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HYPERGEOMETRIC TRANSFORMATIONS OF LINEAR FORMS IN ONE LOGARITHM

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On the 70th birthday of Władysław Narkiewicz

Abstract: We discuss hypergeometric constructions of rational approximations to values of the logarithm function.
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1. Introduction

Until now, the best approximations to values of the logarithm function at some rational or algebraic points are constructed by means of hypergeometric integrals (or hypergeometric series). An essential ingredient to improve the quality of such approximations consists in an arithmetic technique based on a careful p-adic analysis of the approximants. Perhaps the simplest way to do this is to provide relations showing that the quotient of two approximations is related to a quotient of factorials, for which the p-adic study is an easy routine. Such 'structural' approximations gave rise to many important applications, and can usually be expressed through the action of a certain finite group on a family of integrals (or series). In this paper we discuss an interesting structure of rational approximations to logarithm values which does not allow us to obtain improvements on the known irrationality results or irrationality measures, but which leads to several remarks and questions that might be useful in further study of the arithmetic problem.

2. Hypergeometric database

We shall require the following hypergeometric identities: the Euler–Pochhammer integral [4, p. 20, (1.6.6)]

$$ {}_2F_1\left(\begin{matrix} A,\, B \\ C \end{matrix}\,\middle|\, z\right) = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 \frac{t^{B-1}(1-t)^{C-B-1}}{(1-zt)^A}\,\mathrm{d}t, \tag{2.1} $$

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provided Re$C >$ Re$B > 0$; the Euler transformation [4, p. 31, (1.7.1.3)]

$$ {}_2F_1\left(\begin{matrix} A,\ B \\ C \end{matrix} \,\middle|\, z\right) = (1-z)^{-B} \cdot {}_2F_1\left(\begin{matrix} C-A,\ B \\ C \end{matrix} \,\middle|\, \frac{-z}{1-z}\right); \tag{2.2} $$

its iterate — the Kummer transformation

$$ {}_2F_1\left(\begin{matrix} A,\ B \\ C \end{matrix} \,\middle|\, z\right) = (1-z)^{C-A-B} \cdot {}_2F_1\left(\begin{matrix} C-B,\ C-A \\ C \end{matrix} \,\middle|\, z\right); \tag{2.3} $$

the Gauss quadratic transformation [1, p. 88, (2)]

$$ {}_2F_1\left(\begin{matrix} A,\ B \\ A+B+\frac{1}{2} \end{matrix} \,\middle|\, 4z(1-z)\right) = {}_2F_1\left(\begin{matrix} 2A,\ 2B \\ A+B+\frac{1}{2} \end{matrix} \,\middle|\, z\right), \tag{2.4} $$

and the quadratic transformation [4, p. 80, (2.5.33)]

$$ {}_2F_1\left(\begin{matrix} A,\ B \\ A+B+\frac{1}{2} \end{matrix} \,\middle|\, -\frac{z^2}{4(1-z)}\right) = (1-z)^{A} \cdot {}_2F_1\left(\begin{matrix} 2A,\ A+B \\ 2A+2B \end{matrix} \,\middle|\, z\right). \tag{2.5} $$

3. A hypergeometric series

Throughout the paper d_n stands for the least common multiple of the numbers $1, 2, \dots, n$; we also set $d_0 = 1$.

Let a, b, c be integers satisfying $b \ge 0$ and $c \ge 0$, and let λ be an algebraic number, $|\lambda| > 1$ and $\lambda \in K$. Our examples include the fields $K = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{D})$, where $D > 1$ is a square-free even integer; the corresponding rings of integers are $\mathbb{Z}_K = \mathbb{Z}$ and $\mathbb{Z}_K = \mathbb{Z}[\sqrt{D}]$, respectively.

Consider the hypergeometric series

$$ \begin{aligned} H = H(a,b,c;\lambda) &= \frac{1}{\lambda^{2c+2}} \frac{\Gamma\big(a+\frac{1}{2}\big)\Gamma(b+1)}{\Gamma\big(a+b+\frac{3}{2}\big)} \cdot {}_2F_1\left(\begin{matrix} c+1,\ a+\frac{1}{2} \\ a+b+\frac{3}{2} \end{matrix} \,\middle|\, \frac{1}{\lambda^2}\right) \\ &= \sum_{l=0}^{\infty} R(l)\lambda^{-2(l+c+1)} = \sum_{l=-c}^{\infty} R(l)\lambda^{-2(l+c+1)}, \end{aligned} \tag{3.1} $$

where

$$ R(l) = R(a,b,c;l) = \frac{\prod_{j=1}^{c}(l+j)}{c!} \cdot \frac{b!}{\prod_{j=a}^{a+b}(l+j+\frac{1}{2})}. \tag{3.2} $$

