mentioned result from [7] we may suppose that $n \geqslant 5$. Assume that we have an arithmetic progression of the indicated form, with $t = 4$. By the help of Lemmas 3.2 and 3.3 we get that there cannot be three consecutive terms with exponents $(n, 3, n)$, and $(3, 3, 3)$ or (n, n, n), respectively. Hence a simple calculation yields that the only possibilities (except for the ones listed in the theorem) are given by

$$(l_1, l_2, l_3, l_4) = (3, n, 3, 3),\ (3, 3, n, 3).$$

Then Lemma 3.4 yields that $3 \mid x_2$ and $3 \mid x_3$, respectively. However, looking at the progressions modulo 9 and using that $x^3 \equiv 0, \pm 1 \pmod 9$ for all $x \in \mathbb{Z}$ we get a contradiction with the primitivity condition in both cases.

Finally, one can easily check that the extensions of the four-term sequences corresponding to the exponents listed in the statement to five-term ones, yield cases which have been treated already. Hence the proof of the theorem is complete. ■

4. Proof of Theorem 2.2

To prove this theorem we need some lemmas, obtained by the help of elliptic Chabauty's method. To prove the lemmas we used the program package Magma [3]. The transcripts of computer calculations can be downloaded from the URL-s `www.math.klte.hu/~tengely/Lemma4.1` and `www.math.klte.hu/~tengely/Lemma4.2`, respectively.

Lemma 4.1. *Let $\alpha = \sqrt[5]{2}$ and put $K = \mathbb{Q}(\alpha)$. Then the equations*

$$C_1: \ \alpha^4X^4 + \alpha^3X^3 + \alpha^2X^2 + \alpha X + 1 = (\alpha - 1)Y^2 \tag{2}$$

and

$$C_2: \ \alpha^4X^4 - \alpha^3X^3 + \alpha^2X^2 - \alpha X + 1 = (\alpha^4 - \alpha^3 + \alpha^2 - \alpha + 1)Y^2 \tag{3}$$

in $X \in \mathbb{Q}$, $Y \in K$ have the only solutions

$$(X, Y) = (1, \pm(\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1)), \ \left(-\frac{1}{3}, \pm\frac{3\alpha^4 + 5\alpha^3 - \alpha^2 + 3\alpha + 5}{9}\right)$$

and $(X, Y) = (1, \pm 1)$, respectively.

Proof. Using the so-called elliptic Chabauty's method (see [4], [5]) we determine all points on the above curves for which X is rational. The algorithm is implemented by N. Bruin in Magma, so here we indicate the main steps only, the actual computations can be carried out by Magma. We can transform C_1 to Weierstrass form

$$E_1: \ x^3 - (\alpha^2 + 1)x^2 - (\alpha^4 + 4\alpha^3 - 4\alpha - 5)x + (2\alpha^4 - \alpha^3 - 4\alpha^2 - \alpha + 4) = y^2.$$

The torsion subgroup of E_1 consists of two elements. Moreover, the rank of E_1 is two, which is less than the degree of the number field K. Applying elliptic Chabauty (the procedure "Chabauty" of Magma) with $p = 3$, we obtain that $X \in \{1, -1/3\}$.

In case of C_2 a similar procedure works. Now the corresponding elliptic curve E_2 is of rank two. Applying elliptic Chabauty this time with $p = 7$, we get that $X = 1$, and the lemma follows. ■

Lemma 4.2. *Let $\beta = (1+\sqrt{5})/2$ and put $L = \mathbb{Q}(\beta)$. Then the only solutions to the equation*

$$C_3: \ X^4 + (8\beta - 12)X^3 + (16\beta - 30)X^2 + (8\beta - 12)X + 1 = Y^2 \tag{4}$$

in $X \in \mathbb{Q}$, $Y \in L$ are $(X,Y) = (0, \pm 1)$.

Proof. The proof is similar to that of Lemma 4.1. We can transform C_3 to Weierstrass form

$$E_3: \ x^3 - (\beta - 1)x^2 - (\beta + 2)x + 2\beta = y^2.$$

The torsion group of E_3 consists of four points and $(x,y) = (\beta - 1, 1)$ is a point of infinite order. Applying elliptic Chabauty with $p = 13$, we obtain that $(X,Y) = (0, \pm 1)$ are the only affine points on C_3 with rational first coordinates. ∎

Now we can give the

Proof of Theorem 2.2. Suppose that we have a four-term progression of the desired form. Then by Lemmas 3.1, 3.3 and the result of Fermat and Euler we obtain that all the possibilities (except for the ones given in the statement) are

$$\begin{aligned}(l_1, l_2, l_3, l_4) = (2,2,5,5),\ (5,5,2,2),\ (2,5,5,2),\\ (5,2,2,5),\ (2,2,5,2),\ (2,5,2,2).\end{aligned}$$

