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ON THE REDUCED LENGTH OF A POLYNOMIAL WITH REAL COEFFICIENTS, II

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To Jean-Marc Deshouillers
at the occasion of his 60th birthday

Abstract: The length $L(P)$ of a polynomial P is the sum of the absolute values of the coefficients. For $P \in \mathbb{R}[x]$ the properties of $l(P)$ are studied, where $l(P)$ is the infimum of $L(PG)$ for G running through monic polynomials over $\mathbb{R}$.
Keywords: length of a polynomial, unit circle.

The aim of this paper is to complement for cubic polynomials the results of [2] and the notation of that paper is retained. Thus for

$$P(x) = \sum_{i=0}^{d} a_i x^{d-i} \in \mathbb{R}[x]$$

we set

$$L(P) = \sum_{i=0}^{d} |a_i|, \quad l(P) = \inf_{G \in \Gamma} L(PG), \quad \hat{l}(P) = \min\{l(P), l(P^*)\},$$

where

$$\Gamma = \Big\{ x^n + \sum_{i=1}^{n} b_i x^{n-i} : n \in \mathbb{N} \cup \{0\}, b_i \in \mathbb{R} \Big\}, \quad P^* = x^{\deg P} P(x^{-1}).$$

In the paper [2] it has been shown how to compute $l(P)$, if no zero of P is on the boundary of the unit disk, or all are inside the closed disk, or if there is just one zero, possibly multiple, on the boundary and all zeros outside the disk are real of the same sign. Now, we shall show

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Theorem 1. *Let* $P(x)=\prod_{i=1}^{3}(x-\alpha_i)$, *where* α_i *distinct,* $|\alpha_1|\geqslant|\alpha_2|>|\alpha_3|=1$. *Then* $l(P)$ *is effectively computable.*

Corollary. $\hat{l}(P)$ *can be effectively computed for every cubic polynomial* $P\in\mathbb{R}[x]$.

The corollary is of interest, since $\hat{l}(P)$, rather than $l(P)$ occurs in applications given in [1].

Theorem 2. *Let* $P(x)=(x-\alpha)(x^2-\varepsilon)$, *where* $|\alpha|>1$, $\varepsilon=\pm1$. *Then*

$$l(P)=2\big(|\alpha|+1-|\alpha|^{-1}\big).$$

The proof of Theorem 2 can be extended to cover the case $P(x)=\prod_{i=1}^{3}(x-\alpha_i)$, where $|\alpha_1|>1$ and α_2,α_3 are roots of unity. This however requires a tedious consideration of cases and gives no key to the case where α_2,α_3 lie on the unit circle and are not roots of unity, therefore we refrain from doing it here.

The proof of Theorem 1 is based on four lemmas, similar to Lemmas 4, 14, 15 and 16 of [2].

Definition 1. For $P(x)=\prod_{i=1}^{d}(x-\alpha_i)$, α_i distinct, $1\leqslant\delta\leqslant d$, $n_0>n_1>\ldots>n_\delta\geqslant 0$ we put

$$C_\nu(P;n_\nu,\ldots,n_{\delta-1+\nu})=\big(\alpha_i^{n_j}\big)_{\substack{1\leqslant i\leqslant d\\ \nu\leqslant j<\nu+\delta}}\qquad(\nu=0,1).$$

Definition 2. For $P(x)=\prod_{i=1}^{d}(x-\alpha_i)$, α_i distinct $\neq 0$, we put

$$\begin{aligned}T_d(P)=\Big\{Q=x^{n_0}+\sum_{j=1}^{\delta}b_jx^{n_j},\quad &1\leqslant\delta\leqslant d,\\ &n_0>n_1>\ldots>n_\delta=0,\quad\prod_{j=1}^{\delta}b_j\neq0,\ Q\equiv0\,(\operatorname{mod}P),\\ &\operatorname{rank}C_0(P;n_0,\ldots,n_{\delta-1})=\delta=\operatorname{rank}C_1(P;n_1,\ldots,n_\delta),\ L(Q)\leqslant L(P)\Big\}.\end{aligned}$$

Remark. Definition 1 is a special case of Definition 1 in [2], Definition 2 is a modification, in the special case, of Definition 2 in [2], see Corrigendum at the end of the paper.

Lemma 1. *For* P *as in Theorem* 1

$$l(P)=\inf_{Q\in T_3(P)}L(Q).$$

Proof. This is a special case of Lemma 4 in [3], the corrected version of [2]. ∎

Definition 3. For P as in Theorem 1, let m be the order of α_1/α_2, if α_1/α_2 is a root of unity, $m=0$ otherwise,

$$d_1 = \frac{\log(L(P)-1)}{\log|\alpha_1|}, \qquad D_1 = \min_{0<e\leqslant d_1,\, e\not\equiv 0 \,(\mathrm{mod}\, m)} |\alpha_1^e - \alpha_2^e|,$$
$$d_2 = \frac{\log 2D_1^{-1}(L(P)-1)}{\log|\alpha_2|}.$$

Lemma 2. *For every* $Q \in T_3(P)$, $Q = x^{n_0} + \sum_{j=1}^{\delta} b_j x^{n_j}$, $n_0 > \ldots > n_\delta = 0$, $\prod_{j=1}^{\delta} b_j \neq 0$, *we have* $\delta \geqslant 2$,

$$n_0 - n_1 \leqslant d_1, \tag{1}$$

and

$$\textit{either } n_0 \equiv n_1 \,(\mathrm{mod}\, m) \quad \textit{or } n_1 - n_2 \leqslant d_2. \tag{2}$$

