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POLYNOMIAL PARAMETRIZATION OF THE SOLUTIONS OF DIOPHANTINE EQUATIONS OF GENUS 0

Sophie Frisch, Günter Lettl

Dedicated to Prof. Władysław Narkiewicz
on the occasion of his 70$^{\text{th}}$ birthday

Abstract: Let $f \in \mathbb{Z}[X, Y, Z]$ be a non-constant, absolutely irreducible, homogeneous polynomial with integer coefficients, such that the projective curve given by $f = 0$ has a function field isomorphic to the rational function field $\mathbb{Q}(T)$. We show that all integral solutions of the Diophantine equation $f = 0$ (up to those corresponding to some singular points) can be parametrized by a single triple of integer-valued polynomials. In general, it is not possible to parametrize this set of solutions by a single triple of polynomials with integer coefficients.
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Recently, the first author and L. Vaserstein proved that the set of all Pythagorean triples can be parametrized by a single triple of integer-valued polynomials, but not by a single triple of polynomials with integer coefficients (in any number of variables) [2]. We denote by $\mathrm{Int}\,(\mathbb{Z}^m)$ the ring of integer-valued polynomials in m variables,

$$\mathrm{Int}\,(\mathbb{Z}^m) = \{\varphi \in \mathbb{Q}[X_1, \ldots, X_m] \mid \varphi(\mathbb{Z}^m) \subset \mathbb{Z}\}\,.$$

In this paper we will generalize the affirmative part of [2] to such homogeneous equations as define a (plane) projective curve with a rational function field.

Throughout this paper, $f \in \mathbb{Z}[X, Y, Z] \setminus \{0\}$ denotes an irreducible polynomial with integer coefficients, which is homogeneous of degree $n \geq 1$. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and $C_f \subset \mathbb{P}^2(\overline{\mathbb{Q}})$ the plane projective curve defined by $f = 0$,

$$C_f = \left\{(x : y : z) \in \mathbb{P}^2(\overline{\mathbb{Q}}) \mid f(x, y, z) = 0\right\}\,.$$

We will further suppose that the function field $K = \mathbb{Q}(C_f)$ of C_f over $\mathbb{Q}$ is isomorphic to the rational function field $\mathbb{Q}(T)$. This implies that f is absolutely irreducible (i.e., irreducible in $\overline{\mathbb{Q}}[X, Y, Z]$). Our assumption is satisfied, for instance, if C_f has genus 0 and possesses a regular point defined over $\mathbb{Q}$.

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Recall that a point $(x : y : z) \in C_f$ is singular if and only if the local ring $R_{(x:y:z)} \subset K$ of all rational functions of C_f that are defined at $(x : y : z)$ is not a discrete valuation ring (cf. [3, pp. 56-57]). In this case, there are finitely many discrete valuation rings $\mathcal{O}_{P_i} \subset K$ above $R_{(x:y:z)}$ (meaning $R_{(x:y:z)} \subset \mathcal{O}_{P_i}$ and $\mathfrak{m}_{(x:y:z)} \subset P_i$, where $\mathfrak{m}_{(x:y:z)}$ and P_i denote the corresponding maximal ideals). Let C_f^{bad} denote the set of those singular points $(x : y : z) \in C_f$ for which there exists no discrete valuation ring $\mathcal{O}_P$ above $R_{(x:y:z)}$ with $\mathcal{O}_P/P \simeq \mathbb{Q}$. These points will be "bad" for our main theorem.