The partial fraction decomposition of the latter rational function gives us

$$ R(l) = \sum_{j=a}^{a+b} \frac{A_j}{l+j+\frac{1}{2}} + P(l), \tag{3.3} $$

where

$$ A_j = \big(R(l)(l+j+\tfrac{1}{2})\big)\big|_{l=-j-1/2} \in 2^{-2c}\mathbb{Z} \tag{3.4} $$

and $2^{2c'}d_{2c'}P(l)$ is an integer-valued polynomial of degree at most $c'-1$ with $c'=\max\{c-b,0\}$ (cf. [6, Lemma 1]). Writing

$$P(l)=B_0+\sum_{k=1}^{c'-1}B_k\frac{\prod_{j=0}^{k-1}(l+c-j)}{k!}, \qquad 2^{2c'}d_{2c'}B_k\in\mathbb{Z} \tag{3.5}$$

(this corresponds to the expansion in a basis of integer-valued polynomials), we obtain

$$\begin{aligned}
H&=\sum_{l=-c}^{\infty}\lambda^{-2(l+c+1)}\biggl(\sum_{j=a}^{a+b}\frac{A_j}{l+j+\frac12}+\sum_{k=0}^{c'-1}B_k\frac{\prod_{j=0}^{k-1}(l+c-j)}{k!}\biggr)\\
&=\sum_{j=a}^{a+b}A_j\lambda^{-2(c-j)}\sum_{l=-c}^{\infty}\frac{\lambda^{-2(l+j+1)}}{l+j+\frac12}\\
&\qquad+\sum_{k=0}^{c'-1}B_k\lambda^{-2(k+1)}\sum_{l=-c}^{\infty}\frac{\prod_{j=0}^{k-1}(l+c-j)}{k!}\lambda^{-2(l+c-k)}\\
&=\sum_{j=a}^{a+b}A_j\lambda^{-2(c-j)}\sum_{m=j-c}^{\infty}\frac{\lambda^{-2(m+1)}}{m+\frac12}+\sum_{k=0}^{c'-1}B_k\lambda^{-2(k+1)}\sum_{m=0}^{\infty}\frac{\prod_{j=0}^{k-1}(m-j)}{k!}\lambda^{-2(m-k)}\\
&=\sum_{j=a}^{a+b}A_j\lambda^{-2(c-j)}\biggl(\sum_{m=0}^{\infty}-\sum_{m=0}^{j-c-1}\biggr)\frac{\lambda^{-2(m+1)}}{m+\frac12}\\
&\qquad+\sum_{k=0}^{c'-1}B_k\lambda^{-2(k+1)}\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z^k}\sum_{m=0}^{\infty}z^m\bigg|_{z=\lambda^{-2}}
\end{aligned}$$

(the sum $-\sum_{m=0}^{j-c-1}$ should be interpreted as $\sum_{m=j-c}^{-1}$ if $j-c\le-1$)

$$\begin{aligned}
&=\sum_{j=a}^{a+b}A_j\lambda^{-2(c-j)}\sum_{m=0}^{\infty}\frac{\lambda^{-2(m+1)}}{m+\frac12}-\sum_{j=a}^{a+b}A_j\lambda^{-2(c-j)}\sum_{m=0}^{j-c-1}\frac{\lambda^{-2(m+1)}}{m+\frac12}\\
&\qquad+\sum_{k=0}^{c'-1}B_k\lambda^{-2(k+1)}\cdot\frac{1}{(1-\lambda^{-2})^{k+1}}\\
&=\sum_{j=a}^{a+b}A_j\lambda^{-2c+2j-1}\cdot\log\frac{\lambda+1}{\lambda-1}-\sum_{j=a}^{a+b}A_j\lambda^{-2(c-j)}\sum_{m=0}^{j-c-1}\frac{\lambda^{-2(m+1)}}{m+\frac12}\\
&\qquad+\sum_{k=0}^{c'-1}\frac{B_k}{(\lambda^2-1)^{k+1}}.
\end{aligned}$$

Denote by λ_0 and λ_1 the denominators of the algebraic numbers λ^{-1} and $(\lambda^2-1)^{-1}$, respectively; both λ_0 and λ_1 are positive integers. Then

$$2^{2c}\lambda_0^{2c-2a+1}\cdot\sum_{j=a}^{a+b}A_j\lambda^{-2c+2j-1}\in\mathbb{Z}_K,$$

$$2^{2c+1}\lambda_0^{2c-2a+2}d_{2\max\{c-a,a+b-c\}}\cdot\sum_{j=a}^{a+b}A_j\lambda^{-2(c-j)}\sum_{m=0}^{j-c-1}\frac{\lambda^{-2(m+1)}}{m+\frac12}\in\mathbb{Z}_K,$$

$$2^{2c'}\lambda_1^{c'}d_{2c'}\cdot\sum_{k=0}^{c'-1}\frac{B_k}{(\lambda^2-1)^{k+1}}\in\mathbb{Z}_K,$$

where we use (3.4) and (3.5). Combining these inclusions we finally obtain

$$2^{2c+1}\lambda_0^{2c-2a+2}\lambda_1^{c'}d_{2\max\{c-a,c-b,a+b-c\}}\cdot H(a,b,c;\lambda)\in\mathbb{Z}_K\log\frac{\lambda+1}{\lambda-1}+\mathbb{Z}_K. \quad (3.6)$$

