

CONSTRUCTION OF GENERALIZED MODULAR INTEGRALS

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Abstract: In this paper, we find a functional equation that characterizes the series involved in the Fourier coefficients of generalized modular forms of large negative real weights.

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1. Introduction and Definitions

In this paper, we construct a basis for the space of automorphic integrals associated to weakly parabolic generalized modular forms on subgroups of the full modular groups of large real weights. In our construction, we follow closely Niebur [5] construction of automorphic integrals. We also find a functional equation that characterizes a series that appear in the Fourier expansion of generalized modular forms of large negative weights [7]. The importance of the results afore mentioned lies in its use to prove Eichler isomorphism theorems for generalized modular forms of arbitrary large real weights.

By a generalized modular form [3] $F(\tau)$ belonging to a subgroup Γ of a finite index in the full modular group of real weight k and multiplier system (MS) v we mean that $F(\tau)$ is analytic in the upper half plane \mathbb{H} and that $F(\tau)$ satisfies a transformation law

$$F(M\tau) = v(M)(c\tau + d)^k F(\tau), \quad (1)$$

where $|v|$ is not necessarily 1 that depends only of the transformation, and

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Note that the multiplier system satisfy the consistency condition

$$v(M_1 M_2)(c_3\tau + d_3)^k = v(M_1)v(M_2)(c_1 M_2\tau + d_1)^k (c_2\tau + d_2)^k, \quad (2)$$

where

$$M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma$$

for $i = 1, 2, 3$ and $M_3 = M_1M_2$. We shall assume that our generalized modular forms are weakly parabolic generalized modular forms which means that the $|v(Q)| = 1$ for all parabolic matrices P .

Let $S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda > 0$, generate the subgroup Γ_∞ of translations in Γ . Since F satisfies (1), then in particular

$$F(z + \lambda) = v(S)F(z) = e^{2\pi i\kappa}F(z),$$

with $0 \leq \kappa < 1$. Thus if F is meromorphic in H and its poles do not accumulate at infinity, F has the Fourier expansion at ∞

$$F(z) = \sum_{m=-m_0}^{\infty} a_m e^{2\pi i(m+\kappa)z/\lambda}, \tag{3}$$

valid for $y = \text{Im } z > y_0$. Γ has also $s \geq 0$ inequivalent parabolic classes. Each of these classes corresponds to a cyclic subgroup of parabolic elements in Γ leaving fixed a parabolic cusp on the boundary of R , the fundamental region of Γ . Let q_1, q_2, \dots, q_s be the inequivalent parabolic cusps (other than infinity) on the boundary of R and let Γ_i be the cyclic subgroup of Γ fixing q_j , $1 \leq j \leq s$. Suppose also that

$$Q_j = \begin{pmatrix} * & * \\ c_j & d_j \end{pmatrix}$$

is a generator of Γ_i ; Q_j is necessarily parabolic. For $1 \leq j \leq s$; put $v(Q_j) = e^{2\pi i\kappa_j}$, $0 \leq \kappa_j < 1$. Also F has the following Fourier expansion at q_j :

$$F(z) = (z - q_j)^{-k} \sum_{m=-m_j}^{\infty} a_m(j) e^{-2\pi i(m+\kappa_j)/\lambda_j(z-q_j)}, \tag{4}$$

valid for $y = \text{Im } z > y_j$. Here λ_j is a positive real number called the width of the cusp q_j and defined as follows. Let

$$A_j = \begin{pmatrix} 0 & -1 \\ 1 & -q_j \end{pmatrix}$$

so that A_j has determinant 1 and $A_j(q_j) = \infty$. Then $\lambda_j > 0$ is chosen so that

$$A_j^{-1} \begin{pmatrix} 1 & \lambda_j \\ 0 & 1 \end{pmatrix} A_j$$

generates Γ_j , the stabilizer of q_j . We let $C^+(\Gamma, k, v)$ denote the space of entire generalized modular forms of real weight k and multiplier system v on Γ which in addition to being holomorphic in H , it has only terms with $m + \kappa \geq 0$ in (3) and $m + \kappa_j \geq 0$ in (4) for all $1 \leq j \leq s$. Also, let $C^0(\Gamma, k, v)$ denote the subspace of generalized cusps forms which is a subspace of $C^+(\Gamma, k, v)$ but it has only terms with $m + \kappa > 0$ in (3) and $m + \kappa_j > 0$ in (4) for all $1 \leq j \leq s$.