Proof. $Q \equiv 0 \,(\mathrm{mod}\, P)$ implies

$$\alpha_i^{n_0} + \sum_{j=1}^{\delta} b_j \alpha_i^{n_j} = 0, \quad (1 \leqslant i \leqslant d) \tag{3}$$

hence $\delta \geqslant 2$ and

$$|\alpha_1|^{n_0} \leqslant |\alpha_1|^{n_1} \sum_{j=1}^{\delta} |b_j| \leqslant |\alpha_1|^{n_1}(L(P)-1),$$

which gives the inequality (1). Since α_1/α_2 is either not a root of unity or a root of unity of order m, we have $D_1 > 0$. Moreover

$$\alpha_2^{n_1}\left(\alpha_1^{n_0} + b_1\alpha_1^{n_1}\right) - \alpha_1^{n_1}\left(\alpha_2^{n_0} + b_1\alpha_2^{n_1}\right) = (\alpha_1\alpha_2)^{n_1}\left(\alpha_1^{n_0-n_1} - \alpha_2^{n_0-n_1}\right),$$

hence either $n_0 \equiv n_1 \,(\mathrm{mod}\, m)$ or

$$\max\left\{\left|\alpha_2^{n_1}\left(\alpha_1^{n_0} + b_1\alpha_1^{n_1}\right)\right|, \left|\alpha_1^{n_1}\left(\alpha_2^{n_0} + b_1\alpha_2^{n_1}\right)\right|\right\} \geqslant \frac{1}{2}\,|\alpha_1\alpha_2|^{n_1} D_1,$$

which gives

$$\max\left\{\left|\alpha_1^{n_0-n_1} + b_1\right|, \left|\alpha_2^{n_0-n_1} + b_1\right|\right\} \geqslant \frac{1}{2}\,D_1.$$

In the latter case we obtain from (3) for an $i \in \{1,2\}$

$$\frac{1}{2}\,D_1|\alpha_i|^{n_1} \leqslant \left|\sum_{j=2}^{\delta} b_j \alpha_i^{n_j}\right| \leqslant |\alpha_i|^{n_2}(L(P)-1),$$

which gives (2). ■

Lemma 3. *Assume that, in the notation of Definition* 3, $n_1 \not\equiv n_2 \,(\mathrm{mod}\, m)$ *and* $\alpha_3 = 1$. *If for* $n \in \mathbb{N}$, $a \in \mathbb{R}$,

$$n_2 > \frac{\log\bigl(12n\bigl|\alpha_1^{n_2-n_1} - \alpha_2^{n_2-n_1}\bigr|^{-1} \max\bigl\{1, \frac{2n|a|}{3n-1}\bigr\}\bigr)}{\log|\alpha_2|}, \tag{4}$$

then there exists a polynomial $R \in \mathbb{R}[x]$, $R(x) = r_1x^{n_1} + r_2x^{n_2} + r_3$ *such that*

$$(x-\alpha_1)(x-\alpha_2) \mid R(x) - a, \tag{5}$$

$$x - 1 \mid R(x), \tag{6}$$

$$L(R) < \frac{1}{n}. \tag{7}$$

Proof. Put $n_3 = 0$,

$$R(x) = \sum_{j=1}^{3} r_j x^{n_j}, \quad r_j \in \mathbb{C}.$$

Since $\alpha_1 \neq \alpha_2$, the conditions (5) and (6) are equivalent to the system of linear equations

$$\sum_{j=1}^{3} r_j \alpha_i^{n_j} = 0 \qquad (1 \leqslant i \leqslant 3). \tag{8}$$

The determinant of this system equals

$$\Delta_0 = (\alpha_1\alpha_2)^{n_2} \det\bigl(\alpha_i^{n_j-n_2}\bigr)_{1\leqslant i,j\leqslant 3}.$$

Developing the last determinant according to the last column we obtain

$$\begin{vmatrix} \alpha_1^{n_1-n_2} & 1 & \alpha_1^{-n_2} \\ \alpha_2^{n_1-n_2} & 1 & \alpha_2^{-n_2} \\ 1 & 1 & 1 \end{vmatrix}$$
$$= \alpha_1^{n_1-n_2} - \alpha_2^{n_1-n_2} - \alpha_2^{-n_2}\bigl(\alpha_1^{n_1-n_2} - 1\bigr) + \alpha_1^{-n_2}\bigl(\alpha_2^{n_1-n_2} - 1\bigr),$$

hence

$$\Bigl|\Delta_0(\alpha_1\alpha_2)^{-n_2} - \bigl(\alpha_1^{n_1-n_2} - \alpha_2^{n_1-n_2}\bigr)\Bigr| \leqslant 4|\alpha_2|^{-n_2}|\alpha_1|^{n_1-n_2}$$

and, by (4),

$$\bigl|\Delta_0 - \bigl(\alpha_1^{n_1}\alpha_2^{n_2} - \alpha_1^{n_2}\alpha_2^{n_1}\bigr)\bigr| \leqslant \frac{1}{3n}\bigl|\alpha_1^{n_1}\alpha_2^{n_2} - \alpha_1^{n_2}\alpha_2^{n_1}\bigr|,$$

thus

$$|\Delta_0| > \Bigl(1 - \frac{1}{3n}\Bigr)\bigl|\alpha_1^{n_1}\alpha_2^{n_2} - \alpha_1^{n_2}\alpha_2^{n_1}\bigr| > 0 \tag{9}$$

and the system (8) is uniquely solvable. Since on replacing α_i by their complex conjugates we obtain the same system, r_j are real.