4. A certain integral

In fact, the choice of the 'denominator' of the linear form $H(a,b,c;\lambda)$ in (3.6) is not optimal. But it will suffice for our practical purposes, since we will consider just special cases of it when the linear form may be represented by a more familiar integral [2], [5] with already known sharp denominator.

A method of providing sharp denominators of the form (3.1) is given in the recent work [3] of E. Sal'nikova (who in turn attributes this idea to V. Salikhov), but it works only if all the three parameters a, b, and c are non-negative integers (recall that we do not pose the condition $a\ge 0$ in Section 3). Using the Euler–Pochhammer integral (2.1) in this case for the series in (3.1) and then performing the change of variable $x^2=t$ we find that

$$\begin{aligned} H(a,b,c;\lambda)&=\frac{1}{\lambda^{2c+2}}\int_0^1\frac{t^{a-1/2}(1-t)^b}{(1-t/\lambda^2)^{c+1}}\,\mathrm{d}t=\int_0^1\frac{t^{a-1/2}(1-t)^b}{(\lambda^2-t)^{c+1}}\,\mathrm{d}t \\ &=2\int_0^1\frac{x^{2a}(1-x)^b(1+x)^b}{(\lambda-x)^{c+1}(\lambda+x)^{c+1}}\,\mathrm{d}x. \end{aligned} \quad (4.1)$$

The integrand (admitting the symmetry $x\mapsto -x$) can be decomposed to derive the arithmetic properties of the integral H in a way different from our approach in Section 3. But, again, this method works when $a\ge 0$ while we will require the arithmetic in the case $a<0$ as well.

An advantage of the series representation of $H(a,b,c;\lambda)$ consists in the fact that we can easily use hypergeometric transformations in order to sharpen additionally the arithmetic of the linear form. For example, applying the Kummer transformation (2.3) we arrive at

$$H = \frac{1}{\lambda^{2c+2}}\left(1-\frac{1}{\lambda^2}\right)^{b-c} \frac{\Gamma\big(a+\frac12\big)\Gamma(b+1)}{\Gamma\big(a+b+\frac32\big)} \cdot {}_2F_1\left(\begin{matrix} b+1,\ a+b-c+\frac12 \\ a+b+\frac32 \end{matrix}\ \middle|\ \frac{1}{\lambda^2}\right)$$

$$= (\lambda^2-1)^{b-c}\,\frac{\Gamma\big(a+\frac12\big)\Gamma(b+1)}{\Gamma\big(a+b-c+\frac12\big)\Gamma(c+1)}\,H(a+b-c,c,b;\lambda) \tag{4.2}$$

while using the formulae, for a positive integer m,

$$\frac{\Gamma(m+\frac12)}{\Gamma(\frac12)} = (\tfrac12)_m = 2^{-2m}\frac{(2m)!}{m!}, \qquad \frac{\Gamma(-m+\frac12)}{\Gamma(\frac12)} = (-1)^m 2^{2m}\frac{m!}{(2m)!}, \tag{4.3}$$

we can write the quotient of the gamma factors as a quotient of factorials. In the case $a \ge 0$ and $a+b-c \ge 0$ we obtain

$$H(a,b,c;\lambda) = \big(4(\lambda^2-1)\big)^{b-c}\frac{(2a)!\,(a+b-c)!\,b!}{a!\,(2a+2b-2c)!\,c!}H(a+b-c,c,b;\lambda), \tag{4.4}$$

which is exactly the transformation used in [3] to derive nice irrationality measures for some logarithms.

5. A special case

We now switch to a particular case of the above construction, when $c = a$:

$$I = I(a,b;\lambda) = H(a,b,a;\lambda) \tag{5.1}$$

$$= \frac{1}{\lambda^{2a+2}}\frac{\Gamma\big(a+\frac12\big)\Gamma(b+1)}{\Gamma\big(a+b+\frac32\big)} \cdot {}_2F_1\left(\begin{matrix} a+1,\ a+\frac12 \\ a+b+\frac32 \end{matrix}\ \middle|\ \frac{1}{\lambda^2}\right),$$

where both a and b are non-negative integers. The transformation rule (4.4) applied to (5.1) yields

$$I(a,b;\lambda) = \big(4(\lambda^2-1)\big)^{b-a}\frac{(2a)!\,b!^2}{a!^2(2b)!}I(b,a;\lambda). \tag{5.2}$$

Remark 1. The condition posed on the parameters looks quite restrictive. But it happens that it becomes an actual condition in applications to irrationality measures for the numbers $\log(1-1/\mu)$ with $\mu > 1$ an integer (cf. [3]). In particular, the choice of the parameters $a = c = 4n$ and $b = 3n$, where n is an increasing integer parameter, yields the best known estimate for the irrationality measure of $\log 2$ due to E. Rukhadze [2].