Define also $\Gamma_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c = 0 \right\}$.

2. Construction of Automorphic Integrals

Remark. Notice that by Proposition 1 of [4], we see that if \bar{v} is a MS in weight $k + 2$, then v is a multiplier system of weight $-k$ and vice versa. This fact will be used throughout the whole section.

To construct a basis for the space of automorphic integrals associated to weakly parabolic generalized modular forms, we follow similar construction carried in [5].

Let ν be an integer such that $\nu + \kappa > 0$, $k > \alpha'$ where α' is a positive integer to be defined later and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Define

$$g_M(z) = (-2\pi i(\nu + \kappa)/\lambda)^{k+1} e^{2\pi i(\nu + \kappa)Mz/\lambda} / v(M)(cz + d)^{k+2}.$$

Notice that $(\gamma z + \delta)^{-k-2} g_M(Lz) = v(m)g_{ML}(z)$ for $L = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma$. Since the multiplier system is not unitary, we need to worry about the bound for the multiplier system when we take an infinite sum of $g_M(z)$ over all lower rows of all matrices $M \in \Gamma/\Gamma_0$. Suppose that the generators of Γ are the hyperbolic elements $A_1, B_1, A_2, B_2, \dots, A_g, B_g$, the elliptic elements E_1, \dots, E_t and the parabolic elements Q_1, \dots, Q_s . To find a bound for our multiplier system, we need not to worry about the parabolic elements since we are dealing with weakly parabolic generalized modular forms and as a result

$$|\epsilon(Q_i)| = 1$$

for all $i = 1, \dots, s$. So what is left to consider are the hyperbolic and elliptic generators. Suppose now we write $M = M_1 \dots M_L$ where each M_i is a section. Each section is either a nonparabolic generator of Γ or a power of a parabolic generator of Γ . Also assume L is minimal. The importance of this factorization into sections lies in the result of Eichler [1], that for any $M \in \Gamma$, the factorization can be carried out so that

$$L(M) \leq m_1 \log \mu(M) + m_2 \tag{5}$$

where $m_1, m_2 > 0$ are independent of M and

$$\mu(M) = a^2 + b^2 + c^2 + d^2.$$

Assume now that $|v(A_1)| = a_1, |v(B_1)| = b_1, \dots, |v(A_g)| = a_g, |v(B_g)| = b_g$. Since elliptic elements are of finite order and since $|v|$ is a multiplicative character, it is easy to see that $|v(E_i^{n_i})| = 1$ where $E_i^{n_i} = I$ hence, $|v(E_i)| = 1$ for all $i = 1, \dots, t$. Note that if the genus is zero, then there is no hyperbolic generators so the parabolic generalized modular form will be classical. As a result, we let $K_1 = \max(a_i, b_i, 1)$. We then have

$$|v(M)| \leq K_1^{L(M)}. \tag{6}$$

As a result, we see that

$$|v(M)| \leq K_2 \mu(M)^\alpha \tag{7}$$

where $\alpha = m_1 \log K_1$. Notice also that M can be written after reusing the same variables for convenience as $M = \begin{pmatrix} a & \frac{ad-1}{c} \\ c & d \end{pmatrix}$ where $0 \leq a, -d < c$. From (7) and the fact that $0 \leq a, -d < c$, we see that

$$|v(M)| \leq K_3 c^{2\alpha} \tag{8}$$

where K_3 is independent of M . Notice that using (2) that $v(M^{-1}) = \kappa v^{-1}(M)$ where $|\kappa| = 1$. Notice also that $L(M) = L(M^{-1})$. As a result we derived a bound for the multiplier system depending on c . Because the sum over all distinct rows (c, d) of matrices in Γ of $|cz + d|^{-k-2}$ converges on strips of the form $\{x + iy : |x| \leq A, y \geq \epsilon > 0\}$, we see that for $k > 2\alpha - 2$, $G_\nu(z) = \sum_{M \in \Gamma_0/\Gamma} g_M(z) \in C^0(\Gamma, k+2, \nu)$. As a result, we define $\alpha' = [\alpha - 2] + 1$ where $[x]$ denotes the integer part of x .