The determinant Δ_k obtained by substituting in $\left(\alpha_i^{n_j}\right)_{1\leqslant i\leqslant 3}$ for the k-th column the column $[a,a,0]^t$ satisfies for $k<3$

$$\Delta_k = \pm\det\left(\alpha_i^{n_j}\right)_{\substack{i<3\\ j\neq k}} a,$$

hence, by (4),

$$\begin{aligned}|\Delta_k| = |a|\left|\alpha_1^{n_{3-k}} - \alpha_2^{n_{3-k}}\right| < 2|a|\,|\alpha_1|^{n_1} &\leqslant 2|a|\,|\alpha_1\alpha_2|^{n_1}|\alpha_2|^{-n_2} \\ &\leqslant \frac{3n-1}{12n^2}\left|\alpha_1^{n_1}\alpha_2^{n_2} - \alpha_1^{n_2}\alpha_2^{n_1}\right|.\end{aligned} \tag{10}$$

It follows from (9) and (10) that for $k<3$, $r_k = \Delta_k/\Delta_0$ satisfies

$$|r_k| < \frac{1}{4n}\,. \tag{11}$$

It remains to consider $k=3$. In this case

$$|\Delta_k| = |a|\left|\alpha_1^{n_2} + \alpha_2^{n_1} - \alpha_1^{n_1} - \alpha_2^{n_2}\right| \leqslant 4|a|\,|\alpha_1|^{n_1}$$

and we obtain similarly to (11)

$$|r_k| < \frac{1}{2n}\,.$$

It follows now from (11) that

$$L(R) = \sum_{j=1}^{3}|r_j| < \frac{1}{n}\,. \qquad \blacksquare$$

Lemma 4. *Assume, under the assumptions of Theorem* 1, *that* $\alpha_3 = 1$, $P_0 = (x-\alpha_1)(x-\alpha_2)$. *Then*

$$l(P) \leqslant \inf_{Q\in T_2(P_0)}\bigl(L(Q) + |Q(1)|\bigr).$$

Proof. Let

$$Q = x^{q_0} + \sum_{j=1}^{2} c_j x^{q_j},$$

where $q_0 > q_1 > q_2 = 0$, but not necessarily $\prod_{j=1}^{2} c_j \neq 0$.

If $q_1 \not\equiv 0 \pmod m$ and

$$n_2 > \frac{\log\bigl(12n|\alpha_1^{-q_1} - \alpha_2^{q_1}|^{-1} \max\bigl\{1, \frac{2n|Q(1)|}{3n-1}\bigr\}\bigr)}{\log|\alpha_2|},$$

by Lemma 3 with $a = Q(1)$, $n_j = n_2 + p_j$ $(1 \leqslant j < 3)$, there exists a polynomial $R \in \mathbb{R}[x]$, $R(x) = r_1 x^{n_1} + r_2 x^{n_2} + r_3$ satisfying (5)–(7). We consider the polynomial

$$S(x) = Q(x)x^{n_2} + R(x) - Q(1).$$

It follows from (5)–(6) that

$$P_0 \mid S, \quad x - 1 \mid S, \quad \text{thus } P \mid S$$

and, since S is monic,

$$l(P) \leqslant L(S).$$

On the other hand, by (7),

$$L(S) \leqslant L(Q) + |Q(1)| + \frac{1}{n}.$$

Since n is arbitrary, the lemma follows. ■

Proof of Theorem 1. The idea of the proof is to indicate for every n in $\mathbb{N}$ a finite set S_n of monic polynomials divisible by P such that $0 \geqslant l(P) - \min_{Q \in S_n} L(Q) > -\frac{1}{n}$. Since, by Proposition (iii) of [2], $l(P(-x)) = l(P(x))$, we may assume that $\alpha_3 = 1$. Consider first the case where α_1/α_2 is not a root of unity, hence $m = 0$. In order to prove the theorem in this case it suffices to show that for every $n \in \mathbb{N}$

$$\begin{aligned} 0 \geqslant l(P) - \min\bigl\{&\min{}^* L(Q(P; n_0, n_1, 0)), \min{}^{**} L(Q(P; n_0, n_1, n_2, 0)), \\ &\min{}^{**}\bigl(L(Q(P_0; n_0 - n_2, n_1 - n_2, 0)) + \bigl|Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1)\bigr|\bigr)\bigr\} \\ > -\frac{1}{n}, \end{aligned} \tag{12}$$

where the $\min^*$ is taken over all integers $n_0 > n_1 > 0$ satisfying the conditions (1) and (2) such that $x^{n_0} + b_1 x^{n_1} + b_2 \equiv 0 \pmod P$ and

$$\operatorname{rank} C_1(P; n_1, 0) = 2, \tag{13}$$

while the $\min^{**}$ is taken over all integers $n_0 > n_1 > n_2 > 0$ satisfying the conditions (1) and (2), such that

$$n_2 \leqslant \frac{\log\bigl(12n|\alpha_1^{n_2-n_1} - \alpha_2^{n_2-n_1}|^{-1} \max\bigl\{1, \frac{2n}{3n-1}\bigl|Q(P_0; n_0 - n_2, n_1 - n_2, 0)\bigr|\bigr\}\bigr)}{\log|\alpha_2|} \tag{14}$$

and

$$|C_1(P; n_1, n_2, 0)| \neq 0. \tag{15}$$

The condition (13) implies that there is at most one polynomial $Q = x^{n_0} + b_1 x^{n_1} + b_2$ divisible by P, denoted in (12) by $Q(P; n_0, n_1, 0)$; if such a polynomial does not exist for any pair $\langle n_0, n_1 \rangle$, then we take $\min^* = \infty$. The condition (15) implies that there is a unique polynomial $Q = x^{n_0} + \sum_{j=1}^{2} b_j x^{n_j} + b_3$ divisible by P, denoted in (12) by $Q(P; n_0, n_1, n_2, 0)$. Similarly, $Q(P_0; n_0 - n_2, n_1 - n_2, 0)$ is the unique polynomial

$$Q = x^{n_0 - n_2} + \sum_{j=1}^{2} b_j x^{n_j - n_2}$$

divisible by P (note that $\det(\alpha_i^{n_j - n_2})_{1 \leqslant i,j \leqslant 2} \neq 0$). The inequality

$$l(P) \leqslant \min\bigl\{\min^* L\bigl(Q(P; n_0, n_1, 0)\bigr), \min^{**} L\bigl(Q(P; n_0, n_1, n_2, 0)\bigr)\bigr\}$$

is clear and the inequality

$$l(P) \leqslant \min^{**}\bigl(L(Q(P_0, n_0 - n_2, n_1 - n_2, 0)) + |Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1)|\bigr)$$

follows from Lemma 4. This shows the first of inequalities (12).