Applying to the series in (5.1) the Euler transformation (2.2) we find that

$$I = \frac{1}{\lambda^{2a+2}}\Bigl(1-\frac{1}{\lambda^2}\Bigr)^{-(a+1/2)} \frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)} \cdot {}_2F_1\biggl(\begin{matrix} b+\frac12,\ a+\frac12 \\ a+b+\frac32 \end{matrix}\biggm| -\frac{1}{\lambda^2-1}\biggr) \tag{5.3}$$

$$= \frac{1}{\lambda(\lambda^2-1)^a\sqrt{\lambda^2-1}} \frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)} \cdot {}_2F_1\biggl(\begin{matrix} b+\frac12,\ a+\frac12 \\ a+b+\frac32 \end{matrix}\biggm| -\frac{1}{\lambda^2-1}\biggr).$$

We now choose

$$z = \frac12 - \frac{\lambda}{2\sqrt{\lambda^2-1}} = -\frac{\lambda-\sqrt{\lambda^2-1}}{2\sqrt{\lambda^2-1}}, \tag{5.4}$$

in order to satisfy the relation

$$4z(1-z) = -\frac{1}{\lambda^2-1}\,,$$

and apply the Gauss quadratic transformation (2.4) to (5.3):

$$I = \frac{1}{\lambda(\lambda^2-1)^a\sqrt{\lambda^2-1}} \frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)} \cdot {}_2F_1\biggl(\begin{matrix} 2b+1,\ 2a+1 \\ a+b+\frac32 \end{matrix}\biggm| -\frac{\lambda-\sqrt{\lambda^2-1}}{2\sqrt{\lambda^2-1}}\biggr). \tag{5.5}$$

Interchanging the parameters $2b+1$ and $2a+1$ and applying the Euler transformation (2.2) we find that

$$\begin{aligned} I &= \frac{1}{\lambda(\lambda^2-1)^a\sqrt{\lambda^2-1}} \biggl(\frac{\lambda+\sqrt{\lambda^2-1}}{2\sqrt{\lambda^2-1}}\biggr)^{-(2b+1)} \frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)} \\ &\qquad \times {}_2F_1\biggl(\begin{matrix} b-a+\frac12,\ 2b+1 \\ a+b+\frac32 \end{matrix}\biggm| \frac{\lambda-\sqrt{\lambda^2-1}}{\lambda+\sqrt{\lambda^2-1}}\biggr) \\ &= \frac{2^{2b+1}(\lambda-\sqrt{\lambda^2-1})^{2b+1}}{\lambda(\lambda^2-1)^{a-b}} \frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)} \\ &\qquad \times {}_2F_1\biggl(\begin{matrix} b-a+\frac12,\ 2b+1 \\ a+b+\frac32 \end{matrix}\biggm| \frac{1}{(\lambda+\sqrt{\lambda^2-1})^2}\biggr) \end{aligned} \tag{5.6}$$

and, again, interchanging the parameters $b-a+\frac12$ and $2b+1$ and recalling the definition (3.1) we obtain

$$I = \frac{2^{2b+1}(\lambda+\sqrt{\lambda^2-1})^{2b+1}}{\lambda(\lambda^2-1)^{a-b}} \frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(b-a+\frac12\bigr)\Gamma(2a+1)} \cdot H(b-a, 2a, 2b; \lambda+\sqrt{\lambda^2-1}). \tag{5.7}$$

Since a and b enter the construction symmetrically (see also formula (5.9) below) we may assume, without loss of generality, that $a \ge b$ and use the formulae in (4.3) to write the latter transformation as follows:

$$I(a,b;\lambda) = \frac{2^{4b-4a+1}(\lambda+\sqrt{\lambda^2-1})^{2b+1}}{\lambda(\lambda^2-1)^{a-b}} \frac{(2a-2b)!\,b!}{a!\,(a-b)!} \cdot H(b-a, 2a, 2b; \lambda+\sqrt{\lambda^2-1}). \tag{5.8}$$