We investigate the set of integer solutions of the Diophantine equation $f(X, Y, Z) = 0$,

$$\mathcal{L}_f := \left\{(x, y, z) \in \mathbb{Z}^3 \mid f(x, y, z) = 0\right\},$$

up to those solutions which correspond to the "bad" points of the curve. We set

$$\mathcal{L}_f^{\mathrm{bad}} = \{(x, y, z) \in \mathcal{L}_f \mid (x : y : z) \in C_f^{\mathrm{bad}}\}.$$

Theorem 1. *Let $f \in \mathbb{Z}[X, Y, Z] \setminus \{0\}$ be an irreducible, homogeneous polynomial of degree $n \geq 1$ such that the function field $K = \mathbb{Q}(C_f)$ is isomorphic to $\mathbb{Q}(T)$. Then there exist polynomials $g_1, g_2, g_3 \in Int(\mathbb{Z}^m)$ for some $m \in \mathbb{N}$ such that*

$$\mathcal{L}_f \setminus \mathcal{L}_f^{bad} = \left\{\left(g_1(\underline{x}), g_2(\underline{x}), g_3(\underline{x})\right) \;\middle|\; \underline{x} \in \mathbb{Z}^m\right\};$$

in other words, up to the "bad" solutions, all solutions of the Diophantine equation

$$f(X, Y, Z) = 0 \tag{1}$$

can be parametrized by one triple of integer-valued polynomials.

The suppositions of Theorem 1 imply that for $n \leq 2$ the curve C_f has no singular point. For $n = 1$, C_f is just a line and the result of Theorem 1 is obvious (even with $g_i \in \mathbb{Z}[U, V]$). For $n = 2$, we immediately obtain

Corollary 2. *Let $f \in \mathbb{Z}[X, Y, Z]$ be an absolutely irreducible quadratic form. Then there exist polynomials $g_1, g_2, g_3 \in Int(\mathbb{Z}^m)$ for some $m \in \mathbb{N}$ such that*

$$\mathcal{L}_f = \left\{\left(g_1(\underline{x}), g_2(\underline{x}), g_3(\underline{x})\right) \;\middle|\; \underline{x} \in \mathbb{Z}^m\right\}.$$

For the proof of Theorem 1 we will use the resultant of polynomials and therefore recall some well-known results on it (cf. [5, Chap. I, §9-10]).
Given polynomials $g, h \in \mathbb{Z}[U, V]$ in the variables U, V, let $\mathrm{Res}_V(g, h) \in \mathbb{Z}[U]$ denote the resultant of g, h when considered as polynomials in the variable V over the ring $\mathbb{Z}[U]$, and, vice versa, $\mathrm{Res}_U(g, h) \in \mathbb{Z}[V]$ the resultant of g, h as polynomials in U.

Lemma 3. *Let $g, h \in \mathbb{Z}[U, V]$ be relatively prime polynomials.*

a) Then $Res_U(g,h) \neq 0$ *and* $Res_V(g,h) \neq 0$*, and there exist polynomials* $r, s, r', s' \in \mathbb{Z}[U,V]$ *with*

$$gr + hs = Res_U(g,h) \quad and \quad gr' + hs' = Res_V(g,h)\,.$$

b) If g *and* h *are homogeneous of degree* d_1 *and* d_2*, resp., then* $Res_U(g,h)$ *and* $Res_V(g,h)$ *are each homogeneous of degree* d_1d_2*, and consequently*

$$Res_U(g,h) = a\,V^{d_1d_2} \quad and \quad Res_V(g,h) = b\,U^{d_1d_2} \quad with \quad a,b \in \mathbb{Z}\setminus\{0\}\,.$$

We will also use the implication (D)⇒(B) of the main theorem of [1], which for the sake of completeness we state in the following

Proposition 4. *Let* $k \in \mathbb{N}$ *and suppose that* $S \subset \mathbb{Z}^k$ *is the set of integer* k*-tuples in the range of a* k*-tuple of polynomials with rational coefficients, as the variables range through the integers, i.e., there exist* $h_1, \dots, h_k \in \mathbb{Q}[X_1, \dots, X_r]$ *for some* $r \in \mathbb{N}$ *such that*

$$S = \{(h_1(\underline{x}), \dots, h_k(\underline{x})) \mid \underline{x} \in \mathbb{Z}^r\} \cap \mathbb{Z}^k\ .$$