Definition 1. Let ν be an integer such that $\nu + \kappa > 0$ and let $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$. Let $\mu + \alpha = -\nu - \kappa$ and $k > \alpha'$. Put

$$s_M(\tau) = e^{2\pi i(\mu+\alpha)M\tau/\lambda} / \bar{v}(M)(c\tau + d)^{-k},$$

and

$$t_M = p(\tau, M^{-1}\infty, g_M),$$

where

$$\Gamma(k+1)p(\tau, M^{-1}\infty, g_M) = \overline{\int_{M^{-1}\infty}^{i\infty} g_M(z)(z - \bar{\tau})^k dz}.$$

Notice that $s_M(\tau)$ and $t_M(\tau)$ are holomorphic in \mathbb{H} . Notice also that for $L = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma$

$$(\gamma\tau + \delta)^k s_M(L\tau) = \bar{v}(L) s_{ML}(\tau)$$

and

$$(\gamma\tau + \delta)^k t_M(L\tau) = \bar{v}(L)[t_{ML}(\tau) - p(\tau, L^{-1}\infty, g_{ML})].$$

As in [5], to define a series analogous to the series G_ν we need the following lemma.

Lemma 1. Suppose that $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ and $c > 0$, then

$$s_M(\tau) + t_M(\tau) = A(\bar{v}(M))^{-1} e^{2\pi i(\mu+\alpha)a/\lambda c} c^{-1-k} (c\tau+d)^{-1} + O(c^{-2-k+2\alpha'} |c\tau+d|^{-2}),$$

where $A = (-2\pi i(\mu + \alpha)/\lambda)^{k+1} / \Gamma(k+2)$.

Proof The proof of this lemma follows the steps of the proof of Lemma 3.1 in [5] with a change in the multiplier system and a change in the estimate of the sum. Notice that

$$\begin{aligned} & \Gamma(k+1)(2\pi i(\nu+\kappa)/\lambda)^{-k-1}t_M(\tau) \\ &= (\bar{v}(M))^{-1} \int_{-d/c}^{i\infty} e^{-2\pi i(\nu+\kappa)M\bar{z}/\lambda}(c\bar{z}+d)^{-2-k}(\bar{z}-\tau)^k d\bar{z} \end{aligned}$$

The above equation shows why $(\bar{v}(M))^{-1}$ appears in the expression $s_M(\tau)+t_M(\tau)$. Also in the proof of Lemma 3.1 [5, p.377], we have the simple estimate $b(\beta) = \beta/\Gamma(2+k) + O(|\beta|^2 e^{|\beta|})$. Thus plugging in the following equation, derived in the same way as in [5],

$$s_M(\tau) + t_M(\tau) = (\bar{v}(M))^{-1} e^{2\pi i(\mu+\alpha)a/c} (-2\pi i(\mu+\alpha)/c\lambda)^k b(\beta)$$

and using the estimate of the multiplier system (8), we get

$$s_M(\tau)+t_M(\tau) = A(\bar{v}(M))^{-1} e^{2\pi i(\mu+\alpha)a/\lambda c} c^{-1-k} (c\tau+d)^{-1} + O(c^{-2-k+2\alpha'} |c\tau+d|^{-2}).$$

This explains the appearance of α' in the exponent of c inside the O notation.