In order to prove the second one we notice that by Lemmas 1 and 2

$$l(P) = \inf L\bigl(Q(P; n_0, \ldots, n_{\delta-1}, 0)\bigr), \tag{16}$$

where $\langle n_0, \ldots, n_{\delta-1} \rangle$ runs through all strictly decreasing sequences of $\delta \in \{2, 3\}$ positive integers satisfying (1)–(2),

$$Q(P; n_0, \ldots, n_{\delta-1}, 0) \neq 0 \tag{17}$$

and

$$\operatorname{rank} C_1(P; n_1, 0) = 2 \text{ if } \delta = 2, \qquad |C_1(P; n_1, n_2, 0)| \neq 0 \text{ if } \delta = 3. \tag{18}$$

Clearly

$$L\bigl(Q(P; n_0, n_1, 0)\bigr) \geqslant \min^* L\bigl(Q(P; n_0, n_1, 0)\bigr). \tag{19}$$

If $\delta = 3$ and (14) holds, then

$$L\bigl(Q(P; n_0, n_1, n_2, 0)\bigr) \geqslant \min^{**} L\bigl(Q(P; n_0, n_1, n_2, 0)\bigr) \tag{20}$$

and if not, then by Lemma 3 there exists a polynomial $R \in \mathbb{R}[x]$, $R(x) = r_1 x^{n_1} + r_2 x^{n_2} + r_3$ such that (5)–(7) hold with

$$a = Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1).$$

Then the polynomial

$$S(x) = Q(P_0; n_0 - n_2, n_1 - n_2, 0)x^{n_2} + R(x) - Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1)$$

is monic, satisfies

$$P \mid S(x),$$

and by (15)

$$S(x) = Q(P; n_0, n_1, n_2, 0). \tag{21}$$

By (7)

$$\begin{aligned} L(S) > L\bigl(Q(P_0; n_0 - n_2, n_1 - n_2, 0)\bigr) \\ + \bigl|Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1)\bigr| - \frac{1}{n}\,. \end{aligned} \tag{22}$$

The formulae (16)–(22) imply

$$\begin{aligned} &L\bigl(Q(P; n_0, \ldots, n_{\delta-1}, 0)\bigr) \\ &\quad > \min\bigl\{\min\nolimits^{*} L(Q(P; n_0, n_1, 0)), \min\nolimits^{**} L(Q(P; n_0, n_1, n_2, 0)), \\ &\min\nolimits^{**}\bigl(L(Q(P_0, n_0 - n_2, n_1 - n_2, 0)) + \bigl|Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1)\bigr|\bigr)\bigr\} - \frac{1}{n}\,, \end{aligned}$$

for all sequences $\langle n_0, \ldots, n_{\delta-1}\rangle$ satisfying (1), (2) and (18), hence by (16) the second of the inequalities (12) follows. The conditions (1), (2) and (14) are for a given n satisfied by only finitely many sequences $\langle n_0, \ldots, n_{\delta-1}\rangle$ $(2 \leqslant \delta \leqslant 3)$, since for $\delta = 3$

$$n_j - n_2 \leqslant \sum_{\mu=j+1}^{2} d_\mu$$

and for all such sequences satisfying (18) b_j can be effectively determined or shown not to exist (for $\delta = 2$), hence $l(P)$ can be effectively computed.

Consider now the case where α_1/α_2 is a root of unity of order m. Since $P \in \mathbb{R}[x]$, α_1, α_2 are either real or complex conjugate, thus we have either $\alpha_1 = \pm r$, $\alpha_2 = \mp r$, or

$$\alpha_1 = r \exp\frac{\pi i l}{m}\,, \quad \alpha_2 = r \exp\frac{-\pi i l}{m}\,, \quad (l, m) = 1, \quad m > 1, \tag{23}$$

where $r \in \mathbb{R}$, $r > 0$. We shall show that

$$\begin{aligned} 0 \geqslant l(P) - \min\Bigl\{ &\min\nolimits_1 L(Q(P; n_0, n_1, 0)), \min\nolimits_2 L(Q(P; n_0, n_1, 0)), \\ &\min\nolimits_2\bigl(L(Q(P_0; n_0 - n_2, n_1 - n_2, 0)) \\ &\qquad + \bigl|Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1)\bigr|\bigr), \\ &\min\nolimits_3 L(Q(P; n_0, n_1, n_2, 0)), \min\nolimits_4 L(Q(P; n_0, n_1, n_2, 0)), \\ &\min\nolimits_4\bigl(L(Q(P_0; n_0 - n_2, n_1 - n_2, 0)) \\ &\qquad + \bigl|Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1)\bigr|\bigr), 2r^m\Bigr\} > -\frac{1}{n}\,, \end{aligned} \tag{24}$$

where the $\min_1$ is taken over all integers $n_0 > n_1 > 0$ satisfying $n_0 - n_1 \leqslant d_1$, $n_0 \not\equiv n_1 \pmod{m}$, $n_1 \leqslant d_2$,

$$\operatorname{rank} C_1(P; n_1, 0) = 2, \tag{25}$$

the $\min_2$ is taken over all integers $n_0 > n_1 > n_2 > 0$ satisfying $n_0 - n_1 \leqslant d_1$, $n_0 \not\equiv n_1 \pmod{m}$, $n_1 - n_2 \leqslant d_2$, $n_1 \not\equiv n_2 \pmod{m}$,