Then S *is parametrizable by a* k*-tuple of integer-valued polynomials, i.e., there exist* $g_1, \dots, g_k \in Int(\mathbb{Z}^m)$ *for some* $m \in \mathbb{N}$ *such that*

$$S = \{(g_1(\underline{x}), \dots, g_k(\underline{x})) \mid \underline{x} \in \mathbb{Z}^m\}\ .$$

Proof of Theorem 1. Let f be as in the statement of the theorem. Then there exist homogeneous polynomials $h_1, h_2, h_3 \in \mathbb{Q}[U,V]$ such that

$$(X,Y,Z) = \Big(h_1(U,V), h_2(U,V), h_3(U,V)\Big)$$

defines a birational (projective) isomorphism between C_f and the projective line. We may assume $h_1, h_2, h_3 \in \mathbb{Z}[U,V]$ and $\gcd(h_1,h_2,h_3) = 1$ (see, for instance, [4, Sect. 2]).

For every $\mathbb{Q}$-rational point $(u:v) \in \mathbb{P}^1(\mathbb{Q})$, $\big(h_1(u,v) : h_2(u,v) : h_3(u,v)\big)$ is the evaluation of the birational isomorphism at this point. This means that $\big(h_1(u,v) : h_2(u,v) : h_3(u,v)\big)$ is a $\mathbb{Q}$-rational point of C_f and its local ring is contained in some discrete valuation ring of K of degree 1. Therefore

$$\mathcal{L}_{\mathbb{Q}} := \Big\{ \big(w\,h_1(u,v), w\,h_2(u,v), w\,h_3(u,v)\big) \;\Big|\; u,v,w \in \mathbb{Q} \Big\} = \\ \Big\{ \big(w\,h_1(u,v), w\,h_2(u,v), w\,h_3(u,v)\big) \;\Big|\; w \in \mathbb{Q},\ u,v \in \mathbb{Z} \text{ with } \gcd(u,v) = 1 \Big\}$$

is exactly the set of all rational solutions of (1) except for those corresponding to points of C_f^{bad}, and $\mathcal{L}_f \setminus \mathcal{L}_f^{\text{bad}} = \mathcal{L}_{\mathbb{Q}} \cap \mathbb{Z}^3$ is just the set of all integral triples of $\mathcal{L}_{\mathbb{Q}}$.

We claim that there exists some $d \in \mathbb{N}$ such that for all $u,v \in \mathbb{Z}$ with $\gcd(u,v) = 1$ it follows that

$$\gcd\big(h_1(u,v),h_2(u,v),h_3(u,v)\big) \;\big|\; d\ .$$

Let $\gcd(h_1,h_2) = t \in \mathbb{Z}[U,V]$ and put $h_i = t\,h_i'$ with $h_i' \in \mathbb{Z}[U,V]$, $i = 1,2$. Since h_1', h_2' are relatively prime, we obtain that $\mathrm{Res}_V(h_1',h_2') = a\,U^\delta$ with some $0 \neq a \in \mathbb{Z}$ and $\delta \geq 0$, and polynomials $\rho_1,\rho_2 \in \mathbb{Z}[U,V]$ with $\rho_1 h_1 + \rho_2 h_2 = atU^\delta$. Since h_1, h_2, h_3 were assumed to be relatively prime, $\gcd(atU^\delta, h_3) = cU^\alpha$ with $c \in \mathbb{Z}$ and $0 \leq \alpha \leq \delta$. Dividing both atU^δ and h_3 by cU^α and applying the same reasoning as above we finally obtain that there are $0 \neq a_1 \in \mathbb{Z}$, $\delta_1 \geq 0$ and polynomials $\varphi_1,\varphi_2,\varphi_3 \in \mathbb{Z}[U,V]$ with

$$\varphi_1 h_1 + \varphi_2 h_2 + \varphi_3 h_3 = a_1 U^{\delta_1}\ . \tag{2}$$

Using Res_U in the same way, we obtain polynomials $\psi_1,\psi_2,\psi_3 \in \mathbb{Z}[U,V]$, $0 \neq a_2 \in \mathbb{Z}$ and $\delta_2 \geq 0$ such that