Lemma 1 will guarantee that the sum $f(\tau) = \sum_{M \in \Gamma_0 \backslash \Gamma} s_M(\tau) + t_M(\tau)$ converges to a function holomorphic in \mathbb{H} for $k > \alpha'$. Although as in [5], the convergence is not absolute, it follows exactly the same proof that for certain family of parallelograms $\{P_N\}$, the sum over such parallelograms will converge to f as $N \rightarrow \infty$ where \sum_{P_N} means a sum over coset representatives with lower row in P_N . This will also hold for transformed family $\{P_N L\}$, where $L \in \Gamma$ and $P_N L = \{(c, d)L : (c, d) \in P_N\}$. Also notice that for $L = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$

$$\begin{aligned} (\gamma\tau + \delta)^k f(L\tau) &= (\bar{v}(L)) \lim_{N \rightarrow \infty} \sum_{P_N} (s_{ML}(\tau) + t_{ML}(\tau) - p(\tau, L^{-1}\infty, g_{ML})) \\ &= (\bar{v}(L)) \lim_{N \rightarrow \infty} \sum_{P_N L} (s_M(\tau) + t_M(\tau) - p(\tau, L^{-1}\infty, g_M)) \\ &= (\bar{v}(L)) [f(\tau) - p(\tau, L^{-1}\infty, G_\nu)]. \end{aligned}$$

Notice that for the justification of

$$\lim_{N \rightarrow \infty} p\left(\tau, L^{-1}\infty, \sum_{P_N} g_M\right) = p\left(\tau, L^{-1}\infty, \lim_{N \rightarrow \infty} \sum_{P_N} g_M\right),$$

we use the same justification used in [5] (Lemma 4.4 and Corollary 4.5 in [5]). Also we see that the above equation is not true if f has a nonzero constant in its Fourier expansion. As a result we twist the definition of f to introduce F_μ .

Definition 2. Let μ be a negative integer and put $\nu = -\mu - \alpha - \kappa$. Define the m th Poincare series $F_\mu(\tau)$ for weight $-k$ where $k > \alpha'$ as

$$F_\mu(\tau) = \frac{1}{2}a_0(\mu + \alpha) + \sum_{c>0} \lim_{N \rightarrow \infty} \sum_{|d| \leq N} s_M(\tau) + t_M(\tau), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 \setminus \Gamma$$

where

$$a_0(\mu) = |2\pi\mu/\lambda|^{k+1} \frac{2\pi}{\lambda\Gamma(k+2)} e^{2\pi i k/4} \sum_{c>0} c^{-2-k} S(\mu; 0; c, v),$$

for $\alpha = 0$ and $a_0(\mu + \alpha) = 0$ for $\alpha > 0$. Here, S is the Kloostermann sum defined by

$$S(m, n; c, v) = \sum_M (\bar{v}(M))^{-1} e^{2\pi i(a(m+\alpha)+d(n+\alpha))/\lambda c},$$

where M runs over a finite set $\left\{ \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma : 0 \leq a < \lambda|c|, 0 \leq d < \lambda|c| \right\}$.

We now present a Lemma which states that F_μ is holomorphic in \mathbb{H} , and that its Fourier coefficients are given by Rademacher formula.

Lemma 2. For $k > \alpha'$, we have $F_\mu(\tau)$ is holomorphic in \mathbb{H} and $F_\mu(\tau) = e^{2\pi i(\mu+\alpha)\tau/\lambda} + \sum_{m=0}^\infty a_m(\mu + \alpha) e^{2\pi i(m+\alpha)\tau/\lambda}$ where

$$a_m(\mu + \alpha) = (2\pi/\lambda) e^{2\pi i k/4} |(\mu + \alpha)/(m + \alpha)|^{(k+1)/2} \times \sum_{c>0} c^{-1} S(\mu, m; c, v) I_{k+1}(4\pi|(\mu + \alpha)(m + \alpha)|^{1/2}/\lambda c)$$

where I_{k+1} is the modified Bessel function of the first kind.

The proof of Lemma 2 goes in the same way as Lemma 4.2 in [5] with a slight difference in the convergence of the sums. Thus it is crucial to impose the condition $k > \alpha'$ to insure the convergence of the infinite sum once the estimate of the multiplier system is used. Also note that to show where $(\bar{v}(M))^{-1}$ shows up in the Kloostermann sum, we use the sum in Lemma 1 and the estimation equation from [5, p.380] to obtain

$$\sum_{|d| \leq N} s_M(\tau) + t_M(\tau) = \sum_d^* (\bar{v}(M))^{-1} e^{-2\pi i \mu_0 a/c} (2\pi i \mu_0/c)^k \times \sum_{|n| \leq N/c\lambda} e^{2\pi i n \alpha} \sum_{j=1}^\infty \frac{(2\pi i \mu_0/c(c\tau + d - n c\lambda))^j}{\Gamma(j+1+k)} + O(N^{-1}).$$