If we proceed in the same way with the series in (5.5) without interchanging the parameters $2b+1$ and $2a+1$, we get

$$\begin{aligned} I &= \frac{2^{2a+1}(\lambda-\sqrt{\lambda^2-1})^{2a+1}}{\lambda}\,\frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)} \\ &\qquad\times {}_2F_1\biggl(\begin{matrix} a-b+\frac12,\ 2a+1 \\ a+b+\frac32 \end{matrix}\biggm| \frac{1}{(\lambda+\sqrt{\lambda^2-1})^2}\biggr) \\ &= \frac{2^{2a+1}(\lambda+\sqrt{\lambda^2-1})^{2a+1}}{\lambda}\,\frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a-b+\frac12\bigr)\Gamma(2b+1)}\cdot H(a-b,2b,2a;\lambda+\sqrt{\lambda^2-1}) \\ &= \frac{2^{2a-2b+1}(\lambda+\sqrt{\lambda^2-1})^{2a+1}}{\lambda}\,\frac{(2a)!\,(a-b)!\,b!}{a!\,(2a-2b)!\,(2b)!}\cdot H(a-b,2b,2a;\lambda+\sqrt{\lambda^2-1}), \end{aligned} \tag{5.9}$$

where we assume $a \ge b$ as above.

6. Relation to the old construction

With the notation (5.1) of Section 5 (in particular, choosing $a=c$ and using (5.3)), we take $\mu=\frac12(\lambda+1)$ with the motive

$$\frac{(1/\mu)^2}{4(1-1/\mu)} = \frac{1}{\lambda^2-1} \tag{6.1}$$

and apply the quadratic transformation (2.5):

$$\begin{aligned} I &= \frac{(1-1/\mu)^{b+1/2}}{\lambda(\lambda^2-1)^{a+1/2}}\cdot\frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)}\cdot{}_2F_1\biggl(\begin{matrix} 2b+1,\ a+b+1 \\ 2a+2b+2 \end{matrix}\biggm|\frac{1}{\mu}\biggr) \\ &= \frac{(\mu-1)^{b-a}}{2^{2a+1}\mu^{a+b+1}(2\mu-1)}\cdot\frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)}\cdot{}_2F_1\biggl(\begin{matrix} 2b+1,\ a+b+1 \\ 2a+2b+2 \end{matrix}\biggm|\frac{1}{\mu}\biggr). \end{aligned} \tag{6.2}$$

Using the integral representation (2.1) we obtain

$$I = \frac{(\mu-1)^{b-a}}{2^{2a+1}\mu^{a-b}(2\mu-1)}\cdot\frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)}\,\frac{\Gamma(2a+2b+2)}{\Gamma(a+b+1)^2}\int_0^1\frac{x^{a+b}(1-x)^{a+b}}{(\mu-x)^{2b+1}}\,\mathrm{d}x. \tag{6.3}$$

If we do the same for the series in (6.2) after interchanging the two parameters $2b+1$ and $a+b+1$ (which does not affect the hypergeometric series) we get

$$I = \frac{(\mu-1)^{b-a}}{2^{2a+1}(2\mu-1)}\cdot\frac{\Gamma\bigl(a+\frac12\bigr)\Gamma(b+1)}{\Gamma\bigl(a+b+\frac32\bigr)}\,\frac{\Gamma(2a+2b+2)}{\Gamma(2a+1)\,\Gamma(2b+1)}\int_0^1\frac{x^{2b}(1-x)^{2a}}{(\mu-x)^{a+b+1}}\,\mathrm{d}x. \tag{6.4}$$

It remains to use (4.3) to deduce

$$\begin{aligned} I(a,b;\lambda) &= \frac{\bigl(4\mu(\mu-1)\bigr)^{b-a}}{2\mu-1}\cdot\frac{b!\,(2a)!}{a!\,(a+b)!}\cdot J(a+b,a+b,2b;\mu) \\ &= \frac{\bigl(4(\mu-1)\bigr)^{b-a}}{2\mu-1}\cdot\frac{b!\,(a+b)!}{a!\,(2b)!}\cdot J(2b,2a,a+b;\mu), \end{aligned} \tag{6.5}$$

where

$$J(A,B,C;\mu)=\int_0^1 \frac{x^A(1-x)^B}{(\mu-x)^C}\,\frac{\mathrm{d}x}{\mu-x}. \tag{6.6}$$

7. Arithmetic of linear forms

From now on we assume that $\mu=\frac{1}{2}(\lambda+1)$ is an integer greater than 1 (hence $\lambda=2\mu-1$ is also an integer greater than 1); the integer parameters a and b are chosen to satisfy $a\ge b>0$.