$$\psi_1 h_1 + \psi_2 h_2 + \psi_3 h_3 = a_2 V^{\delta_2}\ . \tag{3}$$

For any $u,v \in \mathbb{Z}$ with $\gcd(u,v) = 1$, (2) and (3) imply that $\gcd\big(h_1(u,v), h_2(u,v), h_3(u,v)\big)$ divides both $a_1 u^{\delta_1}$ and $a_2 v^{\delta_2}$. It follows that

$$\gcd\big(h_1(u,v),h_2(u,v),h_3(u,v)\big) \;\big|\; \mathrm{lcm}(a_1,a_2) := d\ .$$

So we obtain polynomials $k_i = \frac{1}{d} h_i \in \mathbb{Q}[U,V]$ with rational coefficients such that

$$\mathcal{L}_f \setminus \mathcal{L}_f^{\mathrm{bad}} = \Big\{ \big(w\,k_1(u,v), w\,k_2(u,v), w\,k_3(u,v)\big) \;\Big|\; u,v,w \in \mathbb{Z} \Big\} \cap \mathbb{Z}^3\ .$$

Now we apply Proposition 4, which yields the assertion of Theorem 1. ■

Remarks. If the integers a_1, a_2 appearing in (2) and (3) in the proof of Theorem 1 are both equal to 1, then $k_i = h_i \in \mathbb{Z}[U,V]$ and $\mathcal{L}_f \setminus \mathcal{L}_f^{\mathrm{bad}}$ can actually be parametrized by a triple of polynomials with integral coefficients (compare Example 2 below).

When applying Proposition 4, we have no information about the number m of variables of the integer-valued polynomials g_i appearing in Theorem 1.

Example 1. This example shows that for $n \geq 3$ "bad" singular points may appear. Consider

$$f = X^3 + Y^3 + X^2 Z - 2Y^2 Z \in \mathbb{Z}[X,Y,Z]\,.$$

Then $(0:0:1) \in C_f$ is a singular point. Only one discrete valuation ring lies over the local ring $R_{(0:0:1)}$, and this valuation ring has residue class field isomorphic to $\mathbb{Q}(\sqrt{2})$. A birational (projective) isomorphism between C_f and the projective line is given by

$$(X:Y:Z) = \Big((V(2U^2 - V^2)) : (U(2U^2 - V^2)) : (V^3 + U^3) \Big)\,,$$

but there is no $\mathbb{Q}$-rational point $(u:v) \in \mathbb{P}^1(\mathbb{Q})$ corresponding to the singular point $(0:0:1)$. Indeed, the corresponding point $(u:v) = (1:\sqrt{2})$ is only defined over $\mathbb{Q}(\sqrt{2})$.

Example 2. In contrast to the Pythagorean triples (corresponding to the unit circle, see [2]), we know that for the equilateral hyperbola the set $\mathcal{L}_f$ can be parametrized by a single triple of polynomials with integer coefficients. Let

$$f = XY - Z^2 \in \mathbb{Z}[X, Y, Z]\,.$$

All $\mathbb{Q}$-rational points of C_f are given by $(u^2 : v^2 : uv)$ with $(u : v) \in \mathbb{P}^1(\mathbb{Q})$. If $u, v \in \mathbb{Z}$ with $\gcd(u, v) = 1$ then also $\gcd(u^2, v^2, uv) = 1$. So the set of all integral solutions of $XY - Z^2 = 0$ is given by

$$\{(u^2w, v^2w, uvw) \mid u, v, w \in \mathbb{Z}\}\,.$$

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Addresses: Sophie Frisch, Institut für Mathematik A, Technische Universität Graz, Steyrergasse 30, A-8010 Graz, Austria
Günter Lettl, Institut für Mathematik und wissenschaftliches Rechnen, Karl-Franzens-Universität, Heinrichstraße 36, A-8010 Graz, Austria

E-mail: frisch@blah.math.tu-graz.ac.at, guenter.lettl@uni-graz.at

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