Note that the sum over the right coset of representatives for Γ over Γ_0 will be written as a sum over $I \cup \left\{ \begin{pmatrix} a & * \\ c & d + n\lambda c \end{pmatrix} : c > 0, 0 \leq a < c\lambda, 0 \leq d \leq c\lambda, n \in \mathbb{Z} \right\}$ and the sum over this set will be represented by $\sum_{c>0} \sum_d^* \sum_n$. By I we denote the identity matrix.

We now show that with F_μ defined as in Lemma 2, it can be written as a limit of finite double sums.

Lemma 3. *Let $k > \alpha'$ and let F_μ be defined as in Lemma 2. Suppose that $L \in \Gamma$ and for each integer N , $P_N L = \{(x, y)L : |x| \leq N, |y| \leq N^2\}$. If $\tau \in \mathbb{H}$, then*

$$F_\mu(\tau) = \frac{1}{2}a_0(\mu + \alpha) + \lim_{N \rightarrow \infty} \sum_{M \in A_N L} s_M(\tau) + t_M(\tau)$$

where $A_N L = I \cup \{M\Gamma_0 \setminus \Gamma : c > 0, (c, d) \in P_N L\}$.

As in Lemma 2, the condition that $k > \alpha'$ will guarantee the convergence of the sums. Notice also that since $R_M(\tau) = O(c^{-2-k-\alpha'}|c\tau + d|^{-2})$ we see that the convergence of $\sum_{M \in \Gamma_0 \setminus \Gamma} R_M(\tau)$ is also guaranteed by the condition that $k > \alpha'$. Now as in [5], if we define

$$T = \sum_{c>0} \lim_{N \rightarrow \infty} \sum_{|d| \leq N} \frac{(\bar{v}(M))^{-1} e^{-2\pi i \mu_0 a/c}}{c^{1+k}(c\tau + d)} \quad \text{and} \quad T_N(L) = \sum_{A_N L} \frac{(\bar{v}(M))^{-1} e^{-2\pi i \mu_0 a/c}}{c^{1+k}(c\tau + d)}$$

then similarly as in [5, pp. 381-382] we see that $\lim_{N \rightarrow \infty} T_N(L) = T$.

Theorem 1. *For $k > \alpha'$, $F_\mu(\tau)$ is an automorphic integral with principal part $e^{2\pi i(\mu+\alpha)\tau/\lambda}$ at ∞ , and its associated cusp form is G_ν .*

Notice that Lemma 2 shows that $F_\mu(\tau)$ is holomorphic for $\tau \in \mathbb{H}$ and it has principal part $e^{2\pi i(\mu+\alpha)\tau/\lambda}$ at ∞ . Also, Lemma 3 proves that

$$F_\mu(\tau) = 1/2a_0(\mu + \alpha) + \lim_{N \rightarrow \infty} \sum_{M \in A_N L^{-1}} s_M(\tau) + t_M(\tau).$$

Notice also that the functional equation of $s_M(\tau)$ and $t_M(\tau)$ implies

$$\begin{aligned} (\bar{v}(L))^{-1}(\gamma\tau + \delta)^k F_\mu(L\tau) &= 1/2(\bar{v}(L))^{-1}(\gamma\tau + \delta)^k a_0(\mu + \alpha) \\ &+ \lim_{N \rightarrow \infty} \sum_{M \in A_N I} s_M(\tau) + t_M(\tau) - p(\tau, L^{-1}\infty, g_M). \end{aligned}$$

As we mentioned before, Corollary 4.5 in [5, p.384] shows that the right hand of the above equation is

$$1/2a_0(\mu + \alpha) + \lim_{N \rightarrow \infty} \sum_{M \in A_N I} s_M(\tau) + t_M(\tau) - p(\tau, L^{-1}\infty, G_\nu).$$

Hence, Lemma 3 implies that F_μ satisfies the functional equation of an automorphic integral with associated cusp form G_ν .

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