$$|C_1(P; n_1, n_2, 0)| \neq 0, \tag{26}$$

the $\min_3$ is taken over all integers $n_0 > n_1 > n_2 > 0$ satisfying

$$n_0 \not\equiv n_1 \equiv n_2 \not\equiv 0 \pmod{m}, \tag{27}$$

$$n_0 \leqslant \frac{\log\Big((L(P)-1)\big|\sin \frac{\pi l(n_0-n_2)}{m}\big|^{-1}\Big)}{\log r}, \tag{28}$$

the $\min_4$ is taken over all integers $n_0 > n_1 > n_2 > 0$ satisfying

$$n_0 - n_1 \leqslant \frac{\log(L(P)-1)}{\log r}, \qquad n_0 \equiv n_1 \not\equiv n_2 \pmod{m},$$

$$n_1 - n_2 \leqslant \frac{\log\big(2n(L(P)-1)\big)}{\log r}, \tag{29}$$

$$|C_1(P; n_1, n_2, 0)| \neq 0. \tag{30}$$

The condition (25) warrants that $Q(P; n_0, n_1, 0)$ occurring under $\min_1$, if it exists, is determined uniquely, otherwise we take $\min_1 = \infty$, the condition (26) warrants that $Q(P; n_0, n_1, n_2, 0)$ occurring under $\min_2$ is determined uniquely, the condition $n_1 \not\equiv n_2 \pmod{m}$ warrants that $Q(P_0; n_0-n_2, n_1-n_2, 0)$ occurring under $\min_2$ is determined uniquely, the condition (27) warrants that $Q(P; n_0, n_1, n_2, 0)$ occurring under $\min_3$ is determined uniquely, the condition (30) warrants that $Q(P; n_0, n_1, n_2, 0)$ occurring under $\min_4$ is determined uniquely, the condition $n_1 \not\equiv n_2 \pmod{m}$ warrants that $Q(P_0; n_0 - n_2, n_1 - n_2, 0)$ occurring under $\min_4$ is determined uniquely. Clearly

$$\begin{aligned} l(P) \leqslant \min\big\{&\min\nolimits_1 L\big(Q(P; n_0, n_1, 0)\big), \min\nolimits_2 L\big(Q(P; n_0, n_1, n_2, 0)\big), \\ &\min\nolimits_3 L\big(Q(P; n_0, n_1, n_2, 0)\big), \min\nolimits_4 L\big(Q(P; n_0, n_1, n_2, 0)\big)\big\} \end{aligned}$$

and the inequality

$$\begin{aligned} l(P) \leqslant \min\big\{&\min\nolimits_2\big(L(Q(P_0; n_0-n_2, n_1-n_2, 0)) \\ &\qquad + \big|Q(P_0; n_0-n_2, n_1-n_2, 0)(1)\big|\big), \\ &\min\nolimits_4\big(L(Q(P_0; n_0-n_2, n_1-n_2, 0)) \\ &\qquad + \big|Q(P_0; n_0-n_2, n_1-n_2, 0)(1)\big|\big)\big\} \end{aligned}$$

follows from Lemma 4. Finally, $l(P) \leqslant 2r^m$, since by Proposition (i) and (iv) from [2]

$$l(P) \leqslant l\big((x^m - \alpha_1^m)(x^m-1)\big) = l\big((x-\alpha_1^m)(x-1)\big)$$

and by Theorem 6 from [2] the right-hand side is $2r^m$. This shows the first of the inequalities 24. In order to prove the second one we again use (16) and (18).

If $\delta = 2$ and $n_0 \not\equiv n_1 \pmod m$, then

$$L\big(Q(P;n_0,n_1,0)\big) \geqslant \min\nolimits_1 L\big(Q(P;n_0,n_1,0)\big).$$

If $\delta = 2$ and $n_0 \equiv n_1 \not\equiv 0 \pmod m$, then from

$$Q(P;n_0,n_1,0) = x^{n_0} + b_1 x^{n_1} + b_2 \tag{31}$$

we infer that

$$\alpha_i^{n_1}(\alpha_i^{n_0-n_1} + b_1) + b_2 = 0 \quad (i=1,2),$$

hence

$$\alpha_i^{n_0-n_1} + b_1 = b_2 = 0,$$

contrary to (17).

If $\delta = 2$ and $n_0 \equiv n_1 \equiv 0 \pmod m$, we infer from (31) that either

$$\alpha_1 = \pm r, \qquad \alpha_2 = \mp r, \qquad b_1 = -\frac{r^{n_0}-1}{r^{n_1}-1}, \qquad b_2 = -1 + \frac{r^{n_0}-1}{r^{n_1}-1} > 0,$$
$$L(Q) = 2|b_1| \geqslant 2r^{n_0-n_1} \geqslant 2r^m$$

or (23) holds and

$$b_1 = -(-1)^{l(n_0-n_1)/m}\,\frac{r^{n_0}-(-1)^{ln_0/m}}{r^{n_1}-(-1)^{ln_1/m}},$$
$$b_2 = -1 + (-1)^{l(n_0-n_1)/m}\,\frac{r^{n_0}-(-1)^{ln_0/m}}{r^{n_1}-(-1)^{ln_1/m}}.$$

If $l(n_0-n_1) \equiv 0 \pmod{2m}$, then $b_2 > 0$,

$$L(Q) = 2|b_1| \geqslant 2 \inf_{n_1 \geqslant m} \left\{ \frac{r^{n_1+m}-1}{r^{n_1}-1}, \frac{r^{n_1+2m}+1}{r^{n_1}+1} \right\} \geqslant 2r^m.$$