First of all, the connection of the series considered in Section 5 with the integral (6.6) allows us to use the results of [5], since

$$J(a+b,a+b,2b;\mu)=-\mu^{2a}\cdot\mathcal{I}\Bigl(a+b,a+b,a-b;-\frac{1}{\mu}\Bigr), \tag{7.1}$$

where

$$\mathcal{I}(h,j,l;z)=z^{h+j+1}(1+z)^{\max\{0,-l\}}\int_0^1 \frac{x^h(1-x)^j}{(1+xz)^{j-l}}\,\frac{\mathrm{d}x}{1+xz} \tag{7.2}$$

is the integral studied in [5]. In particular, we derive that

$$d_{2a}\cdot J(a+b,a+b,2b;\mu)\in\mathbb{Z}\log\Bigl(1-\frac{1}{\mu}\Bigr)+\mathbb{Z}=\mathbb{Z}\log\frac{\lambda-1}{\lambda+1}+\mathbb{Z} \tag{7.3}$$

and, similarly,

$$d_{2a}\cdot 2^{a-b}(\lambda-1)^{b-a}J(2b,2a,a+b;\mu)\in\mathbb{Z}\log\frac{\lambda-1}{\lambda+1}+\mathbb{Z}. \tag{7.4}$$

Now write the first formula in (6.5) in the form

$$\lambda(\lambda^2-1)^{a-b}I(a,b;\lambda)=\frac{b!\,(2a)!}{a!\,(a+b)!}\cdot J(a+b,a+b,2b;\mu). \tag{7.5}$$

It follows from (3.6) that

$$d_{2a}\cdot 2^{2a+1}\lambda^2(\lambda^2-1)^{a-b}I(a,b;\lambda)\in\mathbb{Z}\log\frac{\lambda-1}{\lambda+1}+\mathbb{Z}. \tag{7.6}$$

The exponent of 2 in the quotient of factorials in (7.5) is at most $O(\log a)$.[1] Therefore, using the equality in (7.5) and comparing the inclusions (7.6) and (7.3) we conclude that

$$d_{2a}\cdot 2^{O(\log a)}\lambda^2(\lambda^2-1)^{a-b}I(a,b;\lambda)\in\mathbb{Z}\log\frac{\lambda-1}{\lambda+1}+\mathbb{Z}. \tag{7.7}$$

[1] By $O(\log a)$ we mean an *integer* bounded by $C\log a$ for a certain explicit constant $C>0$ depending only on a/b.

In a similar way, applying the inclusion (3.6) together with the relations (5.2) and (7.5) yields

$$d_{2a} \cdot 2^{2(b-a)+O(\log a)} \lambda^2 I(b,a;\lambda) \in \mathbb{Z} \log \frac{\lambda-1}{\lambda+1} + \mathbb{Z}. \tag{7.8}$$

Furthermore, write the transformations (5.8) and (7.5) as follows:

$$\begin{aligned} &2^{4(b-a)+1}(\lambda+\sqrt{\lambda^2-1})^{2b+1} \cdot H(b-a,2a,2b;\lambda+\sqrt{\lambda^2-1}) \\ &= \lambda(\lambda^2-1)^{a-b} \frac{a!\,(a-b)!}{(2a-2b)!\,b!} \cdot I(a,b;\lambda) \\ &= \frac{(a-b)!\,(2a)!}{(a+b)!\,(2a-2b)!} \cdot J(a+b,a+b,2b;\mu). \end{aligned} \tag{7.9}$$

From (3.6) we get

$$d_{2(a+b)} \cdot 2^{4b+1} H(b-a,2a,2b;\lambda+\sqrt{\lambda^2-1}) \in \mathbb{Z}_K \log \frac{\lambda-1+\sqrt{\lambda^2-1}}{\lambda+1+\sqrt{\lambda^2-1}} + \mathbb{Z}_K, \tag{7.10}$$

where $K = \mathbb{Q}(\sqrt{\lambda^2-1})$. Noting that

$$\log \frac{\lambda-1+\sqrt{\lambda^2-1}}{\lambda+1+\sqrt{\lambda^2-1}} = \frac{1}{2} \log \frac{\lambda-1}{\lambda+1}$$

we derive from (7.10) that

$$d_{2(a+b)} \cdot 2^{4b+2}(\lambda+\sqrt{\lambda^2-1})^{2b+1} \cdot H(b-a,2a,2b;\lambda+\sqrt{\lambda^2-1}) \in \mathbb{Z}_K \log \frac{\lambda-1}{\lambda+1} + \mathbb{Z}_K. \tag{7.11}$$

In fact, according to (7.9), the expression in (7.11) lies in

$$\mathbb{Q} \log \frac{\lambda-1}{\lambda+1} + \mathbb{Q}. \tag{7.12}$$

Hence we conclude that

$$d_{2(a+b)} \cdot 2^{4b+2}(\lambda+\sqrt{\lambda^2-1})^{2b+1} \cdot H(b-a,2a,2b;\lambda+\sqrt{\lambda^2-1}) \in \mathbb{Z} \log \frac{\lambda-1}{\lambda+1} + \mathbb{Z}. \tag{7.13}$$