We show that these possibilities cannot occur. Observe that by symmetry we may assume that we have

$$(l_1, l_2, l_3, l_4) = (2,2,5,5),\ (2,5,5,2),\ (5,2,2,5),\ (2,2,5,2).$$

In the first two cases the progression has a sub-progression of the shape a^2, b^5, c^5. Note that here $\gcd(b,c) = 1$ and bc is odd. Indeed, if c would be even then we would get $4 \mid a^2, c^5$, whence it would follow that b is even - a contradiction. Taking into consideration the fourth term of the original progression, a similar argument shows that b is also odd. Using this subprogression we obtain the equality $2b^5 - c^5 = a^2$. Putting $\alpha = \sqrt[5]{2}$ we get the factorization

$$(\alpha b - c)(\alpha^4 b^4 + \alpha^3 b^3 c + \alpha^2 b^2 c^2 + \alpha b c^3 + c^4) = a^2 \tag{5}$$

in $K = \mathbb{Q}(\alpha)$. Note that the class number of K is one, $\alpha^4, \alpha^3, \alpha^2, \alpha, 1$ is an integral basis of K, $\varepsilon_1 = \alpha - 1$, $\varepsilon_2 = \alpha^3 + \alpha + 1$ provides a system of fundamental units of K with $N_{K/\mathbb{Q}}(\varepsilon_1) = N_{K/\mathbb{Q}}(\varepsilon_2) = 1$, and the only roots of unity in K are given by ± 1. A simple calculation shows that

$$D := \gcd(\alpha b - c, \alpha^4 b^4 + \alpha^3 b^3 c + \alpha^2 b^2 c^2 + \alpha b c^3 + c^4) \mid \gcd(\alpha b - c, 5\alpha b c^3)$$

in the ring of integers O_K of K. Using $\gcd(b,c) = 1$ and $2 \nmid c$ in $\mathbb{Z}$, we get $D \mid 5$ in O_K. Using e.g. Magma, one can easily check that $5 = (3\alpha^4 + 4\alpha^3 - \alpha^2 - 6\alpha - 3)(\alpha^2 + 1)^5$, where $3\alpha^4 + 4\alpha^3 - \alpha^2 - 6\alpha - 3$ is a unit in K, and $\alpha^2 + 1$ is a prime in O_K with $N_{K/\mathbb{Q}}(\alpha^2+1)=5$. By the help of these information, we obtain that

$$\alpha b - c = (-1)^{k_0}(\alpha-1)^{k_1}(\alpha^3+\alpha+1)^{k_2}(\alpha^2+1)^{k_3}z^2$$

with $k_0,k_1,k_2,k_3 \in \{0,1\}$ and $z \in O_K$. Taking the norms of both sides of the above equation, we get that $k_0=k_3=0$. Further, if $(k_1,k_2)=(0,0),(1,1),(0,1)$ then putting $z=z_4\alpha^4+z_3\alpha^3+z_2\alpha^2+z_1\alpha+z_0$ with $z_i \in \mathbb{Z}$ $(i=0,\dots,4)$ and expanding the right hand side of the above equation, we get $2 \mid b$, which is a contradiction. (Note that to check this assertion, in case of $(k_1,k_2)=(0,1)$ one can also use that the coefficients of α^2 and α^3 on the left hand side are zero.) Hence we may conclude that $(k_1,k_2)=(1,0)$. Thus using (5) we get that

$$\alpha^4b^4+\alpha^3b^3c+\alpha^2b^2c^2+\alpha bc^3+c^4=(\alpha-1)y^2$$

with some $y \in O_K$. Hence after dividing this equation by c^4 (which cannot be zero), we get (2), and then a contradiction by Lemma 4.1. Hence the first two possibilities for (l_1,l_2,l_3,l_4) are excluded.

Assume next that $(l_1,l_2,l_3,l_4)=(5,2,2,5)$. Then we have $2x_1^5+x_4^5=3x_2^2$. Using the notation of the previous paragraph, we can factorize this equation over K to obtain

$$(\alpha x_1+x_4)(\alpha^4x_1^4-\alpha^3x_1^3x_4+\alpha^2x_1^2x_4^2-\alpha x_1x_4^3+x_4^4)=3x_2^2. \tag{6}$$

Observe that the primitivity condition implies that $\gcd(x_1,x_4)=1$, and $2 \nmid x_1x_4$. Hence in the same manner as before we obtain that the greatest common divisor of the terms on the left hand side of (6) divides 5 in O_K. Further, a simple calculation e.g. with Magma yields that $3=(\alpha+1)(\alpha^4-\alpha^3+\alpha^2-\alpha+1)$, where $\alpha+1$ and $\alpha^4-\alpha^3+\alpha^2-\alpha+1$ are primes in O_K with $N_{K/\mathbb{Q}}(\alpha+1)=3$ and $N_{K/\mathbb{Q}}(\alpha^4-\alpha^3+\alpha^2-\alpha+1)=81$, respectively. Using these information we can write

$$\alpha x_1+x_4=(-1)^{k_0}(\alpha-1)^{k_1}(\alpha^3+\alpha+1)^{k_2}(\alpha+1)^{k_3}(\alpha^4-\alpha^3+\alpha^2-\alpha+1)^{k_4}z^2$$