If $l(n_0-n_1) \equiv m \pmod{2m}$, then $b_2 < 0$,

$$L(Q) = 2(|b_1|+1) \geqslant 2\left(\frac{r^{2m}-1}{r^m+1} + 1\right) = 2r^m.$$

If $\delta = 3$, $n_0 \not\equiv n_1 \not\equiv n_2 \pmod m$ and (14) holds, then

$$L\big(Q(P;n_0,n_1,n_2,0)\big) \geqslant \min\nolimits_2 L\big(Q(P;n_0,n_1,n_2,0)\big).$$

If $\delta = 3,\ n_0 \not\equiv n_1 \not\equiv n_2 \,(\mathrm{mod}\, m)$ and (14) does not hold, then by Lemma 3 there exists a polynomial $R \in \mathbb{R}[x],\ R(x) = \sum_{j=1}^{2} r_j x^{n_j} + r_3$ such that (5)–(7) hold with

$$a = Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1).$$

Then the polynomial

$$S(x) = Q(P_0; n_0 - n_2, n_1 - n_2, 0)x^{n_2} + R(x) - Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1)$$

satisfies (21) and, by (7),

$$L\big(Q(P_0; n_0, n_1, n_2, 0)\big) > \min\nolimits_2\big(L(Q(P_0; n_0 - n_2, n_1 - n_2, 0)) + \big|Q(P_0; n_0 - n_2, n_1 - n_2, 0)(1)\big|\big) - \frac{1}{n}\,.$$

If $\delta = 3,\ n_0 \not\equiv n_1 \equiv n_2 \,(\mathrm{mod}\, m)$, then from

$$Q(P_0; n_0, n_1, n_2, 0) = x^{n_0} + b_1 x^{n_1} + b_2 x^{n_2} + b_3, \qquad \sum_{j=1}^{3} |b_j| \leqslant L(P) - 1$$

we infer that

$$\alpha_i^{n_0} + \alpha_i^{n_2}(b_1 \alpha_i^{n_1 - n_2} + b_2) + b_3 = 0 \qquad (i = 1, 2),$$

hence

$$n_2 \not\equiv 0 \,(\mathrm{mod}\, m), \qquad b_3 = \frac{\alpha_1^{n_0}\alpha_2^{n_2} - \alpha_1^{n_1}\alpha_2^{n_0}}{\alpha_1^{n_2} - \alpha_2^{n_2}}$$

and either $r^{n_0} \leqslant L(P) - 1$, if $\alpha_1 = -\alpha_2$, or by (23)

$$r^{n_0}\Big|\sin\frac{\pi(n_0 - n_2)l}{m}\Big| \leqslant (L(P) - 1)\Big|\sin\frac{\pi n_2 l}{m}\Big| \leqslant L(P) - 1,$$

which gives (27) and

$$L\big(Q(P; n_0, n_1, n_2, 0)\big) \geqslant \min\nolimits_3 L\big(Q(P; n_0, n_1, n_2, 0)\big).$$

If $\delta = 3,\ n_0 \equiv n_1 \not\equiv n_2 \,(\mathrm{mod}\, m)$ and (29) holds, then

$$L\big(Q(P; n_0, n_1, n_2, 0)\big) \geqslant \min\nolimits_4 L\big(Q(P; n_0, n_1, n_2, 0)\big).$$

If $\delta = 3,\ n_0 \equiv n_1 \not\equiv n_2 \,(\mathrm{mod}\, m)$ and (29) does not hold, then

$$\alpha_i^{n_1}\big(\alpha_i^{n_0 - n_1} + b_1\big) + b_2 \alpha_i^{n_2} + b_3 = 0 \qquad (i = 1, 2),$$

hence

$$|\alpha_i|^{n_1}|\alpha_i^{n_0-n_1}+b_1|\leqslant|\alpha_i|^{n_2}(L(P)-1);$$
$$|\alpha_i|^{n_1-n_2}|\alpha_i^{n_0-n_1}+b_1|\leqslant L(P)-1;$$
$$|\alpha_i^{n_0-n_1}+b_1|<\frac{1}{2n},$$
$$|b_1|>|\alpha_i^{n_0-n_1}|-\frac{1}{2n}\geqslant r^m-\frac{1}{2n},\qquad |b_3|=|1+b_1+b_2|\geqslant|b_1|-1-|b_2|.$$

It follows that

$$L\big(Q(P;n_0,n_1,n_2,0)\big)=1+\sum_{j=1}^{3}|b_j|\geqslant 2|b_1|>2r^m-\frac{1}{n}.$$

If $\delta=3,\ n_0\equiv n_1\equiv n_2\not\equiv 0\,(\operatorname{mod} m)$, then from

$$\alpha_i^{n_0}+b_1\alpha_i^{n_1}+b_2\alpha_i^{n_2}+b_3=0\qquad(i=1,2)$$

we infer that

$$\alpha_i^{n_0}+b_1\alpha_i^{n_1}+b_2=b_3=0,$$

contrary to (17).

Finally, if $\delta=3,\ n_0\equiv n_1\equiv n_2\equiv 0\,(\operatorname{mod} m)$, then $|C_0(P;n_0,n_1,n_2)|=0$, contrary to the definition of $T_3(P)$. ■

Proof of Corollary. If P has a multiple zero or $P(0)=0$, then $l(P)$ and $l(P^*)$ can be computed using Theorems 2, 4 or 5 of [2]. Otherwise, let

$$P=\prod_{i=1}^{3}(x-\alpha_i),\qquad \text{where}\ \ |\alpha_1|\geqslant|\alpha_2|\geqslant|\alpha_3|>0.$$

If $|\alpha_3|>1$, then $l(P)\geqslant|\alpha_1\alpha_2\alpha_3|$ by Proposition (ii) of [2] and $l(P^*)=|\alpha_1\alpha_2\alpha_3|$, hence $\hat{l}(P)=|\alpha_1\alpha_2\alpha_3|$.

If $|\alpha_2|>1=|\alpha_3|$, then $l(P)$ can be computed from Theorem 1 above and $l(P^*)=2|\alpha_1\alpha_2|$.