For the latter expression in (7.9) the primes $p > 2a$ do not occur in the coefficients of the linear form (viewed as an element of (7.12)) owing to (7.3). In addition, we may compare the 2-adic order of the expressions in (7.9). This finally implies a sharpened version of the inclusion (7.13), namely,

$$\begin{aligned} &d_{2a} \cdot 2^{4(b-a)+1+O(\log a)}(\lambda+\sqrt{\lambda^2-1})^{2b+1} \\ &\qquad \times H(b-a,2a,2b;\lambda+\sqrt{\lambda^2-1}) \in \mathbb{Z} \log \frac{\lambda-1}{\lambda+1} + \mathbb{Z}. \end{aligned} \tag{7.14}$$

In the same vein, using (5.9) and (7.9) we obtain

$$d_{2a} \cdot 2^{4(a-b)+1+O(\log a)}(\lambda^2-1)^{a-b}(\lambda+\sqrt{\lambda^2-1})^{2a+1} \times H(a-b,2b,2a;\lambda+\sqrt{\lambda^2-1}) \in \mathbb{Z}\log\frac{\lambda-1}{\lambda+1}+\mathbb{Z}. \tag{7.15}$$

We now normalize our objects:

$$\begin{aligned}
F_0(a,b) &= d_{2a} \cdot 2^{O(\log a)}\lambda^2(\lambda^2-1)^{a-b} \cdot I(a,b;\lambda),\\
F_1(a,b) &= d_{2a} \cdot 2^{2(b-a)+O(\log a)}\lambda^2 \cdot I(b,a;\lambda),\\
F_2(a,b) &= d_{2a} \cdot 2^{O(\log a)}\lambda \cdot J(a+b,a+b,2b;\mu),\\
F_3(a,b) &= d_{2a} \cdot 2^{a-b+O(\log a)}\lambda(\lambda-1)^{b-a} \cdot J(2b,2a,a+b;\mu),\\
F_4(a,b) &= d_{2a} \cdot 2^{4(b-a)+1+O(\log a)}\lambda(\lambda+\sqrt{\lambda^2-1})^{2b+1}\\
&\qquad \times H(b-a,2a,2b;\lambda+\sqrt{\lambda^2-1}),\\
F_5(a,b) &= d_{2a} \cdot 2^{4(a-b)+1+O(\log a)}(\lambda^2-1)^{a-b}\lambda(\lambda+\sqrt{\lambda^2-1})^{2a+1}\\
&\qquad \times H(a-b,2b,2a;\lambda+\sqrt{\lambda^2-1}),
\end{aligned}$$

where we choose the same (maximal) integer in $O(\log a)$. From (7.7), (7.8), (7.3), (7.4), (7.14), and (7.15) it follows that

$$F_j(a,b) \in \mathbb{Z}\log\frac{\lambda-1}{\lambda+1}+\mathbb{Z}, \qquad j=0,1,\dots,5. \tag{7.16}$$

Remark 2. We can add two more objects to the list, since from the final remark in [5] we have two more related integrals:

$$J(a+b,a+b,2a;\mu) = \big(\mu(\mu-1)\big)^{b-a} \cdot J(a+b,a+b,2b;\mu), \tag{7.17}$$

$$J(2a,2b,a+b;\mu) = \mu^{a-b}(\mu-1)^{b-a} \cdot J(2b,2a,a+b;\mu). \tag{7.18}$$

But these transformations do not provide new quotients of factorials.

Moreover, from (5.2), (6.5), (5.8), and (5.9) we obtain

$$\begin{aligned}
F_0(a,b) &= \frac{(2a)!\,b!^2}{a!^2(2b)!} \cdot F_1(a,b)\\
&= \frac{b!\,(2a)!}{a!\,(a+b)!} \cdot F_2(a,b)\\
&= \frac{b!\,(a+b)!}{a!\,(2b)!} \cdot F_3(a,b)\\
&= \frac{(2a-2b)!\,b!}{a!\,(a-b)!} \cdot F_4(a,b)\\
&= \frac{(2a)!\,(a-b)!\,b!}{a!\,(2a-2b)!\,(2b)!} \cdot F_5(a,b).
\end{aligned} \tag{7.19}$$