with $k_0,k_1,k_2,k_3,k_4 \in \{0,1\}$ and $z \in O_K$. Taking the norms of both sides of the above equation, we get that $k_0=0$ and $k_3=1$. Observe that $k_4=1$ would imply $3 \mid x_1,x_4$. This is a contradiction, whence we conclude $k_4=0$. Expanding the above equation as previously, we get that if $(k_1,k_2)=(0,1),(1,0),(1,1)$ then x_1 is even, which is a contradiction again. (To deduce this assertion, when $(k_1,k_2)=(1,1)$ we make use of the fact that the coefficients of α^3 and α^2 vanish on the left hand side.) So we have $(k_1,k_2)=(0,0)$, which by the help of (6) implies

$$\alpha^4x_1^4-\alpha^3x_1^3x_4+\alpha^2x_1^2x_4^2-\alpha x_1x_4^3+x_4^4=(\alpha^4-\alpha^3+\alpha^2-\alpha+1)y^2$$

with some $y \in O_K$. However, after dividing this equation by x_1^4 (which is certainly non-zero), we get (3), and then a contradiction by Lemma 4.1.

Finally, suppose that $(l_1,l_2,l_3,l_4)=(2,2,5,2)$. Using the identity $x_2^2+x_4^2=2x_3^5$, e.g. by the help of a result of Pink and Tengely [13] we obtain

$$x_2=u^5-5u^4v-10u^3v^2+10u^2v^3+5uv^4-v^5$$

and

$$x_4 = u^5 + 5u^4v - 10u^3v^2 - 10u^2v^3 + 5uv^4 + v^5$$

with some coprime integers u, v. Then the identity $3x_2^2 - x_4^2 = 2x_1^2$ implies

$$(u^2 - 4uv + v^2)f(u, v) = x_1^2 \tag{7}$$

where

$$f(u, v) = u^8 - 16u^7v - 60u^6v^2 + 16u^5v^3 + 134u^4v^4 + 16u^3v^5 - 60u^2v^6 - 16uv^7 + v^8.$$

A simple calculation shows that the common prime divisors of the terms at the left hand side belong to the set $\{2, 5\}$. However, $2 \mid x_1$ would imply $4 \mid x_1^2, x_3^5$, which would violate the primitivity condition. Further, if $5 \mid x_1$ then looking at the progression modulo 5 and using that by the primitivity condition $x_2^2 \equiv x_4^2 \equiv \pm 1 \pmod 5$ should be valid, we get a contradiction. Hence the above two terms are coprime, which yields that

$$f(u, v) = w^2$$

holds with some $w \in \mathbb{Z}$. (Note that a simple consideration modulo 4 shows that $f(u, v) = -w^2$ is impossible.) Let $\beta = (1 + \sqrt{5})/2$, and put $L = \mathbb{Q}(\beta)$. As is well-known, the class number of L is one, $\beta, 1$ is an integral basis of L, β is a fundamental unit of L with $N_{L/\mathbb{Q}}(\beta) = 1$, and the only roots of unity in L are given by ± 1. A simple calculation shows that

$$f(u, v) = g(u, v)h(u, v)$$

with

$$g(u, v) = u^4 + (8\beta - 12)u^3v + (16\beta - 30)u^2v^2 + (8\beta - 12)uv^3 + v^4$$

and

$$h(u, v) = u^4 + (-8\beta - 4)u^3v + (-16\beta - 14)u^2v^2 + (-8\beta - 4)uv^3 + v^4.$$

Further, $\gcd(6, x_1) = 1$ by the primitivity of the progression, and one can easily check modulo 5 that $5 \mid x_1$ is also impossible. Hence we conclude that $g(u, v)$ and $h(u, v)$ are coprime in the ring O_L of integers of L. Thus we have

$$g(u, v) = (-1)^{k_0}\beta^{k_1}z^2$$

with some $k_0, k_1 \in \{0, 1\}$ and $z \in O_L$. Note that as $2 \nmid x_1$, equation (7) implies that exactly one of u, v is even. Hence a simple calculation modulo 4 shows that the only possibility for the exponents in the previous equation is $k_0 = k_1 = 0$. However, then after dividing the equation with v^4 (which cannot be zero), we get (4), and then a contradiction by Lemma 4.2.

There remains to show that a four-term progression with exponents $(l_1, l_2, l_3, l_4) = (2, 2, 2, 5)$ or $(5, 2, 2, 2)$ cannot be extended to a five-term one.

By symmetry it is sufficient to deal with the first case. If we insert a square or a fifth power after the progression, then the last four terms yield a progression which has been already excluded. Writing a fifth power, say x_0^5 in front of the progression would give rise to the identity $x_0^5 + x_4^5 = 2x_2^2$, which leads to a contradiction by Lemma 3.1. Finally, putting a square in front of the progression is impossible by the already mentioned result of Fermat and Euler. ■

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