If $|\alpha_2|>1>|\alpha_3|$, then $l(P)$ can be computed from Theorem 2 of [2] and $l(P^*)=|\alpha_1\alpha_2|(1+|\alpha_3|)$.

If $|\alpha_1|>1=|\alpha_2|=|\alpha_3|$, then, by Theorem 6 of [2], $l(P)\geqslant 2|\alpha_1|$ and, by Theorem 4 of [2], $l(P^*)=2|\alpha_1|$, hence $\hat{l}(P)=2|\alpha_1|$.

If $|\alpha_1|>1=|\alpha_2|>|\alpha_3|$, then, by Theorem 6 of [2], $l(P)=2|\alpha_1|=l(P^*)$, hence $\hat{l}(P)=2|\alpha_1|$.

If $|\alpha_1|>1>|\alpha_2|\geqslant|\alpha_3|$, then $l(P)=1+|\alpha_1|$, $l(P^*)$ can be computed from Theorem 2 of [2].

If $|\alpha_1|=1=|\alpha_2|=|\alpha_3|$, then, by Theorem 4 of [2], $l(P)=2=l(P^*)$, hence $\hat{l}(P)=2$.

If $|\alpha_1| = 1 = |\alpha_2| > |\alpha_3|$, then $l(P) = 2$ and, by Theorem 6 of [2], $l(P^*) \geqslant 2$, hence $\hat{l}(P) = 2$.

If $|\alpha_1| = 1 > |\alpha_2| \geqslant |\alpha_3|$, then $l(P) = 2$, $l(P^*)$ can be computed from Theorem 1 above.

If $1 > |\alpha_1| \geqslant |\alpha_2| \geqslant |\alpha_3|$, then $l(P) = 1$ and $l(P^*) \geqslant 1$ by Proposition (ii) of [2], hence $\hat{l}(P) = 1$. ∎

Proof of Theorem 2. The idea of the proof is to estimate $L(Q)$ for Q in $T_3(P)$ with exponents of a given parity. By virtue of Proposition (iii) of [2] we may assume that $\alpha > 1$. Let, first, $Q \in T_3(P)$ have the form $x^{n_0} + b_1 x^{n_1} + b_2$, where $b_1 b_2 \neq 0$. If $n_0 \equiv 0$, $n_1 \equiv 1 \pmod 2$, from $Q \equiv 0 \pmod{x^2 - \varepsilon}$ we obtain $b_1 = 0$; if $n_0 \equiv 1$, $n_1 \equiv 0 \pmod 2$, then $1 = 0$; if $n_0 \equiv n_1 \equiv 1 \pmod 2$, then $b_2 = 0$, each time a contradiction.

Therefore, $n_0 \equiv n_1 \equiv 0 \pmod 2$ and it follows from $Q(x) \equiv 0 \pmod{x^2 - \varepsilon}$ that $\varepsilon^{n_0/2} + b_1 \varepsilon^{n_1/2} + b_2 = 0$ and from $Q(\alpha) = 0$ that

$$b_1 = -\frac{\alpha^{n_0} - \varepsilon^{n_0/2}}{\alpha^{n_1} - \varepsilon^{n_1/2}}, \qquad b_2 = -\varepsilon^{n_0/2} - b_1 \varepsilon^{n_1/2}.$$

If $\varepsilon^{n_0/2} = \varepsilon^{n_1/2}$, then $|b_2| = |b_1| - 1$ and

$$L(Q) = 2|b_1| \geqslant 2 \inf_{n_1 \geqslant 2} \min\left\{\frac{\alpha^{n_1+2} - 1}{\alpha^{n_1} - 1}, \frac{\alpha^{n_1+4} + 1}{\alpha^{n_1} + 1}\right\} \geqslant 2\alpha^2 > 2(\alpha + 1 - \alpha^{-1}).$$

If $\varepsilon^{n_0/2} = -\varepsilon^{n_1/2}$, then $|b_2| = |b_1| + 1$ and

$$L(Q) = 2(|b_1| + 1) \geqslant 2\Big(\frac{\alpha^4 - 1}{\alpha^2 + 1} + 1\Big) = 2\alpha^2 > 2(\alpha + 1 - \alpha^{-1}).$$

Now, let $Q \in T_3(P)$ have the form $x^{n_0} + b_1 x^{n_1} + b_2 x^{n_2} + b_3$, where $b_1 b_2 b_3 \neq 0$. If only one of the n_i is odd or all are odd, we obtain a contradiction as above. If all n_i are even, then $|C_0(P; n_0, n_1, n_2)| = 0$, contrary to the definition of $T_3(P)$. Therefore, we have three cases

$$n_i \equiv 0 \pmod 2, \qquad n_j \equiv 1 \pmod 2 \quad \text{for } j \neq i,\ i = 0, 1, 2.$$

If $i = 0$ we obtain from $Q \equiv 0 \pmod{x^2 - \varepsilon}$

$$b_2 = -b_1 \varepsilon^{(n_1 - n_2)/2}, \qquad b_3 = -\varepsilon^{n_0/2}$$

and from $Q(\alpha) = 0$

$$b_1 = -\frac{\alpha^{n_0} - \varepsilon^{n_0/2}}{\alpha^{n_1} - \varepsilon^{(n_1 - n_2)/2} \alpha^{n_2}},$$

thus

$$L(Q) - 2 = 2|b_1| \geqslant 2(\alpha - \alpha^{-1}).$$

If $\varepsilon^{(n_1-n_2)/2} = 1$, the inequality is clear, and in the opposite case we have $n_0 \geqslant 4$, hence

$$\alpha^{n_0} - 1 \geqslant \alpha^{n_0} - \alpha^{n_0-4} = (\alpha - \alpha^{-1})(\alpha^{n_0-1} + \alpha^{n_0-3}) \geqslant (\alpha - \alpha^{-1})(\alpha^{n_1} + \alpha^{n_2}).$$