These transformations combined with the inclusions in (7.16) imply that

$$\Phi^{-1} \cdot F_0(a,b) \in \mathbb{Z} \log \frac{\lambda-1}{\lambda+1} + \mathbb{Z}, \tag{7.20}$$

where

$$\Phi = \Phi(a,b) = \prod_{p>\sqrt{2a}} p^{\nu_p},$$

$$\begin{aligned} \nu_p = \max\biggl\{0,\ &\operatorname{ord}_p \frac{(2a)!\, b!^2}{a!^2 (2b)!},\ \operatorname{ord}_p \frac{b!\,(2a)!}{a!\,(a+b)!},\ \operatorname{ord}_p \frac{b!\,(a+b)!}{a!\,(2b)!}, \\ &\operatorname{ord}_p \frac{(2a-2b)!\, b!}{a!\,(a-b)!},\ \operatorname{ord}_p \frac{(2a)!\,(a-b)!\, b!}{a!\,(2a-2b)!\,(2b)!}\biggr\}. \end{aligned} \tag{7.21}$$

An easy analysis shows that, for a prime $p > \sqrt{2a}$, the maximum in (7.21) in fact equals

$$\begin{aligned} &\max\biggl\{0,\ \operatorname{ord}_p \frac{(2a)!\, b!^2}{a!^2 (2b)!}\biggr\} \\ &\quad = \max\biggl\{0,\ \operatorname{ord}_p \frac{(2a)!\, b!^2}{a!^2 (2b)!},\ \operatorname{ord}_p \frac{b!\,(2a)!}{a!\,(a+b)!},\ \operatorname{ord}_p \frac{b!\,(a+b)!}{a!\,(2b)!}\biggr\} \\ &\quad = \max\biggl\{0,\ \operatorname{ord}_p \frac{(2a)!\, b!^2}{a!^2 (2b)!},\ \operatorname{ord}_p \frac{(2a-2b)!\, b!}{a!\,(a-b)!},\ \operatorname{ord}_p \frac{(2a)!\,(a-b)!\, b!}{a!\,(2a-2b)!\,(2b)!}\biggr\}. \end{aligned}$$

This happens because of the equalities

$$\frac{(2a)!\, b!^2}{a!^2 (2b)!} = \frac{b!\,(2a)!}{a!\,(a+b)!} \cdot \frac{b!\,(a+b)!}{a!\,(2b)!} = \frac{(2a-2b)!\, b!}{a!\,(a-b)!} \cdot \frac{(2a)!\,(a-b)!\, b!}{a!\,(2a-2b)!\,(2b)!} \tag{7.22}$$

and the fact that the 'p-adic distance' between the pairs

$$\frac{b!\,(2a)!}{a!\,(a+b)!},\ \frac{b!\,(a+b)!}{a!\,(2b)!} \quad \text{and} \quad \frac{(2a-2b)!\, b!}{a!\,(a-b)!},\ \frac{(2a)!\,(a-b)!\, b!}{a!\,(2a-2b)!\,(2b)!}$$

is at most one:

$$\begin{aligned} &\left| \left(\left\lfloor \frac{b}{p} \right\rfloor + \left\lfloor \frac{2a}{p} \right\rfloor - \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{a+b}{p} \right\rfloor \right) - \left(\left\lfloor \frac{b}{p} \right\rfloor + \left\lfloor \frac{a+b}{p} \right\rfloor - \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{2b}{p} \right\rfloor \right) \right| \le 1, \\ &\left| \left(\left\lfloor \frac{2a-2b}{p} \right\rfloor + \left\lfloor \frac{b}{p} \right\rfloor - \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{a-b}{p} \right\rfloor \right) \right. \\ &\qquad \left. - \left(\left\lfloor \frac{2a}{p} \right\rfloor + \left\lfloor \frac{a-b}{p} \right\rfloor + \left\lfloor \frac{b}{p} \right\rfloor - \left\lfloor \frac{a}{p} \right\rfloor - \left\lfloor \frac{2a-2b}{p} \right\rfloor - \left\lfloor \frac{2b}{p} \right\rfloor \right) \right| \le 1. \end{aligned}$$

This shows why all the required arithmetic information is extracted just from the first transformation in (7.19). It seems quite mysterious to us that the rich transformation structure for linear forms does not provide a better arithmetic.

Question 1. As already mentioned in Remark 1, the special choice of the parameters we deal with is quite natural in several applications. A point which seems rather unnatural to us is the different origin of the objects we use, and a lack of a clear algebraic structure of the quotients of factorials involved in transformations (7.19) (where we may also add two more objects, see Remark 2). It would be nice to 'unify' the different approximations (for instance, to have a sole family of integrals or series) and to settle the quotients of factorials into a more 'algebraic' context (as is done for instance in [5], but also in the case of dilogarithm values, of $\zeta(2)$ and $\zeta(3)$). One might also hope to get from such a unifying approach the 'correct' arithmetic of the approximations (as we noted, the inclusions (3.6) can significantly be improved).

Question 2. We are surprised to see that under a specific choice of the parameters in (3.1) the corresponding linear approximations may have rational coefficients even if λ is a quadratic irrationality (see (7.14) and (7.15)). What is a general setting for this curious phenomenon?

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