If $i = 1$ we obtain from $Q \equiv 0 \,(\mathrm{mod}\, x^2 - \varepsilon)$

$$b_2 = -\varepsilon^{(n_0-n_2)/2}, \qquad b_3 = -b_1\varepsilon^{n_1/2}$$

and from $Q(\alpha) = 0$

$$b_1 = -\frac{\alpha^{n_0} - \varepsilon^{(n_0-n_2)/2}\alpha^{n_2}}{\alpha^{n_1} - \varepsilon^{n_1/2}},$$

thus

$$L(Q) - 2 = 2|b_1| > 2(\alpha - \alpha^{-1}).$$

If either $\varepsilon^{n_1/2} = 1$ or $\varepsilon^{(n_0-n_2)/2} = -1$, the inequality is clear, in the opposite case we have $n_0 - n_2 \geqslant 4$, hence

$$\alpha^{n_0} - \alpha^{n_2} \geqslant \alpha^{n_0} - \alpha^{n_0-4} \geqslant (\alpha - \alpha^{-1})(\alpha^{n_0-1} + 1) \geqslant (\alpha - \alpha^{-1})(\alpha^{n_1} + 1).$$

If $i = 2$ we obtain from $Q \equiv 0 \,(\mathrm{mod}\, x^2 - \varepsilon)$

$$b_1 = -\varepsilon^{(n_0-n_1)/2}, \qquad b_3 = -b_2\varepsilon^{n_2/2}$$

and from $Q(\alpha) = 0$

$$b_2 = -\frac{\alpha^{n_0} - \varepsilon^{(n_0-n_1)/2}\alpha^{n_1}}{\alpha^{n_2} - \varepsilon^{n_2/2}},$$

thus

$$L(Q) - 2 = 2|b_2| > 2(\alpha - \alpha^{-1}).$$

If either $\varepsilon^{n_1/2} = 1$ or $\varepsilon^{(n_0-n_1)/2} = -1$, the inequality is clear, in the opposite case $n_0 = n_1 \geqslant 4$, $n_2 \geqslant 2$, $n_0 \geqslant 7$, hence

$$\begin{aligned}\alpha^{n_0} - \alpha^{n_1} &\geqslant \alpha^{n_0} - \alpha^{n_0-4} = \alpha^{n_0-4}(\alpha^2 - 1)(\alpha^2 + 1)\\ &= \alpha^3(\alpha^2 - 1)(\alpha^{n_0-5} + \alpha^{n_0-7}) > (\alpha - \alpha^{-1})(\alpha^{n_2} + 1).\end{aligned}$$

Thus we have

$$l(P) = \inf_{Q \in T_3(P)} L(Q) \geqslant 2(\alpha + 1 - \alpha^{-1}).$$

On the other hand, taking for $\varepsilon = -1$

$$Q = x^4 - \frac{\alpha^4 - 1}{\alpha^3 + \alpha}\, x^3 - \frac{\alpha^4 - 1}{\alpha^3 + \alpha}\, x - 1$$

we obtain $Q \equiv 0 \,(\mathrm{mod}\, P)$, $L(Q) = 2(\alpha + 1 - \alpha^{-1})$, hence

$$l(P) = 2(\alpha + 1 - \alpha^{-1}). \tag{32}$$

If $\varepsilon = 1$, taking

$$Q_n = x^{2n+1} - \frac{\alpha^{2n+1} - \alpha^{2n-1}}{\alpha^{2n} - 1}\, x^{2n} - x^{2n-1} + \frac{\alpha^{2n+1} - \alpha^{2n-1}}{\alpha^{2n} - 1}$$

we obtain $Q_n \equiv 0 \,(\mathrm{mod}\, P)$, $\lim_{n\to\infty} L(Q_n) = 2(\alpha + 1 - \alpha^{-1})$, hence (32) holds again. ■

In connection with Theorem 2 we propose the following

Problem. Is the inequality $l((x^2+tx+1)P(x)) \geqslant 2M(P)$ where $M(P)$, is the Mahler measure, true for all $t \in [-2,2]$ and all $P \in \mathbb{R}[x]$?

Corrections to [2]. Definition 2 on p. 278 should be modified as follows:

$$T_d(P) = \Big\{Q \in S_d(P) : Q = x^{n_0} + \sum_{j=1}^{\delta} b_j x^{n_j},\ \text{where } n_0 > n_1 > \ldots > n_\delta = 0,$$
$$\prod_{j=1}^{\delta} b_j \neq 0, \delta \leqslant d, \operatorname{rank} C_0(P; n_0, \ldots, n_{\delta-1}) = \delta = \operatorname{rank} C_1(P; n_1, \ldots, n_\delta),$$
$$L(Q) \leqslant L(P)\Big\}.$$

This requires a change in the proof of Lemma 4 and many small modifications in the proofs of Theorems 2, 5 and 6. These corrections are all made in [3].

Moreover, the following mistakes should be corrected:

p. 280, line 3: for '$L(Q_0) =$ ' read '$L(Q_0) = 1 +$ '
p. 282, line -5: for 'Definition 1' read 'Definition 4'
p. 294, line -10 and p. 299, lines 7–8: replace 'of degree at most n_1' by
'$R(x) = \sum_{j=1}^{d-1} r_j x^{n_j} + r_d$'
p. 298, formula (61): for '$\overset{*}{\min}$' read '$\min^*$'
formula (63): replace the right-hand side by
'$\max\{n_1 - n_{d-1}, \psi(\ldots)\}$'
line 10: for 'n_d' read 'n_{d-1}'
p. 299, line -5: for '(64)' read '(62) and (64)'
line -3: for 'd_j' read 'd_μ'
line -2: for 'all such sequences' read 'all such sequences b_j'

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